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Connected components of affine Deligne–Lusztig varieties in mixed characteristic

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In the proof of Theorem 1.1, the following assertion in line 4 of page 1725 is incorrect:

> The kernel of the composition $\pi_1(M)^\Gamma \to \pi_1(G)^\Gamma$ is generated by the elements $\sum_{\beta\in\Omega}\beta^\vee$, where $\Omega \in \Phi_{N,\Gamma}$ satisfies $\Omega \cap C \neq \emptyset$ (C defined as in Proposition 4.1.9).

The mistake consists of a misapplication of Proposition 4.1.9, which asserts only that an element in the kernel is a $\mathbb{Z}$-linear combination of elements in the Galois orbit of C. Although an element in the kernel is Γ-invariant, in general this is not enough to imply that it is a sum of elements of the form $\sum_{\beta\in\Omega}\beta^\vee$, where one sums over a Galois orbit Ω of an element in C. More precisely, the assertion is incorrect in certain cases when there are Galois orbits of coroots of different orders. This was pointed out to the authors by Sian Nie.

We replace the above argument by the proposition below. All other assertions of the paper including the rest of the proof of Theorem 1.1 remain unchanged.

Proposition 0.0.1. *There exists $\Phi^0_{N,\Gamma} \subseteq \Phi_{N,\Gamma}$ such that:*

(i) *the kernel of the map $\pi_1(M)^\Gamma \to \pi_1(G)^\Gamma$ is generated by the elements $\sum_{\beta\in\Omega_0}\beta^\vee$ and $\sum_{\beta\in\Omega}\beta^\vee$, where $\Omega \in \Phi_{N,\Gamma}$ satisfies $\Omega\cap C \neq \emptyset$ and $\Omega_0 \in \Phi^0_{N,\Gamma}$;*

(ii) *for any element $\Omega_0 \in \Phi^0_{N,\Gamma}$, the element $\sum_{\beta\in\Omega_0}\beta^\vee$ is mapped to 1 by the composite $\pi_1(M)^\Gamma \cong \pi_0(X^M_{\mu_x}(b)) \to \pi_0(X^G_\mu(b))$.*

Lemma 0.0.2. *Let $x, x' \in \bar{I}^{M,G}_{\mu,b}$ be such that $x' = x + \alpha^\vee - \alpha^{m\vee}$ with α an adapted positive root in N. Let Ω be the Galois orbit of α. Then, for all $g'M(\mathcal{O}_L) \in X^M_{\mu_{x'}}(b)$, there exist $gM(\mathcal{O}_L) \in X^M_{\mu_x}(b)$ and $n \in \mathbb{Z}$ such that $g \sim g'$ and*

$$w_M(g) - w_M(g') = \sum_{i=0}^{m-1}\alpha^\vee + n\sum_{\beta\in\Omega}\beta^\vee \quad \text{in } \pi_1(M).$$

Proof. If $x' \to x$ is of immediate distance and $x' - x = \alpha^\vee - \alpha^{m\vee}$ is as in Definition 4.4.8, then for any $g'M(\mathcal{O}_L) \in X^M_{\mu_{x'}}(b)$, there exists $gM(\mathcal{O}_L) \in X^M_{\mu_x}(b)$ such that $g \sim g'$ and

$$w_M(g) - w_M(g') = \sum_{i=0}^{m-1}\alpha^{i\vee} \quad \text{in } \pi_1(M)$$

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by Proposition 4.5.4 (in which there is also a sign typo in (4.5.5)) and the fact that $J_b^M(F)$ acts transitively on $X^M_{\mu_x}(b)$. Similarly, if $x \to x'$, then for any $g'M(\mathcal{O}_L) \in X^M_{\mu_{x'}}(b)$, there exists $gM(\mathcal{O}_L) \in X^M_{\mu_x}(b)$ such that $g \sim g'$ and

$$w_M(g') - w_M(g) = \sum_{i=m}^{|\Omega|-1} \alpha^{i\vee} \quad \text{in } \pi_1(M).$$

Hence,

$$w_M(g) - w_M(g') = \sum_{i=0}^{m-1} \alpha^{i\vee} - \sum_{\beta\in\Omega} \beta^\vee \quad \text{in } \pi_1(M).$$

The general case is reduced to the immediate distance case by the proof of Proposition 4.4.10. □

Proof of Proposition 0.0.1. Let $S := \{\sum_{\beta\in\Omega} \beta^\vee \in \pi_1(M)^\Gamma | \Omega \in \Phi_{N,\Gamma} \text{ and } \Omega \cap C \neq \emptyset\}$.

