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On the formal Peterson subalgebra and its dual

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Abstract. In the present notes, we study a generalization of the Peterson subalgebra to an oriented (generalized) cohomology theory which we call the formal Peterson subalgebra. Observe that by recent results of Zhong the dual of the formal Peterson algebra provides an algebraic model for the oriented cohomology of the affine Grassmannian.

Our first result shows that the centre of the formal affine Demazure algebra (FADA) generates the formal Peterson subalgebra. Our second observation is motivated by the Peterson conjecture. We show that a certain localization of the formal Peterson subalgebra for the extended Dynkin diagram of type $\hat{A}_1$ provides an algebraic model for "quantum" oriented cohomology of the projective line. Our last result can be viewed as an extension of the previous results on Hopf algebroids of structure algebras of moment graphs to the case of affine root systems. We prove that the dual of the formal Peterson subalgebra (an oriented cohomology of the affine Grassmannian) is the zeroth Hochschild homology of the FADA.

1 Introduction

Equivariant cohomology of an affine Grassmannian has been a topic of intensive investigations for decades. For the small torus action, it can be identified with a certain commutative subalgebra of the associated nil–Hecke algebra of a Kac–Moody root system called the Peterson subalgebra [16]. One of its remarkable properties says that after taking localization it becomes isomorphic to the (small) quantum cohomology of the respective finite part (flag variety) [12, 16]. A parallel isomorphism for the K-theory was conjectured and discussed in [10, 11] and is known as the Peterson Conjecture. This conjecture was recently proven by Kato in [7] using a language of semi-infinite flag varieties.

In the present notes, we study a generalization of the Peterson subalgebra to an oriented (generalized) cohomology theory $h(\text{-})$, e.g., algebraic cobordism $\Omega(\text{-})$ of Levine–Morel. Such a cohomology theory was first introduced and studied in [15], and extended to the torus-equivariant setup in [6, 8] for arbitrary smooth varieties. As for flag varieties associated with root systems, it can be described using the Kostant–Kumar localization approach (for finite root systems see [3–5], and for

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Kac–Moody see [2]). The respective generalization of the nil Hecke algebra is called the formal affine Demazure algebra (FADA). The generalization of the Peterson algebra introduced recently in [18] which we call a formal Peterson subalgebra is then the centralizer of the equivariant coefficient ring in the small torus FADA.

To state our first result, let $R = h(pt)$ denote the coefficient ring of the oriented theory $h(\text{-})$, let $S = h_T(pt)$ denote the respective small torus T equivariant coefficient ring, let $\mathbf{D}_{W_a}$ denote the small torus FADA, and let $\mathbf{D}_{Q^\vee}$ denote the formal Peterson subalgebra as constructed in [18]. We then obtain the following important property of the centre of FADA.

Theorem 1.1 (cf. Theorem 4.4) *If $\mathbb{Q} \subseteq R$, then the centre $Z(\mathbf{D}_{W_a})$ of the small torus FADA generates the formal Peterson subalgebra $\mathbf{D}_{Q^\vee}$ as an S-module. Moreover, the centre $Z(\mathbf{D}_{W_a})$ generates $\mathbf{D}_{W_a}$ as a $\mathbf{D}_W$-module, where $\mathbf{D}_W$ stands for the FADA associated with the finite part of the Kac–Moody root system.*

Our next result can be viewed as an extension of the Peterson conjecture.

Theorem 1.2 (cf. Theorem 5.6) *The localization $\mathbf{D}_{Q^\vee,\mathrm{loc}}$ of the formal Peterson subalgebra $\mathbf{D}_{Q^\vee}$ with respect to an affine root system of type $\hat{A}_1$ has the following presentation:*

$$\mathbf{D}_{Q^\vee,\mathrm{loc}} \simeq S[\mathfrak{t}, \mathfrak{t}^{-1}][\mathfrak{s}]/(\mathfrak{s}^2 - x_{-1}\mathfrak{s}\mathfrak{t} - \mu\mathfrak{t}),$$

where x_{-1} is a certain characteristic class in h and μ is an element depending on x_{-1}.

