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On conformally flat minimal Legendrian submanifolds in the unit sphere

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This paper is concerned with the study on an open problem of classifying conformally flat minimal Legendrian submanifolds in the $(2n+1)$-dimensional unit sphere $\mathbb{S}^{2n+1}$ admitting a Sasakian structure (φ, ξ, η, g) for $n \geqslant 3$, motivated by the classification of minimal Legendrian submanifolds with constant sectional curvature. First of all, we completely classify such Legendrian submanifolds by assuming that the tensor $K := -\varphi h$ is semi-parallel, which is introduced as a natural extension of C-parallel second fundamental form h. Secondly, such submanifolds have also been determined under the condition that the Ricci tensor is semi-parallel, generalizing the Einstein condition. Finally, as direct consequences, new characterizations of the Calabi torus are presented.

Keywords: Legendrian submanifold; unit sphere; conformally flat; semi-parallel; product manifold; rigidity theorem

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1. Introduction

The study on both extrinsic and intrinsic geometry of submanifolds in the unit sphere is always an interesting topic, in which the classification research under suitable geometric conditions plays a significant role and has attracted many geometers. It is well known that, as a real hypersurface of the complex Euclidean space $\mathbb{C}^{n+1}$, the unit sphere $\mathbb{S}^{2n+1}$ of dimension $(2n+1)$ naturally admits a Sasakian structure

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(φ, ξ, η, g) (cf. [**30**]). Moreover, an m-dimensional submanifold M^m in $\mathbb{S}^{2n+1}$ is said to be *C-totally real* (or equivalently, *integral*) if the contact form η of $\mathbb{S}^{2n+1}$ vanishes when it is restricted to M^m, namely $\eta(X) = 0$ for any $X \in TM^m$. In particular, we call a C-totally real submanifold M^m *Legendrian* if it meets the smallest possible codimension, that is to say, $m = n$ (cf. [**33**]), and related to the classification of such submanifolds in $\mathbb{S}^{2n+1}$, many results have been established in the last few decades, see e.g. [**2**, **9**, **16**–**19**, **23**, **24**, **27**–**29**, **31**, **36**].

Recently, sharpening a theorem of Yamaguchi–Kon–Ikawa [**34**], Cheng–He–Hu [**7**] gave a complete classification of all the n-dimensional minimal Legendrian submanifolds in Sasakian space forms with constant sectional curvature, from which it follows that

THEOREM 1.1 cf. [**7**]. *Let M^n $(n \geqslant 2)$ be a minimal Legendrian submanifold in the unit sphere $\mathbb{S}^{2n+1}$ with constant sectional curvature. Then, either M^n is the totally geodesic sphere, or M^n is the flat Clifford torus.*

Regarding theorem 1.1, we see that a Legendrian submanifold M^n in the unit sphere $\mathbb{S}^{2n+1}$ is called *minimal* if its mean curvature H vanishes identically, while the Clifford torus is given by the immersion $T^n \to \mathbb{S}^{2n+1}$ with the parameterization as in (1.2) of [**7**], where $T^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ and $\mathbb{S}^1$ is a circle of radius 1. It was shown by direct calculation that the Clifford torus is a minimal Legendrian submanifold with flat induced metric.

It is worth mentioning that various attempts to generalize the above theorem have been made by geometers under suitable extrinsic and intrinsic conditions. For instance, except the examples in theorem 1.1, Xing–Zhai [**33**] obtained new ones constructed by the Calabi product (cf. [**22**]), when classifying n-dimensional minimal Legendrian submanifolds with C-parallel second fundamental form in cases $n = 3, 4$, where the second fundamental form h of $M^n \to \mathbb{S}^{2n+1}$ is called *C-parallel* if it satisfies $\bar{\nabla}^{\xi} h = 0$ on M^n. Moreover, compact minimal Legendrian submanifolds in the unit sphere $\mathbb{S}^{2n+1}$ with non-negative sectional curvature have been studied by Dillen–Vrancken [**8**] for $n = 3$ and Zhai–Zhang [**38**] for $n = 4$, and were completely classified by Cheng–Hu [**5**] for $n \geqslant 5$. Meanwhile, a condition on the Ricci tensor Ric called *parallel* Ricci tensor, i.e. $\nabla \mathrm{Ric} = 0$ with ∇ being the Levi-Civita connection, has been applied by Hu–Li–Xing [**12**] to successfully classify natural subclasses of such minimal Legendrian submanifolds in $\mathbb{S}^{2n+1}$ for $n = 3, 4$, where it is clear that the parallel Ricci tensor is a natural extension of the Einstein condition, i.e. $\mathrm{Ric} = \kappa g$ with κ a constant and g the induced metric. In particular, during this process, there is always an open problem that can be stated as follows:

Problem. Classify conformally flat minimal Legendrian submanifolds in the unit sphere $\mathbb{S}^{2n+1}$ for $n \geqslant 3$.

Some facts about the above problem are as follows. First of all, a Riemannian manifold (M^n, g) is said to be *conformally flat* if there exists a coordinate chart $\{(U_\alpha, \phi_\alpha); \alpha \in \Lambda\}$ covering M^n such that $(\phi_\alpha^{-1})^* g = \rho_\alpha \mathrm{d}s^2$ for each $\alpha \in \Lambda$, where $\mathrm{d}s^2$ denotes the Euclidean metric on $\mathbb{R}^n$ and ρ_α is a positive function defined on $\mathbb{R}^n$. It is well known that a Riemannian surface is always conformally flat. In higher dimensions, (M^n, g) of dimension $n \geqslant 4$ is conformally flat if and only if its Weyl curvature tensor vanishes identically, while (M^3, g) is conformally flat if and only if its Schouten tensor is a Codazzi tensor, where the Weyl curvature tensor of M^3 vanishes automatically.

Next, as a conformally flat minimal Legendrian submanifold in the unit sphere $\mathbb{S}^{2n+1}$ with non-constant sectional curvature, the Calabi torus was characterized on the pinching conditions of the sectional curvature by Dillen–Vrancken [**8**], the Ricci tensor by Hu–Xing [**14**], and the scalar curvature by Luo–Sun [**25**], Luo–Sun–Yin [**26**], respectively.

Example 1.2. The Calabi torus in the unit sphere $\mathbb{S}^{2n+1}$ (cf. [**26**]).

Let $\gamma = (\gamma_1, \gamma_2) : \mathbb{S}^1 \to \mathbb{S}^3 \subset \mathbb{C}^2$ be a Legendrian curve, defined by

$$\gamma(t) = \left(\sqrt{\frac{n}{n+1}}\mathrm{e}^{i(1/\sqrt{n})t}, \sqrt{\frac{1}{n+1}}\mathrm{e}^{-i\sqrt{n}t}\right), \tag{1.1}$$

and $\phi : \mathbb{S}^{n-1} \to \mathbb{S}^{2n-1} \subset \mathbb{C}^n$ the totally geodesic Legendrian sphere for $n \geqslant 3$. Then

$$f(t, y) = (\gamma_1\phi, \gamma_2) : \mathbb{S}^1 \times \mathbb{S}^{n-1} \to \mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1} \tag{1.2}$$

is a minimal Legendrian immersion and $f(\mathbb{S}^1 \times \mathbb{S}^{n-1})$ is called the *Calabi torus*.

In this paper, towards the above problem, we study the classification of conformally flat minimal Legendrian submanifolds in $\mathbb{S}^{2n+1}$ under some suitable extrinsic and intrinsic conditions. As the first of our main results, motivated by above results, we completely classify such submanifolds with *semi-parallel* tensor K, namely $R \cdot K = 0$ with R being the Riemannian curvature tensor, as a generalization of C-parallel second fundamental form, where $K : TM^n \times TM^n \to TM^n$ is a $(1, 2)$-tensor defined by $K := -\varphi h$ satisfying $h(X, Y) = \varphi K(X, Y)$ for any $X, Y \in TM^n$ (see § 2.3 for details).

Theorem 1.3. *Let M^n $(n \geqslant 3)$ be a conformally flat minimal Legendrian submanifold in the unit sphere $\mathbb{S}^{2n+1}$. If M^n is of semi-parallel tensor K, then it is locally congruent to one of the following three examples:*

(a) *M^n is the totally geodesic sphere;*

(b) *M^n is the flat Clifford torus;*

(c) *M^n is the Calabi torus.*

Remark 1.4. In order to generalize theorem 1.1, Hu–Li–Xing [**12**] investigated minimal Legendrian submanifolds in $\mathbb{S}^{2n+1}$ with Einstein-induced metric and verified that each of such submanifolds must be of constant sectional curvature in case of $n = 4$.

Recall that, for an n-dimensional Riemannian manifold (M^n, g), the traceless Ricci tensor $\widetilde{\mathrm{Ric}}$ of M^n is defined by $\widetilde{\mathrm{Ric}}(X, Y) = \mathrm{Ric}(X, Y) - (n-1)\chi g(X, Y)$ for $X, Y \in TM^n$, where Ric and χ are the Ricci tensor and normalized scalar curvature of M^n, respectively. Let $\|\widetilde{\mathrm{Ric}}\|$ be the tensorial norm of $\widetilde{\mathrm{Ric}}$ with respect to the metric g. Applying theorem 1.3, we can prove the following two rigidity theorems.

COROLLARY 1.5. *Let M^n $(n \geqslant 3)$ be a minimal Legendrian submanifold in the unit sphere $\mathbb{S}^{2n+1}$ with semi-parallel tensor K. Then, for its traceless Ricci tensor $\tilde{\mathrm{Ric}}$, it holds the pointwise inequality:*

$$\|\tilde{\mathrm{Ric}}\|^2 \leqslant \frac{(n-2)(n+1)}{n+2} S\chi, \tag{1.3}$$

where S and χ are the squared norm of the second fundamental form and the normalized scalar curvature of M^n, respectively. Moreover, the equality in (1.3) holds identically if and only if M^n is locally congruent to one of the examples (a)–(c) as in theorem 1.3.

COROLLARY 1.6. *Let M^n $(n \geqslant 3)$ be a closed minimal Legendrian submanifold in the unit sphere $\mathbb{S}^{2n+1}$ with vanishing Weyl curvature tensor. Then, for its traceless Ricci tensor $\tilde{\mathrm{Ric}}$, it holds the integral inequality:*

$$\int_{M^n} \|\tilde{\mathrm{Ric}}\|^2 \,\mathrm{d}V_{M^n} \geqslant \frac{(n-2)(n+1)}{n+2} \int_{M^n} S\chi \,\mathrm{d}V_{M^n}, \tag{1.4}$$

where S and χ are the squared norm of the second fundamental form and the normalized scalar curvature of M^n, respectively. Moreover, the equality in (1.4) holds if and only if M^n is locally congruent to one of the examples (a)–(c) as in theorem 1.3.

On the contrary, for the Riemannian manifold (M^n, g), its Ricci tensor Ric is said to be *semi-parallel* if and only if $R \cdot \mathrm{Ric} = 0$ on M^n. This condition is obviously weaker than that of parallel Ricci tensor, as stated above. Then, the second main result of this paper can be given by

THEOREM 1.7. *Let M^n $(n \geqslant 3)$ be a conformally flat minimal Legendrian submanifold in the unit sphere $\mathbb{S}^{2n+1}$. If M^n is of semi-parallel Ricci tensor, then it is locally congruent to one of the following three cases:*

(a) *M^n is the totally geodesic sphere;*

(b) *M^n is the flat Clifford torus;*

(c) *M^n is the Calabi torus.*

REMARK 1.8. By means of theorem 1.7 and the calculations given in § 3, we see that, for the conformally flat minimal Legendrian submanifolds in $\mathbb{S}^{2n+1}$, the Ricci tensor Ric is semi-parallel if and only if it is parallel, although such equivalence does not hold for general Riemannian manifolds.

Finally, we further prove the following result, by which we can complete the proofs of theorems 1.3 and 1.7, respectively.

THEOREM 1.9. *Let M^n $(n \geqslant 3)$ be a minimal Legendrian submanifold in the unit sphere $\mathbb{S}^{2n+1}$. If (M^n, g) is locally isometric to a Riemannian product $I \times M_2 =$*

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$(I \times M_2,\, \mathrm{d}t^2 \oplus g_2)$, where $I \subset \mathbb{R}$, g is the induced metric on M^n and $(M_2,\, g_2)$ has constant sectional curvature $c \neq 0$, then M^n is locally congruent to the Calabi torus.

The outline of this paper is as follows: in § 2, we give a brief review of the local theory of Legendrian submanifolds in the unit sphere $\mathbb{S}^{2n+1}$ with the Sasakian structure $(\varphi,\, \xi,\, \eta,\, g)$, and then collect necessary material on the conformally flat structure of Riemannian manifolds. For better illustrating our results, we compute in § 3 the invariants of Calabi torus with details. Section 4 is dedicated to studying the properties of Legendrian submanifolds under some certain geometric conditions. In § 5, applying these properties, we complete the proofs of theorems 1.3–1.9 and corollaries 1.5 and 1.6.

2. Preliminaries

In this section, we first collect some necessary material on Sasakian structure $(\varphi,\, \xi,\, \eta,\, g)$ of the unit sphere $\mathbb{S}^{2n+1}$ that can be regarded as a Sasakian space form with constant φ-sectional curvature 1. Then, we briefly review the local theory of Legendrian submanifolds in $\mathbb{S}^{2n+1}$. Finally, some basic notions and facts relative to conformally flat structure of Riemannian manifolds are presented for later use. For more details, we refer to [**12**, **17**, **32**] and the monographs [**3**, **35**].