If the kernel of the composition $\pi_1(M)^\Gamma \to \pi_1(G)^\Gamma$ is generated by the elements in S, then take $\Phi^0_{N,\Gamma} = \emptyset$ and we are done.

Therefore, it suffices to prove the proposition under the following hypothesis.

(HYP): the kernel of the composition $\pi_1(M)^\Gamma \to \pi_1(G)^\Gamma$ is *not* generated by the elements in S.

Then not all elements in $\Phi_{N,\Gamma}$ have the same cardinality and hence the Dynkin diagram of G is of type A_{2n+1}, D_n or E_6. Let $\Phi^{\text{small}}_{N,\Gamma}$ be the subset of $\Phi_{N,\Gamma}$ consisting of the orbits of smallest cardinality, and let $\Phi^{\text{large}}_{N,\Gamma} = \Phi_{N,\Gamma} \backslash \Phi^{\text{small}}_{N,\Gamma}$. Then the elements of $\Phi^{\text{large}}_{N,\Gamma}$ are all of the same cardinality, which is n_G times the cardinality of the elements in $\Phi^{\text{small}}_{N,\Gamma}$ with $n_G = 2$ or 3. Here, $n_G = 3$ only occurs when the Dynkin diagram of G is of type D_4. Moreover, by Proposition 4.1.9, for every $\Omega \in \Phi^{\text{large}}_{N,\Gamma}$, $\sum_{\beta\in\Omega} \beta^\vee$ is contained in the subgroup of $\pi_1(M)^\Gamma$ generated by S. Hence, we will define $\Phi^0_{N,\Gamma}$ as a subset of $\Phi^{\text{small}}_{N,\Gamma}$.

Case 1: the Dynkin diagram of G is of type A_{2n+1} or of type D_4 with $n_G = 3$. One can easily show that the kernel of the map $\pi_1(M)^\Gamma \to \pi_1(G)^\Gamma$ is generated by the elements in S and $\sum_{\beta\in\Omega'} \beta^\vee$ for any $\Omega' \in \Phi^{\text{small}}_{N,\Gamma}$. Hence, it remains to find $\Omega_0 \in \Phi^{\text{small}}_{N,\Gamma}$ satisfying condition (2) in the statement and to define $\Phi^0_{N,\Gamma} = \{\Omega_0\}$.

Claim 1. *Under hypothesis (HYP), there exist adapted positive roots $\alpha_i \in \Omega_i \in \Phi^{\text{large}}_{N,\Gamma}$ for $i = 1, 2$ such that $\alpha_1 - \alpha_2 = \alpha_1^d - \alpha_2^d$ is still a root in N, where d is the number of connected components of the Dynkin diagram of G and such that*

$$x' = x + \alpha_1^\vee - \alpha_1^{d\vee} = x + \alpha_2^\vee - \alpha_2^{d\vee} \in \bar{I}^{M,G}_{\mu,b}.$$

We only show Claim 1 when the Dynkin diagram is of type A_{2n+1}. The proof for the other case of type D_4 is similar and much easier and is therefore omitted. By Proposition 4.1.9, we can choose a connected component of the Dynkin diagram of G with the following numbering of the simple roots.

$$A_{2n+1}: \quad \underset{\gamma_{-n}}{\circ} - \underset{\gamma_{-n+1}}{\circ} \cdots \underset{\gamma_{-1}}{\circ} - \underset{\gamma_0}{\circ} - \underset{\gamma_1}{\circ} \cdots \underset{\gamma_{n-1}}{\circ} - \underset{\gamma_n}{\circ}$$

such that there exists a pair $(i_0, j_0) \in \mathbb{N}^2$ with

$$\langle \gamma_{-i_0} + \cdots + \gamma_0 + \cdots + \gamma_{j_0}, \mu_x \rangle = -1. \tag{0.0.3}$$

Moreover, by condition (HYP) for all $0 \leqslant i \leqslant n$, $\langle \gamma_{-i} + \cdots + \gamma_0 + \cdots + \gamma_i, \mu_x \rangle \neq -1$; therefore, we may assume that $i_0 > j_0$ (possibly exchanging the notations γ_i and γ_{-i}) and that

$$\langle \gamma_{-i_0} + \cdots + \gamma_{-j_0-1}, \mu_x \rangle = -1, \quad \langle \gamma_{j_0+1} + \cdots + \gamma_{i_0}, \mu_x \rangle = 1.$$

It follows that

$$\alpha_1 = \gamma_{-i_0} + \cdots + \gamma_{j_0}, \quad \alpha_2 = \gamma_{-i_0} + \cdots + \gamma_{-j_0-1}$$

are the desired elements in Claim 1.