For cohomology and K-theory, this presentation gives quantum cohomology and quantum K-theory of $\mathbb{P}^1$, respectively. Hence, $\mathbf{D}_{Q^\vee,\mathrm{loc}}$ can be viewed as the "quantum" oriented cohomology of the projective line $\mathbb{P}^1$.

As for our last result, observe that the S-linear dual $\mathbf{D}^*_{Q^\vee}$ of the formal Peterson subalgebra is a natural model for the (small torus) equivariant oriented cohomology of the affine Grassmannian [18]. We obtain the following "Kac–Moody" analog of results of [13].

Theorem 1.3 (cf. Theorem 6.2) *The S-linear dual $\mathbf{D}^*_{Q^\vee}$ of the formal Peterson subalgebra is isomorphic to the 0-th Hochschild homology of the dual $\mathbf{D}^*_{W_a}$ of the small torus FADA.*

Here, the dual $\mathbf{D}^*_{W_a}$ can be interpreted as a model for the T-equivariant oriented cohomology of the respective affine flag variety. Therefore, it has two commuting actions by the equivariant coefficient ring S. Following the ideas of [13] one defines its zeroth Hochschild homology as the quotient obtained by merging these two S-module structures. To prove this result, we introduce a special filtration on the dual $\mathbf{D}^*_{Q^\vee}$ (to reduce it to finite cases). This approach seems to be new even for cohomology and K-theory.

The article is organized as follows: Section 2 revisits the definition of the formal Peterson subalgebra $\mathbf{D}_{Q^\vee}$ from [18]. In Section 3, we establish some basic properties of $\mathbf{D}_{Q^\vee}$ and study the action of $\mathbf{D}_{W_a}$ on it. In Section 4, we study Borel isomorphisms involving the FADA and the Peterson subalgebra, and prove our first main result Theorem 4.4. In Section 5, we focus on the example of type $\hat{A}^1$ and prove our second result, Theorem 5.6. In the last section, we investigate the dual of the formal Peterson subalgebra, and prove our third main result Theorem 6.2. In the appendix, we prove several combinatorial properties of the affine Weyl group that are used in the proof of Theorem 6.2.

2 The formal Peterson subalgebra

In this section, we recall the definition of a small torus FADA and the formal Peterson algebra following [2, 18].

Given an oriented algebraic cohomology theory $h(\text{-})$, in the sense of Levine–Morel (see [15]) there is an associated formal group law F over a commutative ring R with characteristic 0. Here, $R = h(pt)$ is the coefficient ring, and F is defined from the Quillen formula for the characteristic class of a tensor product of line bundles. For example, for connective K-theory (see, e.g., [17]) we have $F_\beta(x, y) = x + y - \beta xy$ over the polynomial ring $R = \mathbb{Z}[\beta]$. Specializing to $\beta = 1$ (resp. $\beta = 0$), one obtains the usual K-theory (resp. cohomology). In these notes, by usual cohomology, we always mean its algebraic part: the Chow ring (modulo rational equivalence) with rational coefficients.

Given a lattice Λ (free abelian group of finite rank) and a formal group law F, consider the associated formal group algebra S of [1] that is the quotient of the power series ring

$$S = R[[\Lambda]]_F = R[[x_\lambda \colon \lambda \in \Lambda]]/\mathcal{J}_F,$$

where $\mathcal{J}_F$ is the closure of the ideal of relations

$$(x_0, x_{\lambda_1+\lambda_2} - F(x_{\lambda_1}, x_{\lambda_2}) \colon \lambda_1, \lambda_2 \in \Lambda).$$