2.1. Sasakian structure $(\varphi,\, \xi,\, \eta,\, g)$ of the unit sphere $\mathbb{S}^{2n+1}$

As a real hypersurface of the complex Euclidean space $\mathbb{C}^{n+1}$ with canonical complex structure J, the $(2n+1)$-dimensional unit sphere $\mathbb{S}^{2n+1}$ naturally admits a Sasakian structure $(\varphi,\, \xi,\, \eta,\, g)$: $\xi = J\bar{N}$ is the structure vector field with the unit normal vector field $\bar{N}$ of the inclusion $\mathbb{S}^{2n+1} \hookrightarrow \mathbb{C}^{n+1}$; g is the induced metric on $\mathbb{S}^{2n+1}$; $\eta(X) = g(X,\, \xi)$ and $\varphi X = JX - \langle JX,\, \bar{N}\rangle \bar{N}$ for any tangent vector field X on $\mathbb{S}^{2n+1}$, where $\langle\cdot,\, \cdot\rangle$ denotes the standard Hermitian metric on $\mathbb{C}^{n+1}$. In particular, for any tangent vector fields X, Y on $\mathbb{S}^{2n+1}$, the Sasakian structure $(\varphi,\, \xi,\, \eta,\, g)$ of $\mathbb{S}^{2n+1}$ satisfies the following properties:

$$\begin{cases} g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \\ \varphi\xi = 0, \quad \eta(\varphi X) = 0, \quad \operatorname{rank}(\varphi) = 2n, \\ \varphi^2 X = -X + \eta(X)\xi, \quad \mathrm{d}\eta(X, Y) = g(X, \varphi Y), \\ \bar{\nabla}_X \xi = -\varphi X, \quad (\bar{\nabla}_X \varphi)Y = g(X, Y)\xi - \eta(Y)X, \end{cases} \tag{2.1}$$

where $\bar{\nabla}$ is the Levi-Civita connection with respect to the induced metric g on $\mathbb{S}^{2n+1}$.

2.2. Local theory of Legendrian submanifolds in the unit sphere $\mathbb{S}^{2n+1}$

Let M^n be a Legendrian submanifold in the unit sphere $\mathbb{S}^{2n+1}$, i.e. the contact form η restricted to M^n vanishes. Consequently, ξ is a normal vector field over M^n. Denote by N a unit normal vector field along M^n, and by U, X, Y, Z the tangent vector fields on M^n in the subsequent paragraphs. Then, we have the Gauss and

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Weingarten formulas:

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad \bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{2.2}$$

where ∇ is the Levi-Civita connection of the induced metric on M^n, still denoted by g, h (resp. A_N) is the second fundamental form (resp. the shape operator with respect to N) of $M^n \to \mathbb{S}^{2n+1}$, and $\nabla^{\perp}$ is the normal connection in the normal bundle $T^{\perp}M^n$. Then, by means of (2.2) it can be verified that

$$g(h(X,Y),N) = g(A_N X, Y). \tag{2.3}$$

Note from the facts $\eta(X) = 0$ and $\mathrm{d}\eta(X,\, Y) = g(X,\, \varphi Y)$ that φ maps the tangent vector fields of M^n to the normal vector fields in $T^{\perp}M^n$. Applying (2.2), we further have

$$\nabla^{\perp}_X \varphi Y = \varphi \nabla_X Y + g(X,Y)\xi, \quad A_{\varphi X} Y = -\varphi h(X,Y) = A_{\varphi Y} X, \tag{2.4}$$

and thus $g(h(X,\, Y),\, \varphi Z)$ is totally symmetric in X, Y, and Z:

$$g(h(X,Y),\varphi Z) = g(h(X,Z),\varphi Y) = g(h(Y,Z),\varphi X). \tag{2.5}$$

It follows from (2.1), (2.3), and the Weingarten formula that

$$g(h(X,Y),\xi) = g(A_\xi X, Y) = 0. \tag{2.6}$$

Moreover, the equations of Gauss, Ricci, and Codazzi are respectively given by

$$R(X,Y)Z = g(Y,Z)X - g(X,Z)Y + [A_{\varphi X}, A_{\varphi Y}]Z, \tag{2.7}$$

$$R^{\perp}(X,Y)\varphi Z = \varphi[A_{\varphi X}, A_{\varphi Y}]Z, \tag{2.8}$$

$$(\bar{\nabla} h)(X,Y,Z) = (\bar{\nabla} h)(Y,X,Z), \tag{2.9}$$

where, by definitions:

$$\begin{aligned}
&[A_{\varphi X}, A_{\varphi Y}] = A_{\varphi X}A_{\varphi Y} - A_{\varphi Y}A_{\varphi X},\\
&R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,\\
&R^{\perp}(X,Y)\varphi Z = \nabla^{\perp}_X\nabla^{\perp}_Y\varphi Z - \nabla^{\perp}_Y\nabla^{\perp}_X\varphi Z - \nabla^{\perp}_{[X,Y]}\varphi Z,\\
&(\bar{\nabla} h)(X,Y,Z) = \nabla^{\perp}_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).
\end{aligned}$$

From now on, we assume that M^n is a minimal Legendrian submanifold in the sphere $\mathbb{S}^{2n+1}$, unless otherwise stated. Contracting the Gauss equation (2.7) twice,

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we have

$$n(n-1)\chi = n(n-1) - S, \quad S = \|h\|^2, \tag{2.10}$$

where χ is the normalized scalar curvature and $\|\cdot\|^2$ denotes the squared norm relative to the metric g. Furthermore, the Ricci identity reads:

$$\begin{aligned}&(\bar\nabla^2 h)(U,X,Y,Z) - (\bar\nabla^2 h)(X,U,Y,Z) = (\bar R\cdot h)(U,X,Y,Z)\\ &= R^\perp(U,X)h(Y,Z) - h(R(U,X)Y,Z) - h(Y,R(U,X)Z),\end{aligned} \tag{2.11}$$

where $\bar R$ is the curvature tensor of the Van der Waerden–Bortolotti connection and

$$\begin{aligned}(\bar\nabla^2 h)(U,X,Y,Z) =& \nabla^\perp_U((\bar\nabla h)(X,Y,Z)) - (\bar\nabla h)(\nabla_U X,Y,Z)\\ &- (\bar\nabla h)(X,\nabla_U Y,Z) - (\bar\nabla h)(X,Y,\nabla_U Z).\end{aligned} \tag{2.12}$$

As usual, the so-called local *Legendre frame* $\{e_1,\,\dots,\,e_n,\,e_{1^*},\,\dots,\,e_{n^*},\,e_{2n+1}\}$ along M^n can be chosen such that, restricted to M^n, the vector fields $e_1,\,e_2,\,\dots,\,e_n$ are orthonormal and tangent to M^n, whereas $\{e_{1^*} = \varphi e_1,\,\dots,\,e_{n^*} = \varphi e_n,\,e_{2n+1} = \xi\}$ are the orthonormal normal vector fields of M^n in $\mathbb{S}^{2n+1}$. In the sequel, we shall make the following convention on range of indices:

$$\begin{gathered} i,j,k,\ell,m,p = 1,\dots,n; \quad \alpha,\beta = 1,\dots,n+1,\\ i^*,j^*,k^*,\ell^*,m^*,p^* = n+1,\dots,2n; \quad \alpha^* = \alpha+n, \beta^* = \beta+n.\end{gathered}$$

Set $h^{k^*}_{ij} = g(h(e_i,\,e_j),\,\varphi e_k)$ and $h^{2n+1}_{ij} = g(h(e_i,\,e_j),\,e_{2n+1})$. Denote by $h^{\alpha^*}_{ij,\ell}$ and $h^{\alpha^*}_{ij,\ell m}$ the first and the second covariant derivatives of $h^{\alpha^*}_{ij}$ with respect to $\bar\nabla$, respectively. Let $R_{ijk\ell} = g(R(e_i,\,e_j)e_\ell,\,e_k)$ and $R_{ij\alpha^*\beta^*} = g(R^\perp(e_i,\,e_j)e_{\beta^*},\,e_{\alpha^*})$ be the components of the curvature tensors of ∇ and $\nabla^\perp$. Denote by $R_{ij} = \sum_k g(R(e_i,\,e_k)e_k,\,e_j)$ the components of the Ricci tensor of g. As M^n is minimal in $\mathbb{S}^{2n+1}$, it is known from (2.5)–(2.9) that

$$R_{ijk\ell} = \delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk} + \sum_m (h^{m^*}_{ik}h^{m^*}_{j\ell} - h^{m^*}_{i\ell}h^{m^*}_{jk}), \quad R_{ijk^*(2n+1)} = 0, \tag{2.13}$$

$$R_{ij} = (n-1)\delta_{ij} - \sum_{k,\ell} h^{\ell^*}_{ik}h^{\ell^*}_{jk}, \quad R_{ijk^*\ell^*} = \sum_m (h^{m^*}_{ik}h^{m^*}_{j\ell} - h^{m^*}_{i\ell}h^{m^*}_{jk}), \tag{2.14}$$

$$h^{k^*}_{ij} = h^{j^*}_{ik} = h^{i^*}_{kj}, \quad h^{2n+1}_{ij} = 0, \quad h^{\alpha^*}_{ij,k} = h^{\alpha^*}_{ik,j}, \sum_i h^{\alpha^*}_{ii} = 0. \tag{2.15}$$

In this situation, the Ricci identity (2.11) can be rewritten as follows:

$$h^{\alpha^*}_{ij,\ell p} - h^{\alpha^*}_{ij,p\ell} = \sum_m h^{\alpha^*}_{mj}R_{mi\ell p} + \sum_m h^{\alpha^*}_{im}R_{mj\ell p} + \sum_\beta h^{\beta^*}_{ij}R_{\ell p\beta^*\alpha^*}. \tag{2.16}$$

Finally, the following uniqueness theorem for the Legendrian submanifolds in the unit sphere $\mathbb{S}^{2n+1}$ is also needed.

THEOREM 2.1 (cf. [12]). *Let f and $\bar{f} : M^n \to \mathbb{S}^{2n+1}$ be two Legendrian immersions of a connected manifold M^n into the unit sphere $\mathbb{S}^{2n+1}$ with the second fundamental forms h and $\bar{h}$, respectively. Assume that*

$$g(f_*X, f_*Y) = g(\bar{f}_*X, \bar{f}_*Y), \quad g(h(X,Y), \varphi f_*Z) = g(\bar{h}(X,Y), \varphi \bar{f}_*Z), \tag{2.17}$$

where X, Y, Z are any tangent vector fields on M^n. Then there exists an isometry τ of $\mathbb{S}^{2n+1}$ such that $f = \tau \circ \bar{f}$.

2.3. Equivalent properties of parallel or semi-parallel tensor K

Recall that, for the Legendrian submanifold M^n in the unit sphere $\mathbb{S}^{2n+1}$, its second fundamental form h is said to be *parallel* if $\bar{\nabla} h = 0$ on M^n, and as a direct generalization, h is said to be *semi-parallel* if $\bar{R} \cdot h = 0$ on M^n, where $\bar{R}$ denotes the curvature tensor corresponding to the Van der Waerden–Bortolotti connection. For the latter one, it is easy to see from the Ricci identity (2.11) that, for tangent vector fields U, X, Y, Z on M^n,

$$(\bar{\nabla}^2 h)(U, X, Y, Z) = (\bar{\nabla}^2 h)(X, U, Y, Z), \tag{2.18}$$

which, under the Legendre frame $\{e_1, \ldots, e_n, e_{1^*}, \ldots, e_{n^*}, e_{2n+1}\}$ on M^n, is equivalent to

$$h^{\alpha^*}_{ij,\ell p} = h^{\alpha^*}_{ij,p\ell}, \tag{2.19}$$

where $1 \leqslant i,\, j,\, \ell,\, p \leqslant n$ and $\alpha^* = \alpha + n$ for $1 \leqslant \alpha \leqslant n+1$.

In addition, associated with $\bar{\nabla}$ and ξ, a covariant differentiation $\bar{\nabla}^\xi$ can be defined such that it acts on h as

$$(\bar{\nabla}^\xi h)(X, Y, Z) = (\bar{\nabla} h)(X, Y, Z) - g(h(Y, Z), \varphi X)\xi. \tag{2.20}$$

Under the Legendre frame on M^n, setting $(\bar{\nabla}^\xi h)(e_k, e_i, e_j) = \sum_\ell \tilde{h}^{\ell^*}_{ij,k} e_{\ell^*}$ and

$$(\bar{\nabla}^\xi_{e_p}(\bar{\nabla}^\xi h))(e_\ell, e_i, e_j) = ((\bar{\nabla}^\xi)^2 h)(e_p, e_\ell, e_i, e_j) = \sum_k \tilde{h}^{k^*}_{ij,\ell p} e_{k^*}, \tag{2.21}$$

we obtain the following relations (cf. [17]):

$$\tilde{h}^{\ell^*}_{ij,k} = h^{\ell^*}_{ij,k}, \quad h^{k^*}_{ij,\ell p} = \tilde{h}^{k^*}_{ij,\ell p} - h^{\ell^*}_{ij} \delta_{kp}, \quad h^{2n+1}_{ij,\ell p} = 2h^{\ell^*}_{ij,p}. \tag{2.22}$$

In particular, h is called *C-parallel* if it satisfies $\bar{\nabla}^\xi h = 0$ on M^n.