Let $\Omega_0 \in \Phi_{N,\Gamma}^{\mathrm{small}}$ be the Galois orbit of $\alpha_1 - \alpha_2$, and $\Phi_{N,\Gamma}^0 = \{\Omega_0\}$. We want to show that $\sum_{\beta \in \Omega_0} \beta^\vee \in \pi_1(M)^\Gamma$ is mapped to 1 under the composite $\pi_1(M)^\Gamma \cong \pi_0(X_{\mu_x}^M(b)) \to \pi_0(X_\mu^G(b))$. By Lemma 0.0.2, for any $g'M(\mathcal{O}_L) \in X_{\mu_{x'}}^M(b)$, there exist $g_1 M(\mathcal{O}_L), g_2 M(\mathcal{O}_L) \in X_{\mu_x}^M(b)$ and $m_1, m_2 \in \mathbb{Z}$ such that $g_1 \sim g' \sim g_2$ and

$$\begin{aligned} w_M(g_1) - w_M(g') &= \sum_{i=0}^{d-1} \alpha_1^i + m_1 \sum_{\beta \in \Omega_1} \beta^\vee \quad \text{in } \pi_1(M), \\ w_M(g_2) - w_M(g') &= \sum_{i=0}^{d-1} \alpha_2^i + m_2 \sum_{\beta \in \Omega_2} \beta^\vee \quad \text{in } \pi_1(M). \end{aligned}$$

Taking the difference of the above two equalities, we get

$$w_M(g_1) - w_M(g_2) = \sum_{\beta \in \Omega_0} \beta^\vee + (m_1 - m_2) \sum_{\beta \in \Omega_1} \beta^\vee + m_2 n_G \sum_{\beta \in \Omega_0} \beta^\vee \quad \text{in } \pi_1(M).$$

Using $g_1 \sim g_2$ and the fact that $J_b^M(F)$ acts transitively on $X_{\mu_x}^M(b)$, the element

$$w_M(g_2) - w_M(g_1) \in \pi_1(M)^\Gamma$$

is mapped to 1 by the composite $\pi_1(M)^\Gamma \cong \pi_0(X_{\mu_x}^M(b)) \to \pi_0(X_\mu^G(b))$.

On the other hand, as $\Omega_0 \in \Phi_{N,\Gamma}^{\mathrm{small}}$ and $\Omega_1 \in \Phi_{N,\Gamma}^{\mathrm{large}}$, by Proposition 4.1.9, $n_G \sum_{\beta \in \Omega_0} \beta^\vee \in \pi_1(M)^\Gamma$ and $\sum_{\beta \in \Omega_1} \beta^\vee \in \pi_1(M)^\Gamma$ are both contained in the subgroup generated by S and hence by the proof of Theorem 1.1 (more precisely the third paragraph on page 1725, which is not affected by the gap we are discussing here), $n_G \sum_{\beta \in \Omega} \beta^\vee$ and $\sum_{\beta \in \Omega_1} \beta^\vee$ are both mapped to 1 by the composite $\pi_1(M)^\Gamma \cong \pi_0(X_{\mu_x}^M(b)) \to \pi_0(X_\mu^G(b))$. Therefore, so is $\sum_{\beta \in \Omega_0} \beta^\vee$, i.e. Ω_0 satisfies condition (2) of the proposition.

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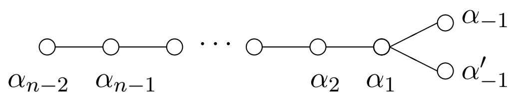
Case 2: the Dynkin diagram of G is of type D_n with $n_G = 2$. Consider any connected component of the Dynkin diagram of G with the following numbering of the simple roots.