In the case $F = F_\beta$, we set S to be the quotient of the polynomial ring

$$S := R[x_\lambda \colon \lambda \in \Lambda]/\langle x_0, x_{\lambda_1+\lambda_2} - F_\beta(x_{\lambda_1}, x_{\lambda_2})\rangle.$$

Let Φ be a finite irreducible root system with a fixed subset $I = \{\alpha_1, \ldots, \alpha_n\}$ of simple roots. Let Q and $Q^\vee$ denote the root and the coroot lattice, respectively. Let W denote the Weyl group, generated by simple reflections s_{α_i}, $\alpha_i \in I$. Consider an affine root system corresponding to the extended Dynkin diagram for Φ with the extra simple root

$$\alpha_0 = -\theta + \delta \in Q \oplus \mathbb{Z}\delta,$$

where $\theta \in \Phi$ is the highest root and δ is the so called null root so that $s_{\alpha_0} = t_{\theta^\vee} s_\theta$ is an extra generator of the respective affine Weyl group $W_{\mathrm{a}} = Q^\vee \rtimes W$. Recall that the latter is generated by reflections $s_{\alpha+k\delta} = t_{-k\alpha^\vee} s_\alpha$, where $s_\alpha \in W$ is a reflection and t_λ, $\lambda \in Q^\vee$ is a translation. The affine Weyl group W_{a} acts on the lattice Q via W that is

$$t_\lambda w(\mu) = w(\mu),\ \mu \in Q,\ w \in W,\ \lambda \in Q^\vee.$$

Therefore, it also acts on the formal group algebra $S = R[[Q]]_F$.

Suppose x_α is a regular (not a zero-divisor) element in S for each $\alpha \in \Phi$. In particular, this holds if 2 is not a zero-divisor in R (see [4, Lemma 2.2]). Let $\mathcal{Q} = S[\frac{1}{x_\alpha} \colon \alpha \in \Phi]$ be the localization of S at x_αs. Consider the twisted group algebra $\mathcal{Q}_{W_{\mathrm{a}}}$ associated with the affine Weyl group W_{a}. By definition, it is a free left $\mathcal{Q}$-module

$\mathfrak{Q}_{W_a} = \mathfrak{Q} \otimes_R R[W_a]$ with basis $\{\eta_u\}_{u \in W_a}$ and the product given by

$$c\eta_u \cdot c'\eta_{u'} = cu(c')\eta_{uu'},\ c, c' \in \mathfrak{Q},\ u, u' \in W_a.$$

For each $\alpha \in \Phi$, define elements

$$\begin{aligned}
\kappa_\alpha &= \tfrac{1}{x_\alpha} + \tfrac{1}{x_{-\alpha}} \text{ (which is an element of } S\text{)},\\
X_\alpha &= \tfrac{1}{x_\alpha}(1 - \eta_{s_\alpha}) \text{ called the Demazure element},\\
Y_\alpha &= \kappa_\alpha - X_\alpha \text{ called the push-pull element},\\
X_{\alpha_0} &= \tfrac{1}{x_{-\theta}}(1 - \eta_{s_0}), \text{ and}\\
Y_{\alpha_0} &= \kappa_\theta - X_{\alpha_0}.
\end{aligned}$$

All these elements satisfy the quadratic relations (e.g., $X_\alpha^2 = \kappa_\alpha X_\alpha$) and the twisted braid relations (see, e.g., [2]). For simplicity of notation, we will omit α or s in the indices, i.e., we will write $x_i = x_{\alpha_i}$, $s_i = s_{\alpha_i}$, $\eta_i = \eta_{s_i}$, $X_i = X_{\alpha_i}$, and $Y_i = Y_{\alpha_i}$.