For our purposes, we introduce the $(1, 2)$-tensor $K : TM^n \times TM^n \to TM^n$ defined by $K := -\varphi h$ satisfying $h(X, Y) = \varphi K(X, Y)$. A straightforward calculation shows that

LEMMA 2.2. *For the tensor K of the Legendrian submanifold M^n in $\mathbb{S}^{2n+1}$, we have*

(1) *$K_X Y = K(X, Y) = A_{\varphi X} Y$ and $g(K(X, Y), Z)$ is totally symmetric;*

(2) *M^n is minimal if and only if* $\operatorname{trace} K_X = 0$ *for any $X \in TM^n$;*

(3) $(\nabla K)(X,\,Y,\,Z)=(\nabla K)(Y,\,X,\,Z)$ *for any* $X,\,Y,\,Z\in TM^n$;

(4) *the Ricci identity for the tensor* K *is given by*

$$\begin{aligned}&(\nabla^2K)(U,X,Y,Z)-(\nabla^2K)(X,U,Y,Z)=(R\cdot K)(U,X,Y,Z)\\&=R(U,X)K(Y,Z)-K(R(U,X)Y,Z)-K(Y,R(U,X)Z),\end{aligned}$$

where $(\nabla^2K)(U,\,X,\,Y,\,Z)=(\nabla_U(\nabla K))(X,\,Y,\,Z)$ *for any* $U,\,X,\,Y,\,Z\in TM^n$.

Furthermore, it can be checked by using (2.20) and (2.21) that

$$\begin{aligned}&(\bar\nabla^\xi h)(X,Y,Z)=\varphi(\nabla K)(X,Y,Z),\\&((\bar\nabla^\xi)^2h)(U,X,Y,Z)-((\bar\nabla^\xi)^2h)(X,U,Y,Z)=\varphi(R\cdot K)(U,X,Y,Z).\end{aligned}\tag{2.23}$$

Consequently, $\nabla K=0$ if and only if $\bar\nabla^\xi h=0$, and $R\cdot K=0$ if and only if

$$((\bar\nabla^\xi)^2h)(U,X,Y,Z)=((\bar\nabla^\xi)^2h)(X,U,Y,Z),\tag{2.24}$$

which, under the Legendre frame $\{e_1,\,\ldots,\,e_n,\,e_{1^*},\,\ldots,\,e_{n^*},\,e_{2n+1}\}$ on M^n, is equivalent to

$$\tilde h^{k^*}_{ij,\ell p}=\tilde h^{k^*}_{ij,p\ell}.\tag{2.25}$$

Here, we shall call the tensor K *parallel* (resp. *semi-parallel*) if $\nabla K=0$ (resp. $R\cdot K=0$) holds on M^n.

LEMMA 2.3. *Let* M^n $(n\geqslant 2)$ *be a Legendrian submanifold in the unit sphere* $\mathbb{S}^{2n+1}$. *Then,* M^n *is of* C*-parallel second fundamental form if and only if* $\nabla K=0$ *on* M^n, *where* ∇ *denotes the Levi-Civita connection, and* M^n *satisfies equation (2.24) if and only if* $R\cdot K=0$ *on* M^n, *where* R *denotes the curvature tensor of the connection* ∇.

Assume that the tensor K of Legendrian submanifold M^n does not vanish at some point $x\in M^n$. We shall consider $U_xM^n=\{v\in T_xM^n\,|\,g(v,\,v)=1\}$ and then define a function F on U_xM^n by $F(u):=g(K(u,\,u),\,u)=g(A_{\varphi u}u,\,u)$ for $u\in U_xM^n$. Since U_xM^n is compact, there exists a unit vector $e_1\in U_xM^n$ at which the function $F(u)$ attains an absolute maximum, denoted by λ_1 and $\lambda_1>0$. As a result, it holds that:

$$g(K_{e_1}e_1,u)=0,\quad g(K_{e_1}e_1,e_1)\geqslant 2g(K_{e_1}u,u),\ u\perp e_1,\ u\in U_xM^n.\tag{2.26}$$

LEMMA 2.4 (cf. [12]; Lemma 5.1 and Corollary 5.1 of [11]). *There exists an orthonormal basis* $\{e_1,\,\ldots,\,e_n\}$ *of* T_xM^n *so that* $K_{e_1}e_i=\lambda_ie_i$ *for* $1\leqslant i\leqslant n$, *where* λ_1 *is the maximum value of* F *on* U_xM^n. *Also,* $\lambda_1\geqslant 2\lambda_i$ *for* $i\geqslant 2$, *and if* $\lambda_1=2\lambda_j$ *for some* $j\geqslant 2$, *then* $F(e_j)=0$. *Moreover, for a unit vector* $u\in T_xM^n$, *if* $K_uu=\lambda u$, *then* λ *is an extremal value of the function* F *on* U_xM^n.

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2.4. Conformally flat Riemannian manifolds and product manifolds

Let (M^n, g) be an n-dimensional connected Riemannian manifold with the normalized scalar curvature χ and the Levi-Civita connection ∇ of the metric g. The Schouten tensor P of (1, 1)-type defined by

$$P = \frac{Q}{n-2} - \frac{n\chi}{2(n-2)}\mathrm{id} \tag{2.27}$$

is a self-adjoint operator with respect to the metric g, where Q and id denote the Ricci operator and the identity transformation, respectively. By definition there holds:

$$g(X, QY) = \mathrm{Ric}(X, Y) = g(QX, Y), \tag{2.28}$$

where Ric denotes the Ricci tensor of M^n and X, Y are vector fields tangent to M^n.

Recall that (M^n, g) is said to be *conformally flat* if around each point of M^n there exists a neighbourhood which can be conformally immersed into the Euclidean space $\mathbb{R}^n$. When $n \geqslant 4$, it is known that (M^n, g) is conformally flat if and only if its Weyl curvature tensor vanishes. In this situation, the curvature tensor R of g can be rewritten as below:

$$R(X, Y)Z = g(Y, Z)PX - g(X, Z)PY + g(PY, Z)X - g(PX, Z)Y, \tag{2.29}$$

where X, Y, Z are tangent vector fields of M^n, and the Schouten tensor P is *Codazzi*, i.e.:

$$(\nabla_X P)Y = (\nabla_Y P)X. \tag{2.30}$$

When $n = 3$, we should remark that the Weyl curvature tensor vanishes automatically, and (M^3, g) is conformally flat if and only if P is a Codazzi tensor as above.

Moreover, the Ricci tensor Ric of M^n is called *parallel* or *semi-parallel* if it satisfies $\nabla\mathrm{Ric} = 0$ or $R \cdot \mathrm{Ric} = 0$. In the latter case, by definition one has:

$$(R \cdot \mathrm{Ric})(X, Y) = R(X, Y)\mathrm{Ric} = \nabla_X\nabla_Y\mathrm{Ric} - \nabla_Y\nabla_X\mathrm{Ric} - \nabla_{[X,Y]}\mathrm{Ric}. \tag{2.31}$$

In addition, we call the Riemannian manifold (M^n, g) *quasi-Einstein* if its Ricci operator Q admits exactly two distinct eigenvalues at each point, one of which is simple, and the traceless Ricci tensor $\tilde{\mathrm{Ric}}$ of M^n is defined by

$$\tilde{\mathrm{Ric}}(X, Y) = \mathrm{Ric}(X, Y) - (n-1)\chi g(X, Y). \tag{2.32}$$

Finally, we recall the following theorem for later use.

THEOREM 2.5 (cf. Theorem 3.7 of [4]). *Let $(M^n, g) = (I \times M_2, \mathrm{d}t^2 \oplus g_2)$ be a Riemannian product, where $I \subset \mathbb{R}$ and $\dim M_2 \geqslant 2$. Then, (M^n, g) is conformally flat if and only if (M_2, g_2) is a space form of constant curvature.*

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3. Geometric invariants of the Calabi torus

In [**25**], Luo and Sun have made some calculations about the Calabi torus in the unit sphere $\mathbb{S}^{2n+1}$. In this section, in order to obtain the exact knowledge about the Calabi torus, we compute its geometric invariants with more details.

PROPOSITION 3.1. *The Calabi torus in the unit sphere $\mathbb{S}^{2n+1}$ with the immersion given by f as in example 1.2 is indeed a minimal Legendrian submanifold with C-parallel second fundamental form and conformally flat induced metric for $n \geqslant 3$, satisfying the identity:*

$$\|\tilde{\mathrm{Ric}}\|^2 = \frac{(n-2)(n+1)}{n+2} S\chi, \tag{3.1}$$

where $\tilde{\mathrm{Ric}}$ is the traceless Ricci tensor, S is the squared norm of the second fundamental form, and χ is the normalized scalar curvature. In particular, the Calabi torus is quasi-Einstein and its Ricci tensor is parallel with respect to the Levi-Civita connection.

Proof. Note from the induced metric of $f(t,\, y) : \mathbb{S}^1 \times \mathbb{S}^{n-1} \to \mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$:

$$f^*(g) = (\mathrm{d}t)^2 + \frac{n}{n+1}[(\mathrm{d}y_1)^2 + \cdots + (\mathrm{d}y_n)^2]$$

that f is an isometric immersion, where $y = (y_1, \ldots, y_n) \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$ and $\sum_{i=1}^n y_i^2 = 1$. Adopting the following local reparametrization:

$$(y_1, y_2, \ldots, y_n) = (\sin\theta_1, \cos\theta_1 \sin\theta_2, \ldots, \cos\theta_1 \cos\theta_2 \cdots \cos\theta_{n-2} \cos\theta_{n-1}),$$

we obtain a local orthonormal frame $\{e_i\}_{i=1}^n$ on $f(\mathbb{S}^1 \times \mathbb{S}^{n-1}) =: M^n$ with respect to the metric g, satisfying the relations:

$$\begin{cases} e_1 = -f_t, \quad e_2 = \sqrt{\dfrac{n+1}{n}} f_{\theta_1}, \\ e_3 = \sqrt{\dfrac{n+1}{n}} \cos^{-1}\theta_1 f_{\theta_2}, \quad \ldots, \quad e_n = \sqrt{\dfrac{n+1}{n}} \displaystyle\prod_{\ell=1}^{n-2} \cos^{-1}\theta_\ell\, f_{\theta_{n-1}}. \end{cases} \tag{3.2}$$

As the unit sphere $\mathbb{S}^{2n+1}$ admits a natural Sasakian structure $(\varphi,\, \xi,\, \eta,\, g)$, by definition we see that $\eta(e_i) = 0$ for $1 \leqslant i \leqslant n$ and thus f is a Legendrian immersion.

Denote by h the second fundamental form of $f : \mathbb{S}^1 \times \mathbb{S}^{n-1} \to \mathbb{S}^{2n+1}$. Then, direct calculations by using the Gauss formula show that (cf. [**12**]):

$$\begin{cases} \nabla_{e_i} e_j = -\sqrt{\dfrac{n+1}{n}} \dfrac{\sin\theta_{j-1}}{\prod_{k=1}^{j-1}\cos\theta_k} e_i, & 2 \leqslant j < i \leqslant n, \\ \nabla_{e_i} e_i = \sqrt{\dfrac{n+1}{n}} \displaystyle\sum_{\ell=2}^{i-1} \dfrac{\sin\theta_{\ell-1}}{\prod_{k=1}^{\ell-1}\cos\theta_k} e_\ell, & 3 \leqslant i \leqslant n, \\ \nabla_{e_i} e_j = 0, & \text{otherwise}, \end{cases} \tag{3.3}$$

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where ∇ denotes the Levi-Civita connection of the metric g, and

$$h(e_1,e_1)=\frac{n-1}{\sqrt{n}}\varphi e_1,\quad h(e_1,e_i)=-\frac{1}{\sqrt{n}}\varphi e_i,\ \ h(e_i,e_j)=-\frac{1}{\sqrt{n}}\delta_{ij}\varphi e_1,\ \ 2\leqslant i,j\leqslant n. \tag{3.4}$$

It is obvious that such an immersion f is minimal. Combining with (3.3) and (3.4), we further conclude from (2.4) and (2.20) that $(\bar{\nabla}^{\xi}h)(e_i,\,e_j,\,e_k)=0$ holds for $1\leqslant i,\,j,\,k\leqslant n$, i.e. the immersion f is of C-parallel second fundamental form.

For the Riemannian curvature tensor of M^n, applying (3.3) again, we easily get:

$$\begin{aligned} &R(e_1,e_i)e_1=R(e_1,e_i)e_j=R(e_i,e_j)e_1=0,\\ &R(e_i,e_j)e_k=\frac{n+1}{n}(\delta_{jk}e_i-\delta_{ik}e_j),\quad 2\leqslant i,j,k\leqslant n. \end{aligned} \tag{3.5}$$

Therefore, we obtain from (3.4) and (3.5) that

$$\begin{aligned} &\mathrm{Ric}(e_1,e_1)=\mathrm{Ric}(e_1,e_i)=0,\qquad \mathrm{Ric}(e_i,e_j)=\frac{(n-2)(n+1)}{n}\delta_{ij},\\ &S=\|h\|^2=\frac{(n-1)(n+2)}{n},\qquad \chi=\frac{(n-2)(n+1)}{n^2},\qquad 2\leqslant i,j\leqslant n, \end{aligned} \tag{3.6}$$

and then (3.1) follows from (2.32). On the contrary, it is also known from (3.6) that

$$Qe_1=0,\qquad Qe_i=\frac{(n-2)(n+1)}{n}e_i,\qquad 2\leqslant i\leqslant n, \tag{3.7}$$

which shows that $(M^n,\,g)$ is quasi-Einstein. Moreover, with the help of (2.27), we conclude that both (2.29) and (2.30) hold. Consequently, the Weyl curvature tensor of M^n vanishes and its Schouten tensor is Codazzi, meaning that $(M^n,\,g)$ is conformally flat. Finally, making use of (3.3) and (3.7), we calculate that:

$$(\nabla_{e_1}Q)e_1=(\nabla_{e_1}Q)e_i=(\nabla_{e_i}Q)e_1=(\nabla_{e_i}Q)e_j=0,\ \ 2\leqslant i,j\leqslant n. \tag{3.8}$$

From this, it is easily seen that Ric is parallel with respect to the Levi-Civita connection ∇, and hence we have completed the proof of proposition 3.1. □

4. Properties of the Legendrian submanifolds in the unit sphere $\mathbb{S}^{2n+1}$

In this section, before completing the proofs of the main results, we will investigate the properties of the Legendrian submanifolds in the unit sphere $\mathbb{S}^{2n+1}$ under some certain geometric conditions.