D_n:



Let Ω_i be the Galois orbit of α_i for $1 \leqslant i \leqslant n-2$. Then the kernel of $\pi_1(M)^\Gamma \to \pi_1(G)^\Gamma$ is generated by the elements in S and $\sum_{\beta\in\Omega_i} \beta^\vee$ for $1 \leqslant i \leqslant n-2$. We will construct subsets

$$\Phi^1_{N,\Gamma} \subseteq \Phi^2_{N,\Gamma} \subseteq \cdots \subseteq \Phi^{n-2}_{N,\Gamma} =: \Phi^0_{N,\Gamma} \subseteq \Phi_{N,\Gamma}$$

such that for any $\Omega \in \Phi^{n-2}_{N,\Gamma}$, the element $\sum_{\beta\in\Omega_0} \beta^\vee$ is mapped to 1 under the composite $\pi_1(M)^\Gamma \cong \pi_0(X^M_{\mu_x}(b)) \to \pi_0(X^G_\mu(b))$ and, for any $1 \leqslant i \leqslant n-2$, $\sum_{\beta\in\Omega_i} \beta^\vee$ is contained in the subgroup of $\pi_1(M)^\Gamma$ generated by the elements in S and $\sum_{\beta\in\Omega} \beta^\vee$, where $\Omega \in \Phi^i_{N,\Gamma}$. Note that the orbits of α_{-1} and α'_{-1} are in $\Phi^{\mathrm{large}}_{N,\Gamma}$ and therefore need not be considered.

We first construct $\Phi^1_{N,\Gamma}$. By Proposition 4.1.9, there exists $\alpha \in C$ such that $\sigma^l(\alpha_1) \preceq \alpha$ for some l. Without loss of generality, we assume that $l = 0$. We may also assume that $\sum_{\beta\in\Omega_1} \beta^\vee$ is not contained in the subgroup of $\pi_1(M)^\Gamma$ generated by S, otherwise let $\Phi^1_{N,\Gamma} = \emptyset$. By (HYP), the Galois orbit of $\alpha \in C$ is in $\Phi^{\mathrm{large}}_{N,\Gamma}$. In particular, $\langle \alpha_1, \mu_x\rangle \geqslant 0$ and $\langle \alpha_{-1}, \mu_x\rangle = -1$ (possibly exchanging α_{-1} and α'_{-1}). By the existence of α and the minimality of μ_x, there are four possibilities:

Case 2.1: $\langle \alpha_1, \mu_x\rangle = 0$, $\langle \alpha'_{-1}, \mu_x\rangle = 1$;
Case 2.2: $\langle \alpha_1, \mu_x\rangle = 0$, $\langle \alpha'_{-1}, \mu_x\rangle = 0$;
Case 2.3: $\langle \alpha_1, \mu_x\rangle = 1$, $\langle \alpha'_{-1}, \mu_x\rangle = -1$;
Case 2.4: $\langle \alpha_1, \mu_x\rangle = 1$, $\langle \alpha'_{-1}, \mu_x\rangle = 0$ and there exist $2 \leqslant i \leqslant n-2$ such that

$$\langle \alpha_j, \mu_x\rangle = \begin{cases} -1, & j = i, \\ 0, & 2 \leqslant j \leqslant i-1. \end{cases}$$

In Cases 2.2 and 2.3, we have $\alpha_1 + \alpha_{-1} + \alpha'_{-1} \in C$. In Case 2.4, $\alpha_i + \cdots + \alpha_1 + \alpha_{-1} + \alpha'_{-1} \in C$ and $\alpha_i + \cdots + \alpha_2 \in C$. Thus, for Cases 2.2–2.4, $\sum_{\beta\in\Omega_1} \beta^\vee$ is contained in the subgroup of $\pi_1(M)^\Gamma$ generated by S and hence these cases will not occur.

It remains to consider Case 2.1. In that case

$$x + \alpha_{-1}^\vee - {\alpha'_{-1}}^\vee = x + (\alpha_{-1} + \alpha_1)^\vee - (\alpha'_{-1} + \alpha_1)^\vee \in \bar{I}^{M,G}_{\mu,b}.$$

The same computation as in Case 1 shows that $\Phi^1_{N,\Gamma} := \{\Omega_1\}$ satisfies the desired properties.

For general i, we apply the same discussion as above. We obtain either $\Phi^i_{N,\Gamma} = \Phi^{i-1}_{N,\Gamma}$ or $\Phi^i_{N,\Gamma} = \Phi^{i-1}_{N,\Gamma} \cup \{\tilde{\Omega}_i\}$ with $\tilde{\Omega}_i$ the Galois orbit of $\alpha_1 + \cdots + \alpha_i$. Altogether, the proposition holds in Case 2.

Case 3: the Dynkin diagram of G is of type E_6. The discussion is very similar to Case 2 and is therefore omitted. □

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