Similarly, consider the twisted group algebra $\mathfrak{Q}_{Q^\vee} = \mathfrak{Q} \otimes_R R[Q^\vee]$. It is a free $\mathfrak{Q}$-module with basis $\{\eta_{t_\lambda}\}_{\lambda \in Q^\vee}$. Observe that $\mathfrak{Q}_{Q^\vee}$ is commutative since $t_\lambda(c) = c$, $c \in \mathfrak{Q}$. Consider two homomorphisms of left $\mathfrak{Q}$-modules

$$\begin{aligned}
\mathrm{pr}\colon \mathfrak{Q}_{W_a} &\to \mathfrak{Q}_{Q^\vee},\ c\eta_{t_\lambda w} \mapsto c\eta_{t_\lambda},\ c \in \mathfrak{Q}, w \in W, \text{ and}\\
\iota\colon \mathfrak{Q}_{Q^\vee} &\to \mathfrak{Q}_{W_a},\ c\eta_{t_\lambda} \mapsto c\eta_{t_\lambda}.
\end{aligned}$$

By definition, ι is a section of pr, and it is a ring homomorphism. Set $\psi = \iota \circ \mathrm{pr}$, so $\psi|_{\iota(\mathfrak{Q}_{Q^\vee})} = \mathrm{id}$. Define elements

$$Z_\alpha = \tfrac{1}{x_{-\alpha}}(1 - \eta_{t_{\alpha^\vee}}),\ \alpha \in \Phi.$$

Lemma 2.1 *We have $\psi(zX_i) = 0$ for any $z \in \mathfrak{Q}_{W_a}$, $\alpha_i \in I$, and $\psi(X_0) = Z_\theta$.*

Proof For $z = c\eta_u$, $c \in \mathfrak{Q}$, $u \in W_a$, and $\alpha_i \in I$, we have

$$\begin{aligned}
\mathrm{pr}(c\eta_u X_i) &= \mathrm{pr}\Big(c\eta_u \tfrac{1}{x_{\alpha_i}}(1 - \eta_{s_i})\Big)\\
&= \mathrm{pr}\Big(\tfrac{c}{u(x_{\alpha_i})}(\eta_u - \eta_{us_i})\Big) = 0.
\end{aligned}$$

As for the X_0, we obtain

$$\begin{aligned}
\psi(X_0) &= \iota \circ \mathrm{pr}\big(\tfrac{1}{x_{-\theta}}(1 - \eta_{s_0})\big)\\
&= \tfrac{1}{x_{-\theta}}\iota \circ \mathrm{pr}\big((1 - \eta_{t_{\theta^\vee} s_\theta})\big)\\
&= \tfrac{1}{x_{-\theta}}(1 - \eta_{t_{\theta^\vee}}) = Z_\theta,
\end{aligned}$$

and the proof is finished. ∎

Denote $I_a = \{\alpha_0, \dots, \alpha_n\}$. Following [2] define the small torus FADA $\mathbf{D}_{W_a}$ to be the subring of $\mathfrak{Q}_{W_a}$ generated by S and the elements X_i, $\alpha_i \in I_a$. Set $\mathbf{D}_{W_a/W} = \mathrm{pr}(\mathbf{D}_{W_a})$ to be its image in $\mathfrak{Q}_{Q^\vee}$. We called it the relative FADA. Assuming $\mathbb{Q} \subset R$ and using the small torus GKM descripition it is proven in [18, Lemma 5.1] that the map ι induces a map $\mathbf{D}_{W_a/W} \to \mathbf{D}_{W_a}$. We then define the formal Peterson subalgebra to be the image of the relative FADA

$$\mathbf{D}_{Q^\vee} := \imath(\mathbf{D}_{W_a/W}).$$

According to [18, Theorem 5.7] the formal Peterson subalgebra $\mathbf{D}_{Q^\vee}$ is a Hopf subalgebra in $\mathfrak{Q}_{Q^\vee}$. Moreover, $\mathbf{D}_{Q^\vee}$ coincides with the centralizer $C_{\mathbf{D}_{W_a}}(S)$ of the formal group algebra S in the FADA $\mathbf{D}_{W_a}$.