4.1. Minimal Legendrian submanifolds

For our purposes, we first calculate the Laplacian of S to derive the following:

LEMMA 4.1. *Let M^n $(n\geqslant 2)$ be a minimal Legendrian submanifold in the unit sphere $\mathbb{S}^{2n+1}$. Then, it holds the identity:*

$$\frac{1}{2}\Delta S=\|\bar{\nabla}^{\xi}h\|^2-\|\mathrm{Rie}\|^2-\|\mathrm{Ric}\|^2+n(n^2-1)\chi, \tag{4.1}$$

where $\|\mathrm{Rie}\|^2$ denotes the squared norm of the Riemannian curvature tensor with respect to the metric g on M^n. Moreover, if it is of semi-parallel tensor K, we

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further have:

$$n(n-1)\Delta\chi + 2\|\bar{\nabla}^{\xi} h\|^2 = 0, \tag{4.2}$$

$$n(n^2-1)\chi = \|\mathrm{Rie}\|^2 + \|\mathrm{Ric}\|^2. \tag{4.3}$$

Proof. Choose the local Legendre frame $\{e_1, \ldots, e_n, e_{1^*}, \ldots, e_{n^*}, e_{2n+1}\}$ along M^n as in § 2. By definition, we have

$$\frac{1}{2}\Delta S = \frac{1}{2}\Delta\Big(\sum_{i,j,\alpha}(h_{ij}^{\alpha^*})^2\Big) = \sum_{i,j,\ell,\alpha}(h_{ij,\ell}^{\alpha^*})^2 + \sum_{i,j,\alpha} h_{ij}^{\alpha^*}\Delta h_{ij}^{\alpha^*}. \tag{4.4}$$

Applying the Ricci identity (2.16), we deduce from (2.15) that

$$\begin{aligned} \Delta h_{ij}^{\alpha^*} &= \sum_{\ell} h_{ij,\ell\ell}^{\alpha^*} = \sum_{\ell} h_{i\ell,j\ell}^{\alpha^*} \\ &= \sum_{\ell} h_{i\ell,\ell j}^{\alpha^*} + \sum_{\ell,m} h_{m\ell}^{\alpha^*} R_{mij\ell} + \sum_{\ell,m} h_{im}^{\alpha^*} R_{m\ell j\ell} + \sum_{\ell,\beta} h_{i\ell}^{\beta^*} R_{j\ell\beta^*\alpha^*} \\ &= \sum_{\ell} h_{\ell\ell,ij}^{\alpha^*} + \sum_{\ell,m} h_{m\ell}^{\alpha^*} R_{mij\ell} + \sum_{m} h_{im}^{\alpha^*} R_{mj} + \sum_{\ell,m} h_{i\ell}^{m^*} R_{j\ell m^*\alpha^*}. \end{aligned} \tag{4.5}$$

This together with (2.13)–(2.15) yields that

$$\begin{aligned} \sum_{i,j,\alpha} h_{ij}^{\alpha^*}\Delta h_{ij}^{\alpha^*} = &\sum_{i,j,k,\ell,m} h_{ij}^{k^*} h_{m\ell}^{k^*} R_{mij\ell} + \sum_{i,j,k,\ell,m} h_{ij}^{k^*} h_{i\ell}^{m^*} R_{j\ell mk} \\ &+ \sum_{i,j,k,m} h_{ij}^{k^*} h_{im}^{k^*} R_{mj} - S. \end{aligned} \tag{4.6}$$

To go on from (4.6), we make use of the relation

$$\sum_{i,j,k,\ell,m} h_{ij}^{k^*} h_{i\ell}^{m^*} R_{j\ell mk} = -\sum_{i,j,k,\ell,m} h_{ij}^{k^*} h_{i\ell}^{m^*} R_{kmj\ell} = -\sum_{i,j,k,\ell,m} h_{mj}^{k^*} h_{i\ell}^{k^*} R_{mij\ell}, \tag{4.7}$$

and (2.13) to calculate that:

$$\sum_{i,j,k,\ell,m} h_{ij}^{k^*} h_{m\ell}^{k^*} R_{mij\ell} + \sum_{i,j,k,\ell,m} h_{ij}^{k^*} h_{i\ell}^{m^*} R_{j\ell mk} = -\sum_{i,j,\ell,m}(R_{imj\ell})^2 + 2n(n-1)\chi. \tag{4.8}$$

On the contrary, with the help of (2.14), it is easily seen that:

$$\sum_{i,j,k,m} h_{ij}^{k^*} h_{im}^{k^*} R_{mj} = -\sum_{i,j}(R_{ij})^2 + n(n-1)^2\chi. \tag{4.9}$$

Substituting (4.8) and (4.9) into (4.6), we then conclude that:

$$\sum_{i,j,\alpha} h_{ij}^{\alpha^*}\Delta h_{ij}^{\alpha^*} = -\sum_{i,j,k,\ell}(R_{ijk\ell})^2 - \sum_{i,j}(R_{ij})^2 + n(n^2-1)\chi - S. \tag{4.10}$$

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As $h^{2n+1}_{ij,\ell} = h^{\ell^*}_{ij}$ and $h^{k^*}_{ij,\ell} = \tilde{h}^{k^*}_{ij,\ell}$ for any i, j, k, ℓ, it follows that

$$\sum_{i,j,\ell,\alpha} (h^{\alpha^*}_{ij,\ell})^2 = \sum_{i,j,k,\ell} (h^{k^*}_{ij,\ell})^2 + \sum_{i,j,\ell} (h^{\ell^*}_{ij})^2 = \|\bar{\nabla}^\xi h\|^2 + S, \tag{4.11}$$

and thus we obtain (4.1) by substituting (4.10) and (4.11) into (4.4) immediately.

Now, if assuming that M^n has semi-parallel tensor K, then

$$0 = \tilde{h}^{k^*}_{ij,\ell p} - \tilde{h}^{k^*}_{ij,p\ell} = \sum_m h^{k^*}_{mj} R_{mi\ell p} + \sum_m h^{k^*}_{im} R_{mj\ell p} + \sum_m h^{m^*}_{ij} R_{mk\ell p}, \tag{4.12}$$

where we used $h^{k^*}_{ij,\ell p} = \tilde{h}^{k^*}_{ij,\ell p} - h^{\ell^*}_{ij}\delta_{kp}$ and $h^{2n+1}_{ij,\ell p} = 2h^{\ell^*}_{ij,p}$. Similarly, we obtain that

$$\sum_{i,j,\alpha} h^{\alpha^*}_{ij} \Delta h^{\alpha^*}_{ij} = -\sum_{i,j,k,\ell,m} h^{k^*}_{ij} h^{m^*}_{i\ell} (\delta_{jm}\delta_{\ell k} - \delta_{jk}\delta_{\ell m}) = -S, \tag{4.13}$$

which combining with (4.11) shows that

$$\frac{1}{2}\Delta S = \|\bar{\nabla}^\xi h\|^2. \tag{4.14}$$

This gives (4.2) by (2.10), and substituting (4.14) into (4.1) finally yields (4.3). □

Remark 4.2. It is known from (4.2) that, for closed minimal Legendrian submanifolds or minimal Legendrian submanifolds with constant scalar curvature in $\mathbb{S}^{2n+1}$ with $n \geqslant 2$, the semi-parallelism of tensor K and the C-parallelism of second fundamental form are equivalent.

Furthermore, recalling the components of the Weyl curvature tensor W of M^n satisfy

$$W_{ijk\ell} = R_{ijk\ell} - \frac{1}{n-2}(\delta_{ik}R_{j\ell} + \delta_{j\ell}R_{ik} - \delta_{i\ell}R_{jk} - \delta_{jk}R_{i\ell}) + \frac{n\chi}{n-2}(\delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk}), \tag{4.15}$$

we have the following expression for $n \geqslant 3$ (cf. [**15**, **32**]):

$$\|\mathrm{Rie}\|^2 = \|W\|^2 + \frac{4}{n-2}\|\mathrm{Ric}\|^2 - \frac{2n^2(n-1)}{n-2}\chi^2, \tag{4.16}$$

where $\|W\|^2 = \sum_{i,j,k,\ell}(W_{ijk\ell})^2$. From $\tilde{R}_{ij} = R_{ij} - (n-1)\chi\delta_{ij}$, it is easy to see that

$$\|\mathrm{Ric}\|^2 = \|\tilde{\mathrm{Ric}}\|^2 + n(n-1)^2\chi^2. \tag{4.17}$$

Here, $\tilde{\mathrm{Ric}}$ is the traceless part of Ric and $\|\tilde{\mathrm{Ric}}\|^2 = \sum_{i,j}(\tilde{R}_{ij})^2$. From the combination of (2.10), (4.16), and (4.17), we then derive by using (4.1) the following result:

Proposition 4.3. *Let M^n $(n \geqslant 3)$ be a minimal Legendrian submanifold in the unit sphere $\mathbb{S}^{2n+1}$. Then, it holds the identity:*

$$\frac{1}{2}\Delta S = \|\bar{\nabla}^\xi h\|^2 - \|W\|^2 - \frac{n+2}{n-2}\|\tilde{\mathrm{Ric}}\|^2 + (n+1)S\chi. \tag{4.18}$$

4.2. Conformally flat Legendrian submanifolds

Assume that M^n $(n \geqslant 3)$ is a conformally flat Legendrian submanifold in the unit sphere $\mathbb{S}^{2n+1}$. Then, we shall prove that

LEMMA 4.4. *Let M^n $(n \geqslant 3)$ be a conformally flat Legendrian submanifold in the unit sphere $\mathbb{S}^{2n+1}$. Then, for the tensor K and Schouten tensor P of M^n, it holds that*

$$\begin{aligned} \underset{U,X,Y}{\mathfrak{S}}(g(U,Z)K(PX,Y) - g(X,Z)K(PU,Y) \\ + g(PX,K(Y,Z))U - g(PU,K(Y,Z))X) = 0, \end{aligned} \tag{4.19}$$

where U, X, Y, Z are vector fields tangent to M^n and $\mathfrak{S}$ denotes the cyclic summation.

Proof. We begin with taking the covariant derivative of the Codazzi equation for K along a vector field U tangent to M^n:

$$(\nabla^2 K)(U,X,Y,Z) - (\nabla^2 K)(U,Y,X,Z) = (\nabla G)(U,X,Y,Z) = 0, \tag{4.20}$$

where, according to lemma 2.2 (3), $G(X, Y, Z) := (\nabla K)(X, Y, Z) - (\nabla K)(Y, X, Z) = 0$ for tangent vector fields X, Y, Z on M^n. It is obvious from (4.20) that

$$\underset{U,X,Y}{\mathfrak{S}}((\nabla^2 K)(U,X,Y,Z) - (\nabla^2 K)(U,Y,X,Z)) = 0. \tag{4.21}$$

Furthermore, direct calculations by using the Ricci identity show that

$$\begin{aligned} 0 &= \underset{U,X,Y}{\mathfrak{S}}((\nabla^2 K)(U,X,Y,Z) - (\nabla^2 K)(U,Y,X,Z)) \\ &= \underset{U,X,Y}{\mathfrak{S}}((\nabla^2 K)(U,X,Y,Z) - (\nabla^2 K)(X,U,Y,Z)) \\ &= \underset{U,X,Y}{\mathfrak{S}}(R(U,X)K(Y,Z) - K(R(U,X)Y,Z) - K(Y,R(U,X)Z)). \end{aligned} \tag{4.22}$$

Finally, the assertion immediately follows by substituting (2.29) into (4.22). □

REMARK 4.5. The technique used to prove lemma 4.4 is called the *Tsinghua principle*, which was first discovered by H. Li, L. Vrancken, and X. Wang (cf. [**1**]). Recently, this remarkable principle has been widely applied, and turns out to be very useful for various purposes, see e.g. [**6**, **10**, **13**, **20**, **21**, **37**].

Choosing an orthonormal frame $\{e_i\}_{i=1}^n$ over M^n such that e_i is the eigenvector field of the Ricci operator Q with μ_i the corresponding eigenvalue, by (2.27) we easily see that $Pe_i = \nu_i e_i$ and $\nu_i = \mu_i/(n-2) - n\chi/(2(n-2))$. Without loss of generality, we shall suppose that Q has t distinct eigenvalues $\mu_1, \ldots, \mu_t$ with multiplicities $n_1, \ldots, n_t$, respectively. Let $\mathfrak{D}(\mu_s)$ (resp. $\mathfrak{D}(\nu_s)$) denote the distribution such that $\mathfrak{D}(\mu_s)(x)$ (resp. $\mathfrak{D}(\nu_s)(x)$) is the eigenspace of $\mu_s(x)$ (resp. $\nu_s(x)$) at an arbitrary point $x \in M^n$ for $1 \leqslant s \leqslant t$, and $n_1 + \cdots + n_t = n$. For simplicity of notations, we also make the convention that, for $i \leqslant t$ and $j \geqslant t+1$, if $\mu_j = \mu_i$ we shall write $n_j = n_i$, $\mathfrak{D}(\mu_j) = \mathfrak{D}(\mu_i)$ and $\mathfrak{D}(\nu_j) = \mathfrak{D}(\nu_i)$.

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Lemma 4.6. *Let M^n $(n \geqslant 3)$ be a conformally flat Legendrian submanifold in the unit sphere $\mathbb{S}^{2n+1}$. Then, with respect to the orthonormal frame $\{e_i\}_{i=1}^n$ on M^n as above, the tensor K of M^n satisfies the following properties:*

(1) *If $\nu_i \neq \nu_j$ and $n_i, n_j \geqslant 2$, then $K(e_i, e_j) = 0$;*

(2) *If $n_i = 1$ and $n_j \geqslant 2$, then there exist functions λ^i_j depending on the choice of ν_i, ν_j such that $K(e_i, e_j) = \lambda^i_j e_j$;*

(3) *If there are at least two distinct eigenvalues ν_j, ν_k such that $n_j, n_k \geqslant 2$ and $n_i = 1$, then there exists a differentiable function $\bar{\lambda}_i$ such that it satisfies that $(\nu_i - \nu_j)\lambda^i_j = (\nu_i - \nu_k)\lambda^i_k = \bar{\lambda}_i$.*

Proof. We begin with taking the product of equation (4.19) with vector field V and setting $U = e_k$, $X = e_i$, $Y = e_j$, $Z = e_\ell$, and $V = e_m$ to derive the relation:

$$\begin{aligned} 0 = &(\nu_i - \nu_j)(K^\ell_{ij}\delta_{km} + K^m_{ij}\delta_{k\ell}) \\ &+ (\nu_j - \nu_k)(K^\ell_{jk}\delta_{im} + K^m_{jk}\delta_{i\ell}) \\ &+ (\nu_k - \nu_i)(K^\ell_{ik}\delta_{jm} + K^m_{ik}\delta_{j\ell}), \end{aligned} \tag{4.23}$$

where $1 \leqslant i, j, k, \ell, m \leqslant n$, and $K^m_{ij} := g(K(e_i, e_j), e_m)$.

First of all, we assume that $\nu_i \neq \nu_j = \nu_k$ for distinct i, j, k, and then $e_i \in \mathfrak{D}(\nu_i)$ and $e_j, e_k \in \mathfrak{D}(\nu_j)$ for $n_j \geqslant 2$. Taking $m \neq j, k$, we therefore obtain from (4.23) that

$$K^m_{ij}\delta_{k\ell} - K^m_{ik}\delta_{j\ell} = 0. \tag{4.24}$$

Taking $\ell = k$ yields that $K^m_{ij} = 0$, by which we see that $K(e_i, e_j) \in \mathfrak{D}(\nu_j)$ for $n_j \geqslant 2$. Similarly, $K(e_i, e_j) \in \mathfrak{D}(\nu_i)$ for $n_i \geqslant 2$. Combining with the assumption $\nu_i \neq \nu_j$, we can conclude that $K(e_i, e_j) = 0$ provided that $n_i, n_j \geqslant 2$. Hence, we get assertion (1).

Next, if $n_i = 1$ and $n_j \geqslant 2$, some $e_k \in \mathfrak{D}(\nu_j)$ different from e_j can be chosen to satisfy $\nu_i \neq \nu_j = \nu_k$ for distinct i, j, k. In this situation, we take $m = k$ in (4.23) to deduce that

$$K^\ell_{ij} + K^k_{ij}\delta_{k\ell} - K^k_{ik}\delta_{j\ell} = 0. \tag{4.25}$$

It follows that $K^i_{jj} = K^i_{kk}$ for $\ell = j$, $K^k_{ij} = 0$ for $\ell = k$, and moreover $K^\ell_{ij} = 0$ for $\ell \neq j, k$. Consequently, assertion (2) follows by putting $\lambda^i_j = K^i_{jj}$, where λ^i_j does not depend on the choice of $e_k \in \mathfrak{D}(\nu_j)$.

Finally, for $n_i = 1$ and $\nu_j \neq \nu_k$ with $n_j, n_k \geqslant 2$, by taking $\ell = j$ in (4.23) and applying assertion (1), we easily get:

$$(\nu_i - \nu_j)K^j_{ij}\delta_{km} + (\nu_j - \nu_k)K^j_{jk}\delta_{im} + (\nu_k - \nu_i)(K^j_{ik}\delta_{jm} + K^m_{ik}) = 0. \tag{4.26}$$

Thus, taking $m = k$, and noting $\lambda^i_j = K^i_{jj} = K^j_{ij}$ and $\lambda^i_k = K^i_{kk} = K^k_{ik}$, we further have

$$(\nu_i - \nu_j)\lambda^i_j = (\nu_i - \nu_k)\lambda^i_k =: \bar{\lambda}_i. \tag{4.27}$$

This verifies assertion (3). Hence, lemma 4.6 has been proved. □

4.3. Conformally flat minimal Legendrian submanifolds with $R \cdot K = 0$

In this subsection, we shall consider that M^n ($n \geqslant 3$) is a conformally flat minimal Legendrian submanifold in the unit sphere $\mathbb{S}^{2n+1}$ with semi-parallel tensor K. Therefore, we obtain from lemma 2.2 (4) that

$$\begin{aligned} 0 &= (R \cdot K)(X, Y, Z, U) \\ &= R(X, Y)K(Z, U) - K(R(X, Y)Z, U) - K(Z, R(X, Y)U), \end{aligned} \tag{4.28}$$

where X, Y, Z, U are vector fields tangent to M^n. According to lemma 2.3, we then present the following lemma involving the number r of distinct eigenvalues of the Schouten tensor P of M^n.

LEMMA 4.7. *Let M^n ($n \geqslant 3$) be a conformally flat minimal Legendrian submanifold in the unit sphere $\mathbb{S}^{2n+1}$ with $R \cdot K = 0$. Then, either M^n is of constant sectional curvature, or it is quasi-Einstein. In the latter case, the Schouten tensor of M^n admits two distinct eigenvalues ν_1 and ν_2 at each point, where one of them is simple, such that $\nu_1 + \nu_2 = 0$.*

Proof. First of all, by taking $X = e_i$, $Y = e_j$, $Z = e_k$, and $U = e_\ell$ in (4.28), we can apply (2.29) to calculate the relation:

$$\begin{aligned} 0 = (\nu_i + \nu_j)&[g(K(e_k, e_\ell), e_j)e_i - g(K(e_k, e_\ell), e_i)e_j] \\ &+ (\nu_i + \nu_j)[\delta_{ik}K(e_j, e_\ell) - \delta_{jk}K(e_i, e_\ell)] \\ &+ (\nu_i + \nu_j)[\delta_{i\ell}K(e_j, e_k) - \delta_{j\ell}K(e_i, e_k)]. \end{aligned} \tag{4.29}$$

For $k = \ell = i \neq j$ in (4.29), it is easy to see that:

$$(\nu_i + \nu_j)[g(K(e_i, e_i), e_j)e_i - g(K(e_i, e_i), e_i)e_j + 2K(e_i, e_j)] = 0. \tag{4.30}$$

Taking the inner product of (4.30) with e_i, we then deduce from lemma 2.2 that:

$$(\nu_i + \nu_j)g(K(e_i, e_i), e_j) = 0, \quad \forall\, i \neq j. \tag{4.31}$$

Similarly, interchanging the roles of e_i and e_j in (4.31) gives

$$(\nu_j + \nu_i)g(K(e_j, e_j), e_i) = 0, \quad \forall\, i \neq j. \tag{4.32}$$

Furthermore, by taking the inner product of (4.30) with e_j, we obtain that

$$(\nu_i + \nu_j)g(K(e_i, e_i), e_i) = 0, \quad \forall\, i \neq j, \tag{4.33}$$

which together with (4.30) and (4.31) implies that

$$(\nu_i + \nu_j)K(e_i, e_j) = 0, \quad \forall\, i \neq j. \tag{4.34}$$

On the contrary, for $k = i \neq j = \ell$ in (4.29), we easily get:

$$\begin{aligned} 0 = (\nu_i + \nu_j)&[g(K(e_i, e_j), e_j)e_i - g(K(e_i, e_j), e_i)e_j] \\ &+ (\nu_i + \nu_j)[K(e_j, e_j) - K(e_i, e_i)]. \end{aligned} \tag{4.35}$$

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This combining with (4.31) and (4.32) yields that

$$(\nu_i + \nu_j)[K(e_i, e_i) - K(e_j, e_j)] = 0, \quad \forall\, i \neq j. \tag{4.36}$$

Next, we shall continue with the proof here by proving the following three claims.

Claim 1. *If $\nu_i \neq 0$ and $n_i \geqslant 2$, then $g(K(u,\, v),\, \omega) = 0$ for any $u,\, v,\, \omega \in \mathfrak{D}(\nu_i)$.*

If $\nu_i \neq 0$ and $n_i \geqslant 2$, taking $\nu_i = \nu_j$ and $e_i = u$ in (4.33), then we easily conclude that $g(K(u,\, u),\, u) = 0$ for any unit vector field $u \in \mathfrak{D}(\nu_i)$. Consequently, the assertion follows from the symmetry given in lemma 2.2.

Claim 2. *If $\nu_i^2 \neq \nu_j^2$, then $K(u,\, v) = 0$ for any $u,\, v \in \mathfrak{D}(\nu_i) \oplus \mathfrak{D}(\nu_j)$.*

If $\nu_i^2 \neq \nu_j^2$, it is obvious that $(\nu_i + \nu_j)(\nu_i - \nu_j) \neq 0$ and we then obtain from (4.33) that $g(K(u,\, u),\, u) = 0$ for any unit vector field $u \in \mathfrak{D}(\nu_i)$. From the result of the symmetry in lemma 2.2, we have $K(u,\, u) \notin \mathfrak{D}(\nu_i)$. Similarly, $g(K(v,\, v),\, v) = 0$ for any unit vector field $v \in \mathfrak{D}(\nu_j)$ and thus $K(v,\, v) \notin \mathfrak{D}(\nu_j)$. Moreover, with the help of (4.34) and (4.36), it can be checked that $K(u,\, v) = 0$ and $K(u,\, u) = K(v,\, v) \notin \mathfrak{D}(\nu_i) \oplus \mathfrak{D}(\nu_j)$. When $r = 2$, claim 2 holds immediately. When $r \geqslant 3$, for an arbitrary eigenvalue ν_k different from ν_i and ν_j, it is known from $\nu_i^2 \neq \nu_j^2$ that either $\nu_i + \nu_k \neq 0$ or $\nu_j + \nu_k \neq 0$. In either case, we can apply (4.31) to obtain $K(u,\, u) = K(v,\, v) = 0$. Hence, claim 2 has been proved.

Claim 3. *If $r \geqslant 2$, there exist two distinct eigenvalues ν_i and ν_j such that $\nu_i + \nu_j = 0$.*

If $r \geqslant 2$, we suppose on the contrary that $\nu_i + \nu_j \neq 0$ holds for any $\nu_i \neq \nu_j$, namely $\nu_i^2 \neq \nu_j^2$. It then follows from claim 2 that $K = 0$ on M^n and hence $h = 0$ by definition. This together with the Gauss equation shows that M^n has constant sectional curvature, which is a contradiction to $r \geqslant 2$, and thus we have verified claim 3.

Now, according to claim 3, we denote by ν_1 and ν_2 the two distinct eigenvalues of P such that $\nu_1 + \nu_2 = 0$. In this situation, we further claim that $r \leqslant 2$. Otherwise, if $r \geqslant 3$, then $(\nu_i + \nu_1)(\nu_i + \nu_2) \neq 0$ for an arbitrary eigenvalue ν_i different from ν_1 and ν_2. Therefore, it satisfies that $\nu_i^2 \neq \nu_1^2$ and $\nu_i^2 \neq \nu_2^2$, and so we obtain from claim 2 that:

$$\begin{aligned} &K(u,u) = K(v,v) = K(\omega,\omega) = 0, \quad \omega \in \mathfrak{D}(\nu_i), \\ &K(u,\omega) = K(v,\omega) = 0, \quad u \in \mathfrak{D}(\nu_1), \quad v \in \mathfrak{D}(\nu_2). \end{aligned} \tag{4.37}$$

Furthermore, we see that $K(u,\, v) = 0$ in terms of the arbitrariness of ν_i, meaning that $h = 0$ identically. This contradiction implies that the number $r \leqslant 2$.

Assume that $r = 1$. It is obvious that M^n has constant sectional curvature for $n \geqslant 3$.