3 Properties of the FADA and the formal Peterson subalgebra

In the present section, we establish several properties of the FADA and the formal Peterson subalgebra. For the K-theory, some of these properties were proven in [7] using different arguments. We start with the following version of the projection formula.

Lemma 3.1 *For any $z, z' \in \mathfrak{Q}_{W_a}$ and $\xi \in \mathfrak{Q}_{Q^\vee}$ we have in $\mathfrak{Q}_{Q^\vee}$*

(i) $\mathrm{pr}(\imath(\xi)z) = \xi\mathrm{pr}(z)$, *and*
(ii) $\mathrm{pr}(z\sigma z') = \mathrm{pr}(z)\mathrm{pr}(\sigma z')$, *where* $\sigma = \sum_{w\in W}\eta_w$.

Observe that for the K-theory the property (ii) played a key role in [7, Theorem 1.7].

Proof (i) Let $\xi = c_1\eta_{t_{\lambda_1}}$ and $z = c_2\eta_{t_{\lambda_2}w}$, where $w \in W$, $c_i \in \mathfrak{Q}$, $\lambda_i \in Q^\vee$. Then, we obtain

$$\begin{aligned}\mathrm{pr}(\imath(\xi)z) &= \mathrm{pr}(c_1\eta_{t_{\lambda_1}}c_2\eta_{t_{\lambda_2}w}) = \mathrm{pr}(c_1t_{\lambda_1}(c_2)\eta_{t_{\lambda_1+\lambda_2}w})\\ &= c_1t_{\lambda_1}(c_2)\eta_{t_{\lambda_1+\lambda_2}} = c_1\eta_{t_{\lambda_1}}c_2\eta_{t_{\lambda_2}} = \xi\mathrm{pr}(z).\end{aligned}$$

(ii) Let $z = c\eta_{t_\lambda v}$ and $z' = c'\eta_{t_{\lambda'}v'}$, $v, v' \in W$. Then, we get

$$\begin{aligned}\mathrm{pr}(z\sigma z') &= \mathrm{pr}\Big(c\eta_{t_\lambda v}\sum_{w\in W}\eta_w c'\eta_{t_{\lambda'}v'}\Big)\\ &= \mathrm{pr}\Big(\sum_{w\in W}ct_\lambda vw(c')\eta_{t_\lambda vwt_{\lambda'}v'}\Big).\end{aligned}$$

Since $t_\lambda vwt_{\lambda'}v' = t_\lambda(vwt_{\lambda'}(vw)^{-1})vwv' = t_\lambda t_{vw(\lambda')}vwv'$, reindexing the sum by $w' = vw$ we obtain

$$\begin{aligned}\mathrm{pr}(z\sigma z') &= \mathrm{pr}\Big(\sum_{w'\in W}ct_\lambda w'(c')\eta_{t_\lambda t_{w'(\lambda')}w'v'}\Big)\\ &= \sum_{w'\in W}ct_\lambda w'(c')\eta_{t_\lambda t_{w'(\lambda')}}.\end{aligned}$$

On the other side, $\mathrm{pr}(z) = c\eta_{t_\lambda}$ and

$$\begin{aligned}\mathrm{pr}(\sigma z') &= \mathrm{pr}\Big(\sum_w\eta_w c'\eta_{t_{\lambda'}v'}\Big) = \mathrm{pr}\Big(\sum_w w(c')\eta_{wt_{\lambda'}v'}\Big)\\ &= \mathrm{pr}\Big(\sum_w w(c')\eta_{t_{w(\lambda')}wv'}\Big) = \sum_w w(c')\eta_{t_{w(\lambda')}}.\end{aligned}$$

The result then follows. ∎

We now extend the Hecke action on the Peterson algebra for the K-theory introduced in [7, Section 2] to the action on the formal Peterson algebra.

We define an action of $\mathfrak{Q}_{W_a}$ on $\mathfrak{Q}_{Q^\vee}$ by

$$z \diamond \xi := \mathrm{pr}(z\imath(\xi)),\ z \in \mathfrak{Q}_{W_a},\ \xi \in \mathfrak{Q}_{Q^\vee}.$$

More explicitly, we have

$$c\eta_{t_\lambda w} \diamond c'\eta_{t_{\lambda'}} = cw(c')\eta_{t_{\lambda+w(\lambda')}},\ c, c' \in \mathfrak{Q},\ w \in W. \tag{3.1}$$

Direct computation shows that W_a is an action.

For $w \in W$, $\xi \in \mathfrak{Q}_{Q^\vee}$, and $\alpha \in \Phi$ define

$$w(\xi) = \eta_w \diamond \xi, \text{ and}$$
$$\Delta_\alpha(\xi) = X_\alpha \diamond \xi = \tfrac{1}{x_\alpha}(\xi - s_\alpha(\xi)).$$

We then have the following.

Lemma 3.2 *For any $z, z' \in \mathfrak{Q}_{W_a}$ and $\xi \in \mathfrak{Q}_{Q^\vee}$ we have*

(i) $\mathrm{pr}(zz') = z \diamond \mathrm{pr}(z')$*, and, in particular,*

$$\mathrm{pr}(X_i z) = X_i \diamond \mathrm{pr}(z) = \Delta_i(\mathrm{pr}(z)),\ \alpha_i \in I,$$

(ii) $X_0 \diamond \xi = \Delta_{-\theta}(\xi) + Z_\theta s_\theta(\xi)$.

Proof (i) Let $z = c\eta_{t_\lambda w}$ and $z' = c'\eta_{t_{\lambda'} w'}$. Then, we obtain

$$\begin{aligned}\mathrm{pr}(zz') &= \mathrm{pr}(c\eta_{t_\lambda w} c'\eta_{t_{\lambda'} w'}) = \mathrm{pr}(ct_\lambda w(c')\eta_{t_{\lambda+w(\lambda')} ww'}) \\ &= ct_\lambda w(c')\eta_{t_{\lambda+w(\lambda')}} = c\eta_{t_\lambda w} \diamond c'\eta_{t_{\lambda'}}.\end{aligned}$$

(ii) For $\xi = c\eta_{t_\lambda}$, we get

$$\begin{aligned}X_0 \diamond (c\eta_{t_\lambda}) &= \tfrac{c}{x_{-\theta}}\eta_{t_\lambda} - \tfrac{s_0(c)}{x_{-\theta}}\eta_{t_{s_\theta(\lambda)} t_{\theta^\vee}} \\ &= \tfrac{c}{x_{-\theta}}\eta_{t_\lambda} - \tfrac{s_\theta(c)}{x_{-\theta}}\eta_{t_{s_\theta(\lambda)} t_{\theta^\vee}} - \tfrac{s_\theta(c)}{x_{-\theta}}\eta_{t_{s_\theta(\lambda)}} + \tfrac{s_\theta(c)}{x_{-\theta}}\eta_{t_{s_\theta(\lambda)}} \\ &= \Delta_{-\theta}(c\eta_{t_\lambda}) + Z_\theta s_\theta(c\eta_{t_\lambda})\end{aligned}$$

and the result follows. ∎

Lemma 3.3 *The $\diamond$-action of $\mathfrak{Q}_{W_a}$ on $\mathfrak{Q}_{Q^\vee}$ induces an action of $\mathbf{D}_{W_a}$ on $\mathbf{D}_{W_a/W}$.*