Assume that $r = 2$. We will denote these two distinct eigenvalues of P by ν_1 and ν_2, whose multiplicities are n_1 and n_2, respectively. Together with claim 3, it follows that $\nu_2 = -\nu_1 \neq 0$. In what follows, we shall argue by contradiction and suppose that $n_1 \geqslant 2$ and $n_2 \geqslant 2$. Let $X_1^i,\, \ldots,\, X_{n_i}^i$ be the orthonormal eigenvector fields of P that span the distribution $\mathfrak{D}(\nu_i)$, $i = 1,\, 2$. Thus, we see from claim 1 and (4.36) that

$$g(K(X^i_p, X^i_q), X^i_s) = 0, \quad K(X^i_p, X^i_p) = K(X^i_q, X^i_q), \quad \forall\, p, q, s, \quad i = 1, 2. \tag{4.38}$$

Since M^n is minimal in $\mathbb{S}^{2n+1}$, by means of (4.38) we have

$$\begin{aligned} 0 &= \sum_{p=1}^{n_1} g(K(X^1_p, X^1_p), v) + \sum_{q=1}^{n_2} g(K(X^2_q, X^2_q), v) \\ &= \sum_{p=1}^{n_1} g(K(X^1_p, X^1_p), v) = n_1 g(K(X^1_1, X^1_1), v), \quad v \in \mathfrak{D}(\nu_2), \end{aligned} \tag{4.39}$$

which implies that $g(K(u,\, u),\, v) = 0$ for any $u \in \mathfrak{D}(\nu_1)$ and $v \in \mathfrak{D}(\nu_2)$. Similarly, there holds that $g(K(v,\, v),\, u) = 0$ and therefore $K(u,\, v) = 0$. According to this and (4.38), we get $h = 0$ identically, a contradiction to $r = 2$. Hence, lemma 4.7 has been proved. □

Remark 4.8. By checking the proof of lemma 4.7 step by step, we see that the assertion still holds for $n = 3$, when the conformally flat condition is replaced by vanishing Weyl curvature tensor.

4.4. Conformally flat minimal Legendrian submanifolds with $R \cdot Q = 0$

In this subsection, assuming that M^n $(n \geqslant 3)$ is a conformally flat minimal Legendrian submanifold in the unit sphere $\mathbb{S}^{2n+1}$ such that the Ricci tensor Ric is semi-parallel, by definition we have

$$\begin{aligned} 0 &= (R \cdot \mathrm{Ric})(X, Y, U, Z) = (R(X, Y)\mathrm{Ric})(U, Z) \\ &= -\mathrm{Ric}(R(X, Y)U, Z) - \mathrm{Ric}(U, R(X, Y)Z) \\ &= g(R(X, Y)QZ, U) - g(QR(X, Y)Z, U) \\ &= g((R(X, Y)Q)Z, U) = g((R \cdot Q)(X, Y, Z), U), \end{aligned} \tag{4.40}$$

where U, X, Y, Z are tangent vector fields on M^n. This implies that Ric is semi-parallel if and only if the Ricci operator Q is semi-parallel, i.e. $R \cdot Q = 0$.

Lemma 4.9. *Let M^n $(n \geqslant 3)$ be a conformally flat minimal Legendrian submanifold in the unit sphere $\mathbb{S}^{2n+1}$ with $R \cdot Q = 0$. Then, either M^n is of constant sectional curvature, or it is quasi-Einstein. In the latter case, the Schouten tensor of M^n admits two distinct eigenvalues ν_1 and ν_2 at each point, where one of them is simple, such that $\nu_1 + \nu_2 = 0$.*

Proof. Under the orthonormal frame $\{e_i\}_{i=1}^n$ on M^n as in § 4.2, direct calculations by using (2.29) yield that

$$\begin{aligned} 0 &= g((R(e_i, e_j)Q)e_j, e_i) \\ &= g(R(e_i, e_j)Qe_j, e_i) - g(QR(e_i, e_j)e_j, e_i) \\ &= (n-2)(\nu_j + \nu_i)(\nu_j - \nu_i), \quad 1 \leqslant i \neq j \leqslant n, \end{aligned} \tag{4.41}$$

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where we used the relation $\mu_j - \mu_i = (n-2)(\nu_j - \nu_i)$. Consequently, the number r of distinct eigenvalues of the Schouten tensor P is at most 2, and if $r = 1$, then we conclude that M^n has constant sectional curvature.

If $r = 2$, then we obtain from (4.41) that $\nu_1 + \nu_2 = 0$ by expressing the two distinct eigenvalues of P by ν_1 and ν_2, respectively. Now, it suffices to prove that either of ν_1 and ν_2 must be simple. For this purpose, we shall suppose on the contrary that $n_i \geqslant 2$, where n_i is the multiplicity of ν_i for $i = 1, 2$. Let $\{X_i\}_{i=1}^{n_1}$ (resp. $\{Y_j\}_{j=1}^{n_2}$) be the orthonormal frame of $\mathfrak{D}(\nu_1)$ (resp. $\mathfrak{D}(\nu_2)$). It then follows from lemma 4.6 (1) that

$$K(X_i, X_i) \in \mathfrak{D}(\nu_1), \quad K(Y_j, Y_j) \in \mathfrak{D}(\nu_2), \quad K(X_i, Y_j) = 0, \ \forall\, i, j. \tag{4.42}$$

Applying lemma 2.2 (1), we deduce from the Gauss equation (2.7) that

$$g(R(X_i, Y_j)Y_j, X_i) = 1. \tag{4.43}$$

Since M^n is conformally flat, it is easy to see from (2.29) that

$$g(R(X_i, Y_j)Y_j, X_i) = \nu_1 + \nu_2, \tag{4.44}$$

which together with (4.43) yields that

$$\nu_1 + \nu_2 = 1. \tag{4.45}$$

This contradicts with the fact $\nu_1 + \nu_2 = 0$, and thus lemma 4.9 has been proved. □

4.5. Conformally flat Legendrian submanifolds with quasi-Einstein metric

In the following, according to lemmas 4.7 and 4.9, we shall deal with the case when M^n $(n \geqslant 3)$ is a conformally flat Legendrian submanifold in $\mathbb{S}^{2n+1}$, such that it is quasi-Einstein with ν_1 and ν_2 the distinct eigenvalues of its Schouten tensor, and $\mathfrak{D}(\nu_1)$ and $\mathfrak{D}(\nu_2)$ the corresponding distributions of eigenspace, where $\nu_1 + \nu_2 = 0$ and ν_1 is simple. Then, for a unit vector field E_1 of $\mathfrak{D}(\nu_1)$ and an orthonormal frame $\{E_i\}_{i=2}^n$ of $\mathfrak{D}(\nu_2)$, we see from (2.27) and (2.29) that

$$\begin{aligned} &R(E_1, E_i)E_1 = R(E_1, E_i)E_j = R(E_i, E_j)E_1 = 0, \\ &R(E_i, E_j)E_k = 2\nu_2(\delta_{jk}E_i - \delta_{ik}E_j), \quad 2 \leqslant i, j, k \leqslant n, \\ &QE_1 = 0, \ QE_i = n\chi E_i, \quad \nu_2 = -\nu_1 = \frac{n}{2(n-2)}\chi \neq 0, \end{aligned} \tag{4.46}$$

where we used the facts $\operatorname{tr} Q = n(n-1)\chi$ and $\nu_i = \mu_i/(n-2) - n\chi/(2(n-2))$ for $i = 1, 2$.

Lemma 4.10. *Let M^n $(n \geqslant 3)$ be a conformally flat Legendrian submanifold in the unit sphere $\mathbb{S}^{2n+1}$. Assume that M^n is quasi-Einstein with ν_1 and ν_2 the distinct eigenvalues of its Schouten operator such that $\nu_1 + \nu_2 = 0$, where ν_1 is simple. Then, it holds that*

$$\begin{aligned} &\nabla_{E_1} E_1 = 0, \quad \nabla_{E_i} E_1 = -\alpha E_i, \quad \alpha = \frac{E_1(\nu_2)}{\nu_2 - \nu_1}, \\ &E_1(\alpha) = \alpha^2, \quad E_i(\alpha) = E_i(\nu_1) = E_i(\nu_2) = 0, \quad 2 \leqslant i \leqslant n. \end{aligned} \tag{4.47}$$

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Proof. For any tangent vector V on M^n, we denote by V^ℓ the projection of V onto $\mathfrak{D}(\nu_\ell)$ for $\ell = 1$, 2, respectively. A straightforward computation shows that

$$\begin{aligned}(\nabla_{E_1}P)E_i &= E_1(\nu_2)E_i + (\nu_2 - \nu_1)(\nabla_{E_1}E_i)^1,\\ (\nabla_{E_i}P)E_1 &= E_i(\nu_1)E_1 + (\nu_1 - \nu_2)(\nabla_{E_i}E_1)^2,\end{aligned} \tag{4.48}$$

where $2 \leqslant i \leqslant n$. As M^n is conformally flat, we have $(\nabla_{E_1}P)E_i = (\nabla_{E_i}P)E_1$ and then it follows that:

$$E_1(\nu_2)E_i - E_i(\nu_1)E_1 - (\nu_1 - \nu_2)[(\nabla_{E_1}E_i)^1 + (\nabla_{E_i}E_1)^2] = 0. \tag{4.49}$$

Multiplying the above equation with E_1 and further with E_k for $k \geqslant 2$, we easily get:

$$E_i(\nu_1) = (\nu_1 - \nu_2)g(\nabla_{E_1}E_1, E_i), \tag{4.50}$$

$$E_1(\nu_2)g(E_i, E_k) = (\nu_1 - \nu_2)g(\nabla_{E_i}E_1, E_k), \tag{4.51}$$

where $2 \leqslant i,\ k \leqslant n$. Similarly, we deduce from $(\nabla_{E_i}P)E_j = (\nabla_{E_j}P)E_i$ that

$$E_i(\nu_2)E_j - E_j(\nu_2)E_i + (\nu_1 - \nu_2)[(\nabla_{E_j}E_i)^1 - (\nabla_{E_i}E_j)^1] = 0, \tag{4.52}$$

where $2 \leqslant i \neq j \leqslant n$, and thus there holds

$$E_i(\nu_2)E_j - E_j(\nu_2)E_i + (\nu_1 - \nu_2)[(\nabla_{E_j}E_i)^1 - (\nabla_{E_i}E_j)^1] = 0, \tag{4.53}$$

which together with the fact $\nu_1 \neq \nu_2$ immediately yields that

$$g(\nabla_{E_i}E_j - \nabla_{E_j}E_i, E_1) = 0, \quad E_i(\nu_2) = 0, \quad 2 \leqslant i \neq j \leqslant n. \tag{4.54}$$

Next, with the help of $\nu_1 \neq \nu_2$, it is easy to see from (4.50) and (4.54) that

$$\nabla_{E_1}E_1 = 0,\ \ (\nabla_{E_i}E_j - \nabla_{E_j}E_i) \in \mathfrak{D}(\nu_2),\ \ E_i(\nu_1) = E_i(\nu_2) = 0,\ \ 2 \leqslant i \leqslant n. \tag{4.55}$$

On the contrary, by applying (4.51) we further have

$$\nabla_{E_i}E_1 = -\alpha E_i, \qquad \alpha = \frac{E_1(\nu_2)}{\nu_2 - \nu_1}. \tag{4.56}$$

Therefore, it can be checked from (4.46), (4.55) and (4.56) that

$$\begin{aligned}0 &= R(E_i, E_1)E_1 = E_1(\alpha)E_i - \alpha^2 E_i,\\ 0 &= R(E_i, E_j)E_1 = E_j(\alpha)E_i - E_i(\alpha)E_j,\end{aligned} \tag{4.57}$$

where we made use of the definition of the curvature tensor R for $2 \leqslant i \neq j \leqslant n$. Hence, $E_1(\alpha) = \alpha^2$ and $E_i(\alpha) = 0$ for $2 \leqslant i \leqslant n$. This completes the proof of lemma 4.10. □

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5. Proofs of the main results

In order to complete the proofs of the main results, we first verify the proposition:

PROPOSITION 5.1. *Let M^n $(n \geqslant 3)$ be a conformally flat minimal Legendrian submanifold in the unit sphere $\mathbb{S}^{2n+1}$. Then, M^n is quasi-Einstein with ν_1 and ν_2 the distinct eigenvalues of its Schouten operator such that $\nu_1 + \nu_2 = 0$, where ν_1 is simple, if and only if (M^n, g) is locally isometric to a Riemannian product $I \times M_2 = (I \times M_2, \mathrm{d}t^2 \oplus g_2)$, where $I \subset \mathbb{R}$, g is the induced metric on M^n and (M_2, g_2) has constant sectional curvature $c \neq 0$.*

Proof. First of all, omitting the upper index $i = 1$ for λ_j^i, by lemma 4.6 (2) we can write

$$K(E_1, E_1) = \lambda_1 E_1, \quad K(E_1, E_j) = \lambda_2 E_j, \quad 2 \leqslant j \leqslant n, \tag{5.1}$$

where E_1 is a unit vector field of $\mathfrak{D}(\nu_1)$ and $\{E_j\}_{j=2}^n$ is an orthonormal frame of $\mathfrak{D}(\nu_2)$. By means of the Gauss equation (2.7) and (4.46), it can be checked that:

$$0 = R(E_i, E_1)E_1 = (1 + \lambda_1\lambda_2 - \lambda_2^2)E_i, \quad 2 \leqslant i \leqslant n. \tag{5.2}$$

Combining with the Codazzi equation for K, we can apply (4.47) to calculate that:

$$\begin{aligned} 0 &= (\nabla K)(E_1, E_i, E_1) - (\nabla K)(E_i, E_1, E_1) \\ &= (E_1(\lambda_2) - \alpha(2\lambda_2 - \lambda_1))E_i - E_i(\lambda_1)E_1, \\ 0 &= (\nabla K)(E_i, E_j, E_1) - (\nabla K)(E_j, E_i, E_1) \\ &= E_i(\lambda_2)E_j - E_j(\lambda_2)E_i, \quad 2 \leqslant i \neq j \leqslant n. \end{aligned} \tag{5.3}$$

Consequently, $E_1(\lambda_2) = \alpha(2\lambda_2 - \lambda_1)$ and $E_i(\lambda_1) = E_i(\lambda_2) = 0$ for all $i \geqslant 2$.

Secondly, noting from the minimality of M^n that $0 = \operatorname{trace} K_{E_1} = \lambda_1 + (n-1)\lambda_2$, we solve from (5.2) to obtain that

$$\lambda_1 = \frac{n-1}{\sqrt{n}}, \quad \lambda_2 = -\frac{1}{\sqrt{n}}. \tag{5.4}$$

where, replacing E_1 by $-E_1$ if necessary, we can always assume that $\lambda_1 \geqslant 0$. Thus, $\alpha = 0$ and it then follows from (4.47) that ν_1, ν_2 are constant and

$$\nabla_{E_1} E_1 = 0, \quad g(\nabla_{E_i} E_j, E_1) = 0, \quad 2 \leqslant i, j \leqslant n. \tag{5.5}$$

Therefore, $\mathfrak{D}(\nu_1)$ and $\mathfrak{D}(\nu_2)$ are both totally geodesic in M^n, and so (M^n, g) is locally a Riemannian product manifold $I \times M_2$, where $I \subset \mathbb{R}$ and M_2 is the integral manifold of $\mathfrak{D}(\nu_2)$. Let R_2 be the Riemannian curvature tensor of M_2. Using (4.46), we easily get:

$$g(R_2(E_i, E_j)E_j, E_i) = g(R(E_i, E_j)E_j, E_i) = 2\nu_2, \quad 2 \leqslant i, j \leqslant n. \tag{5.6}$$

From this, we see that (M_2, g_2) has constant sectional curvature $2\nu_2$ and g_2 is the metric of M_2. As a result, (M^n, g) is locally isometric to the Riemannian product $I \times M_2 = (I \times M_2, \mathrm{d}t^2 \oplus g_2)$, where (M_2, g_2) has constant sectional curvature $c = 2\nu_2 \neq 0$.

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Finally, if assuming that $(M^n,\, g)$ is locally isometric to a Riemannian product $I \times M_2 = (I \times M_2,\, \mathrm{d}t^2 \oplus g_2)$, where $(M_2,\, g_2)$ has constant sectional curvature $c \neq 0$, then

$$g(R(E_1, E_i)E_i, E_1) = 0, \quad g(R(E_i, E_j)E_j, E_i) = c, \quad 2 \leqslant i \neq j \leqslant n, \tag{5.7}$$

where E_1 is a unit vector field tangent to I and $\{E_i\}_{i=2}^n$ is an orthonormal frame on M_2. Applying (5.7), we can verify the assertion by direct calculation. □

Now, by means of proposition 5.1 we are ready to prove the following theorem:

THEOREM 5.2 (cf. theorem 1.9). *Let M^n $(n \geqslant 3)$ be a minimal Legendrian submanifold in the unit sphere $\mathbb{S}^{2n+1}$. If $(M^n,\, g)$ is locally isometric to a Riemannian product $I \times M_2 = (I \times M_2,\, \mathrm{d}t^2 \oplus g_2)$, where $I \subset \mathbb{R}$, g is the induced metric on M^n and $(M_2,\, g_2)$ has constant sectional curvature $c \neq 0$, then M^n is locally congruent to the Calabi torus.*

Proof. According to theorem 2.5, it is known that $(M^n,\, g)$ is conformally flat for $n \geqslant 3$, and therefore we obtain from proposition 5.1 that M^n is quasi-Einstein with ν_1 and ν_2 the distinct eigenvalues of its Schouten operator such that $\nu_1 + \nu_2 = 0$, where ν_1 is simple. Let $E_1 \in \mathfrak{D}(\nu_1)$ be a unit vector field and $\{E_i\}_{i=2}^n$ an orthonormal frame of $\mathfrak{D}(\nu_2)$. Here, $I \subset \mathbb{R}$ and M_2 are the integral manifolds of $\mathfrak{D}(\nu_1)$ and $\mathfrak{D}(\nu_2)$, respectively. Related to the Schouten tensor P of M^n, by calculation we have

$$PE_1 = \nu_1 E_1 = -\frac{c}{2}E_1, \quad PE_i = \nu_2 E_i = \frac{c}{2}E_1, \quad 2 \leqslant i \leqslant n. \tag{5.8}$$

Similar argument as in the proof of proposition 5.1 shows that

$$\begin{aligned} K(E_1, E_1) &= \frac{n-1}{\sqrt{n}}E_1, \quad K(E_1, E_i) = -\frac{1}{\sqrt{n}}E_i, \\ K(E_i, E_j) &= -\frac{1}{\sqrt{n}}\delta_{ij}E_1 + \sum_{k=2}^{n} K_{ij}^k E_k, \quad 2 \leqslant i, j \leqslant n, \end{aligned} \tag{5.9}$$

where $K_{ij}^k := g(K(E_i,\, E_j),\, E_k)$.

Next, we claim that $K_{ij}^k \equiv 0$. To verify this, we argue by supposing on the contrary that there exists some point $x \in M^n$ at which $K_{ij}^k \neq 0$, and then divide the proof into the following four steps.

Step 1. *There exists an orthonormal basis $\{Y_i\}_{i=1}^{n-1}$ of T_xM_2 such that the tensor K of M^n takes the forms:*

$$K(Y_1, Y_1) = -\frac{1}{\sqrt{n}}X + (n-2)\sqrt{\frac{(n+1)-nc}{n(n-1)}}Y_1, \quad K(Y_1, Y_i) = -\sqrt{\frac{(n+1)-nc}{n(n-1)}}Y_j, \tag{5.10}$$

where $X := E_1(x)$, $(n+1) - nc > 0$, and $2 \leqslant j \leqslant n-1$.

Since $K_{ij}^k(x) \neq 0$ for some point $x \in M^n$, the symmetry of K_{ij}^k in all indices implies that there exists a unit vector $Y_1 \in U_xM_2 := \{v \in T_xM_2 \,|\, g(v,\, v) = 1\}$, which is a compact set, such that the function $g(K(u, u), u)$ defined on U_xM_2 attains an absolute maximum $\alpha_1 := g(K(Y_1, Y_1), Y_1) > 0$ and therefore $K(Y_1, Y_1) = -(1/\sqrt{n})X + \alpha_1 Y_1$. Then, a self-adjoint operator $\mathcal{A}(Y) : T_xM_2 \to T_xM_2$ can be defined by

$$\mathcal{A}(Y) := K_{Y_1}Y - g(K(Y_1, Y), X)X. \tag{5.11}$$

It is obvious that $\mathcal{A}(Y_1) = \alpha_1 Y_1$. Choosing the unit eigenvectors $\{Y_j\}_{j=2}^{n-1}$ of $\mathcal{A}$ orthogonal to Y_1 satisfying $\mathcal{A}(Y_j) = \alpha_j Y_j$, we can conclude from lemma 2.4 that $\alpha_1 \geqslant 2\alpha_j$, where $\alpha_j := g(K(Y_1, Y_j), Y_j)$. It then follows that

$$K(Y_1, Y_1) = -\frac{1}{\sqrt{n}}X + \alpha_1 Y_1, \quad K(Y_1, Y_j) = \alpha_j Y_j, \quad 2 \leqslant j \leqslant n-1. \tag{5.12}$$

Therefore, a straightforward calculation by using (2.7) and (2.29) shows that

$$2\nu_2 Y_j = R(Y_j, Y_1)Y_1 = \left(\frac{n+1}{n} + \alpha_1\alpha_j - \alpha_j^2\right)Y_j, \tag{5.13}$$

which together with (5.8) immediately gives

$$\alpha_j^2 - \alpha_1\alpha_j + c - \frac{n+1}{n} = 0, \quad 2 \leqslant j \leqslant n-1. \tag{5.14}$$

On the contrary, by the minimality of M^n we easily get:

$$0 = \operatorname{trace} K_{Y_1} = \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1}. \tag{5.15}$$

This combining with (5.14) yields that

$$(n+1) - nc = \frac{n}{n-2}(\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_{n-1}^2) > 0. \tag{5.16}$$

Consequently, noting the fact $\alpha_1 \geqslant 2\alpha_j$, we solve from (5.14) and (5.15) to obtain that

$$\alpha_1 = (n-2)\sqrt{\frac{(n+1)-nc}{n(n-1)}}, \quad \alpha_2 = \cdots = \alpha_{n-1} = -\sqrt{\frac{(n+1)-nc}{n(n-1)}}. \tag{5.17}$$

Hence, the assertion of step 1 follows immediately.

Step 2. *There exists a vector field $\bar{V}_1$ on a neighbourhood $\bar{U}$ of x such that the tensor K of M^n takes the form:*

$$K(\bar{V}_1, \bar{V}_1) = -\frac{1}{\sqrt{n}}g(\bar{V}_1, \bar{V}_1)E_1 + \alpha_1\bar{V}_1. \tag{5.18}$$

where α_1 is defined as in (5.17).

Choose an arbitrary differentiable orthonormal frame $\{\bar{Y}\}_{i=1}^{n-1}$ on a neighbourhood U of $x \in M^n$ such that $\bar{Y}_i(x) = Y_i$, and define a mapping $\varphi : \mathbb{R}^{n-1} \times U \to$

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$\mathbb{R}^{n-1}$ by

$$\varphi(a_1, a_2, \ldots, a_{n-1}, \tilde{x}) = (b_1, b_2, \ldots, b_{n-1}), \tag{5.19}$$

where

$$b_k := \sum_{i,j=1}^{n-1} a_i a_j g(K(\bar{Y}_i, \bar{Y}_j), \bar{Y}_k) - \alpha_1 a_k, \quad 1 \leqslant k \leqslant n-1, \tag{5.20}$$

are regarded as functions on $\mathbb{R}^{n-1} \times U : b_k = b_k(a_1,\, a_2,\, \ldots,\, a_{n-1},\, \tilde{x})$. By means of (5.10), it is easy to see that $b_k(1,\, 0,\, \ldots,\, 0,\, x) = 0$ for all k, and

$$\left.\frac{\partial b_k}{\partial a_j}\right|_{(1,0,\ldots,0,x)} = \begin{cases} (n-2)\sqrt{\dfrac{(n+1)-nc}{n(n-1)}} > 0, & k = j = 1, \\ -\sqrt{\dfrac{n(n+1)-n^2c}{n-1}} \neq 0, & 2 \leqslant k = j \leqslant n-1, \\ 0, & 1 \leqslant k \neq j \leqslant n-1. \end{cases} \tag{5.21}$$

This implies that $(\partial b_k/\partial a_j)$ is invertible at the point $(1,\, 0,\, \ldots,\, 0,\, x) \in \mathbb{R}^{n-1} \times U$. Consequently, by the implicit function theorem there exist differentiable functions $\{a_i(\tilde{x})\}_{1 \leqslant i \leqslant n-1}$ which are defined on a neighbourhood $\bar{U} \subset U$ of x and satisfy the relations:

$$\begin{aligned} &a_1(x) = 1, \quad a_2(x) = \cdots = a_{n-1}(x) = 0, \\ &b_k(a_1(\tilde{x}), a_2(\tilde{x}), \ldots, a_{n-1}(\tilde{x}), \tilde{x}) \equiv 0, \quad 1 \leqslant k \leqslant n-1, \quad \forall\, \tilde{x} \in \tilde{U}. \end{aligned} \tag{5.22}$$

Now, we put $\bar{V}_1 = \sum_{i=1}^{n-1} a_i \bar{Y}_i$ and thus $\bar{V}_1(x) = Y_1$. Finally, we obtain (5.18) by applying (5.20) and (5.22). Hence, step 2 has been proved.

Step 3. *There exists an orthonormal frame* $\{\tilde{Y}_i\}_{i=1}^{n-1}$ *on a neighbourhood* $\tilde{U} \subset \bar{U}$ *of* x *such that the tensor* K *of* M^n *takes the forms:*

$$K(\tilde{Y}_1, \tilde{Y}_1) = -\frac{1}{\sqrt{n}} E_1 + (n-2)\sqrt{\frac{(n{+}1)-nc}{n(n{-}1)}}\,\tilde{Y}_1, \quad K(\tilde{Y}_1, \tilde{Y}_j) = -\sqrt{\frac{(n{+}1)-nc}{n(n-1)}}\,\tilde{Y}_j, \tag{5.23}$$

where $(n+1) - nc > 0$ *and* $2 \leqslant j \leqslant n-1$.

For our purposes, we shall verify that the set

$$\Lambda_x := \left\{ \tilde{\alpha}_1 \in \mathbb{R} \,|\, \exists\, V_1 \in U_x M_2,\ s.t.\ K(V_1, V_1) = -\frac{1}{\sqrt{n}} X + \tilde{\alpha}_1 V_1 \right\} \tag{5.24}$$

consists of finite numbers, which are independent of the point $x \in M^n$. With the help of (5.10), we first find that Λ_x is non-empty. Furthermore, for an arbitrary $\tilde{\alpha}_1$ associated with $V_1 \in U_x M_2$ satisfying $K(V_1,\, V_1) = -(1/\sqrt{n})X + \tilde{\alpha}_1 V_1$, we can define another self-adjoint operator $\mathcal{B}(Y) : T_x M_2 \to T_x M_2$ by

$$\mathcal{B}(Y) := K_{V_1} Y - g(K(V_1, Y), X) X, \tag{5.25}$$

where $X := E_1(x)$. As a result, $\mathcal{B}(V_1) = \tilde{\alpha}_1 V_1$. Let $\{V_j\}_{j=2}^{n-1}$ be the unit eigenvectors of $\mathcal{B}$, orthogonal to V_1, with the corresponding eigenvalues $\{\tilde{\alpha}_j\}_{j=2}^{n-1}$, respectively.

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Then

$$K(V_1,V_1)=-\frac{1}{\sqrt{n}}X+\tilde{\alpha}_1V_1,\quad K(V_1,V_j)=\tilde{\alpha}_jV_j,\ 2\leqslant j\leqslant n-1. \tag{5.26}$$

In this situation, applying (2.7) and (2.29), we easily get:

$$\tilde{\alpha}_j^2-\tilde{\alpha}_1\tilde{\alpha}_j+c-\frac{n+1}{n}=0,\quad 2\leqslant j\leqslant n-1, \tag{5.27}$$

which combining with $(n+1)-nc>0$ implies that there exists an integer $0\leqslant k\leqslant n-2$ such that, if necessary, after renumbering the basis, we have

$$\begin{aligned}\tilde{\alpha}_2=\cdots=\tilde{\alpha}_{k+1}&=\frac{1}{2}\left(\tilde{\alpha}_1+\sqrt{\tilde{\alpha}_1^2-4\left(c-\frac{n+1}{n}\right)}\right),\\ \tilde{\alpha}_{k+2}=\cdots=\tilde{\alpha}_{n-1}&=\frac{1}{2}\left(\tilde{\alpha}_1-\sqrt{\tilde{\alpha}_1^2-4\left(c-\frac{n+1}{n}\right)}\right).\end{aligned} \tag{5.28}$$

As $0=\operatorname{trace}K_{V_1}=\tilde{\alpha}_1+\tilde{\alpha}_2+\cdots+\tilde{\alpha}_{n-1}$, it can be checked from (5.28) that

$$n\tilde{\alpha}_1-(n-2k-2)\sqrt{\tilde{\alpha}_1^2-4\left(c-\frac{n+1}{n}\right)}=0. \tag{5.29}$$

Hence, Λ_x consists of finite numbers that are independent of the point $x\in M^n$.

According to step 2, we see that $\|\bar{V}_1\|(x)=1$ for $\|\bar{V}_1\|:=\sqrt{g(\bar{V}_1,\bar{V}_1)}$. For this reason, there exists a neighbourhood $\tilde{U}\subset\bar{U}$ of x so that $\bar{V}_1$ does not vanish on $\tilde{U}$. Thus, setting $\tilde{Y}_1=\bar{V}_1/\|\bar{V}_1\|$, we derive from (5.18) that

$$K(\tilde{Y}_1,\tilde{Y}_1)=-\frac{1}{\sqrt{n}}E_1+\frac{\alpha_1}{\|\bar{V}_1\|}\tilde{Y}_1, \tag{5.30}$$

where α_1 is defined as in (5.17). As $\alpha_1/\|\bar{V}_1\|$ changes continuously on $\tilde{U}$, we conclude that $\|\bar{V}_1\|(\tilde{x})=\|\bar{V}_1\|(x)=1$ for any $\tilde{x}\in\tilde{U}$, because Λ_x consists of finite numbers. Similarly, after taking orthonormal vector fields $\tilde{Y}_2,\ldots,\tilde{Y}_{n-1}$ orthogonal to $\tilde{Y}_1$ such that $\{\tilde{Y}_i\}_{i=1}^{n-1}$ forms an orthonormal frame on $\tilde{U}$, we follow the proof of step 1 to obtain (5.23) and finally complete the proof of step 3.

Step 4. *Show that* $g(K(E_i,E_j),E_k)=K_{ij}^k\equiv 0$ *for* $2\leqslant i,j,k\leqslant n$.

Direct calculations by using (5.23) shows that

$$\begin{aligned}&(\nabla_{\tilde{Y}_j}K)(\tilde{Y}_1,\tilde{Y}_1)=(\alpha_1-2\alpha_2)\nabla_{\tilde{Y}_j}\tilde{Y}_1,\\ &(\nabla_{\tilde{Y}_1}K)(\tilde{Y}_j,\tilde{Y}_1)=\alpha_2\nabla_{\tilde{Y}_1}\tilde{Y}_j-K(\nabla_{\tilde{Y}_1}\tilde{Y}_j,\tilde{Y}_1)-K(\tilde{Y}_j,\nabla_{\tilde{Y}_1}\tilde{Y}_1),\end{aligned} \tag{5.31}$$

where α_1 and α_2 are defined as in (5.17) and $2\leqslant j\leqslant n-1$. Based on the relation

$$g((\nabla_{\tilde{Y}_j}K)(\tilde{Y}_1,\tilde{Y}_1),\tilde{Y}_1)=g((\nabla_{\tilde{Y}_1}K)(\tilde{Y}_j,\tilde{Y}_1),\tilde{Y}_1), \tag{5.32}$$

we deduce from (5.23) that $\nabla_{\tilde{Y}_1}\tilde{Y}_1=0$ and thus $(\nabla_{\tilde{Y}_1}K)(\tilde{Y}_j,\tilde{Y}_1)=0$ for $2\leqslant j\leqslant n-1$. Together with the fact $\alpha_1-2\alpha_2\neq 0$, it is seen from (5.31) that $\nabla_{\tilde{Y}_j}\tilde{Y}_1=0$

for all $j \geqslant 2$. By the definition of the curvature tensor, we apply (4.46) and (5.8) to obtain that

$$c = 2\nu_2 = g(R(\tilde{Y}_2, \tilde{Y}_1)\tilde{Y}_1, \tilde{Y}_2) = 0. \tag{5.33}$$

This is a contradiction to $c \neq 0$ and hence we complete the proof of step 4.

Now, for the orthonormal frame $\{E_i\}_{i=1}^n$ on M^n, from (5.9) it follows that

$$\begin{aligned} &h(E_1, E_1) = \frac{n-1}{\sqrt{n}}\varphi E_1, \quad h(E_1, E_i) = -\frac{1}{\sqrt{n}}\varphi E_i, \\ &h(E_i, E_j) = -\frac{1}{\sqrt{n}}\delta_{ij}\varphi E_1, \quad 2 \leqslant i, j \leqslant n. \end{aligned} \tag{5.34}$$

Consequently, by applying theorem 2.1 we conclude from (3.4) and (5.34) that M^n is locally congruent to the Calabi torus. □

5.1. Completion of the proof of theorem 1.3

M^n $(n \geqslant 3)$ be a conformally flat minimal Legendrian submanifold in the unit sphere $\mathbb{S}^{2n+1}$ with semi-parallel tensor K. By means of lemma 4.7, we see that either M^n is of constant sectional curvature, or it is quasi-Einstein with ν_1 and ν_2 the distinct eigenvalues of its Schouten operator such that $\nu_1 + \nu_2 = 0$, where ν_1 is simple. In the former case, theorem 1.1 states that M^n is the totally geodesic sphere or the flat Clifford torus. In the latter case, according to proposition 5.1 and theorem 5.2, we conclude that M^n is locally congruent to the Calabi torus. Conversely, these calculations in § 3 guarantee that examples (a)–(c) are all conformally flat minimal Legendrian submanifolds in $\mathbb{S}^{2n+1}$ with semi-parallel tensor K.

5.2. Completion of the proof of corollary 1.5

Let M^n $(n \geqslant 3)$ be a minimal Legendrian submanifold in the unit sphere $\mathbb{S}^{2n+1}$ with semi-parallel tensor K. By means of lemma 4.1 and proposition 4.3, we calculate that

$$\|\bar{\nabla}^\xi h\|^2 = \frac{1}{2}\Delta S = \|\bar{\nabla}^\xi h\|^2 - \|W\|^2 - \frac{n+2}{n-2}\|\tilde{\mathrm{Ric}}\|^2 + (n+1)S\chi. \tag{5.35}$$

Consequently, we have

$$(n+1)S\chi = \frac{n+2}{n-2}\|\tilde{\mathrm{Ric}}\|^2 + \|W\|^2 \geqslant \frac{n+2}{n-2}\|\tilde{\mathrm{Ric}}\|^2, \tag{5.36}$$

by which we obtain (1.3) and find that the equality holds on M^n if and only if the Weyl curvature tensor of M^n vanishes identically. When $n \geqslant 4$, the assertion immediately follows from theorem 1.3 and proposition 3.1. When $n = 3$, according to lemma 4.7 and remark 4.8, either M^3 has constant sectional curvature, or M^3 is quasi-Einstein with ν_1 and ν_2 the distinct eigenvalues of its Schouten operator so that $\nu_1 + \nu_2 = 0$, where ν_1 is simple. In the former case, we obtain from theorem 1.1 that M^3 is the totally geodesic sphere or the flat Clifford torus. In the latter case, by remark 4.8 and claim 1, we deduce from (4.34) and (4.36) that

$$\begin{aligned} &g(K(E_i,E_j),E_k)=0, \quad 2\leqslant i,j,k\leqslant 3,\\ &K(E_i,E_i)=K(E_j,E_j), \quad K(E_i,E_j)=0, \quad 2\leqslant i\neq j\leqslant 3, \end{aligned} \tag{5.37}$$

where E_1 is a unit vector field of $\mathfrak{D}(\nu_1)$ and $\{E_2, E_3\}$ is an arbitrary orthonormal frame of $\mathfrak{D}(\nu_2)$. As M^3 is minimal in $\mathbb{S}^7$, we further have

$$\begin{aligned} &g(K(E_1,E_1),E_1)=-2g(K(E_i,E_i),E_1), \quad 2\leqslant i\leqslant 3,\\ &g(K(E_1,E_1),E_i)=-\sum_{j=2}^{3}g(K(E_j,E_j),E_i)=0, \quad 2\leqslant i\leqslant 3. \end{aligned} \tag{5.38}$$

Therefore, similar calculations as in the proof of proposition 5.1 show that

$$\begin{aligned} &h(E_1,E_1)=\frac{2}{\sqrt{3}}\varphi E_1, \quad h(E_1,E_i)=-\frac{1}{\sqrt{3}}\varphi E_i,\\ &h(E_i,E_j)=-\frac{1}{\sqrt{3}}\delta_{ij}\varphi E_1, \quad 2\leqslant i,j\leqslant 3. \end{aligned} \tag{5.39}$$

Finally, combining with (3.4) and (5.39), we derive from theorem 2.1 that M^n is locally congruent to the Calabi torus. Conversely, by § 3 these examples (a)–(c) for $n=3$ are all minimal Legendrian submanifolds in $\mathbb{S}^7$ with semi-parallel tensor K.

5.3. Completion of the proof of corollary 1.6

Let M^n ($n\geqslant 3$) be a closed minimal Legendrian submanifold in the unit sphere $\mathbb{S}^{2n+1}$ with vanishing Weyl curvature tensor. By calculation we apply (4.18) to obtain that

$$\frac{1}{2}\Delta S=\|\bar{\nabla}^{\xi}h\|^2-\frac{n+2}{n-2}\|\tilde{\mathrm{Ric}}\|^2+(n+1)S\chi\geqslant-\frac{n+2}{n-2}\|\tilde{\mathrm{Ric}}\|^2+(n+1)S\chi. \tag{5.40}$$

Now, by using the compactness of M^n, we can integrate inequality (5.40) to obtain the integral inequality in (1.4), according to the divergence theorem, where the equality holds on M^n if and only if M^n is of C-parallel second fundamental form, i.e. $\bar{\nabla}^{\xi}h=0$, which implies that $R\cdot K=0$. When $n\geqslant 4$, the assertion follows from theorem 1.3 and proposition 3.1 immediately. When $n=3$, Theorem 1.3 and (1.5) of Xing–Zhai [**33**] state that M^3 is locally congruent to one of the examples (a)–(c) for $n=3$. Conversely, by § 3 the examples (a)–(c) for $n=3$ are closed minimal Legendrian submanifolds in $\mathbb{S}^7$ with C-parallel second fundamental form.

5.4. Completion of the proof of theorem 1.7

Let M^n ($n\geqslant 3$) be a conformally flat minimal Legendrian submanifold in the unit sphere $\mathbb{S}^{2n+1}$ with semi-parallel Ricci tensor. Together with (4.40), it then follows from lemma 4.9 that either M^n has constant sectional curvature, or M^n is quasi-Einstein with ν_1 and ν_2 the distinct eigenvalues of its Schouten operator such that $\nu_1+\nu_2=0$, where ν_1 is simple. In the former case, either M^n is the totally

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geodesic sphere, or M^n is the flat Clifford torus, in terms of theorem 1.1. In the latter case, applying proposition 5.1 and theorem 5.2, we derive that M^n is locally congruent to the Calabi torus. Conversely, it is easy to see from proposition 3.1 that examples (a)–(c) are all conformally flat minimal Legendrian submanifolds in $\mathbb{S}^{2n+1}$ with semi-parallel Ricci tensor.

Data availability

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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Competing interests

None.

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