Proof Let $z, z' \in \mathbf{D}_{W_a}$, and let $\xi = \mathrm{pr}(z')$. Then, we have

$$z \diamond \xi = z \diamond \mathrm{pr}(z') = \mathrm{pr}(zz') \in \mathrm{pr}(\mathbf{D}_{W_a}) = \mathbf{D}_{W_a/W}$$

and the lemma follows. ∎

Identifying the formal Peterson algebra $\mathbf{D}_{Q^\vee}$ (resp. $\imath(\mathfrak{Q}_{Q^\vee})$) with $\mathbf{D}_{W_a/W}$ (resp. $\mathfrak{Q}_{Q^\vee}$) via the ring homomorphism $\imath$ we obtain an action of $\mathbf{D}_{W_a}$ on $\mathbf{D}_{Q^\vee}$ and an action of $\mathfrak{Q}_{W_a}$ on $\imath(\mathfrak{Q}_{Q^\vee})$. From this point on, we write ξ as both an element in $\mathbf{D}_{W_a/W}$ (resp. $\mathfrak{Q}_{Q^\vee}$) and in $\mathbf{D}_{Q^\vee} = \imath(\mathbf{D}_{W_a/W})$ (resp. $\imath(\mathfrak{Q}_{Q^\vee})$). If we consider a product $\xi_1\xi_2$ with $\xi_i \in \mathfrak{Q}_{Q^\vee}$, we may assume it is in $\mathfrak{Q}_{W_a}$. However, for the product $z\xi$ with $z \in \mathfrak{Q}_{W_a}$ and $\xi \in \mathfrak{Q}_{Q^\vee}$, we view ξ as an element in $\mathfrak{Q}_{W_a}$ via the map $\imath$. Following these identifications we obtain

$$z \diamond \xi = \psi(z\xi),\ \text{where } \xi \in \mathbf{D}_{Q^\vee} \subset \imath(\mathfrak{Q}_{Q^\vee}),\ z \in \mathbf{D}_{W_a} \subset \mathfrak{Q}_{W_a}, \tag{3.2}$$

and Lemma 3.1 gives

$$\psi(\xi z) = \xi\psi(z). \tag{3.3}$$

Example 3.4 Consider the affine root system of extended Dynkin type $\hat{A}_2$. It has three simple roots $\alpha_0, \alpha_1, \alpha_2$ and the highest root $\theta = \alpha_1 + \alpha_2$. Denote $X_{ij} = X_i X_j$ for simplicity. Direct computations then give:

$$\begin{aligned}\psi(X_{10}) &= X_1 \diamond X_0 = \tfrac{1}{x_1} Z_{\alpha_1+\alpha_2} - \tfrac{1}{x_1} Z_{\alpha_2},\\ \psi(X_{20}) &= \tfrac{1}{x_2} Z_{\alpha_1+\alpha_2} - \tfrac{1}{x_2} Z_{\alpha_1},\\ \psi(X_{210}) &= X_2 \diamond \psi(X_{10}).\end{aligned}$$

Finally, we describe the centre of FADA.

Lemma 3.5 (i) *For any $\xi \in \mathfrak{Q}_{Q^\vee}$ and $\alpha_i \in I$, we have*

$$\eta_i \diamond \xi = \xi \iff \eta_i \xi = \xi \eta_i.$$

Moreover, if this condition is satisfied, we have

$$c\eta_i \diamond (\xi\xi') = \xi(c\eta_i \diamond \xi').$$

(ii) *The centres of $\mathfrak{Q}_{W_a}$ and $\mathbf{D}_{W_a}$ can be described as follows:*

$$\begin{aligned}Z(\mathfrak{Q}_{W_a}) &= \{\xi \in \mathfrak{Q}_{Q^\vee} : \eta_w \xi = \xi \eta_w, \forall w \in W\} = (\mathfrak{Q}_{Q^\vee})^W,\\ Z(\mathbf{D}_{W_a}) &= \{\xi \in \mathbf{D}_{Q^\vee} : \eta_w \xi = \xi \eta_w, \forall w \in W\} = (\mathbf{D}_{Q^\vee})^W.\end{aligned}$$

(iii) *There are ring homomorphisms*

$$\mathfrak{Q}_{W_a} \to \mathrm{End}_{(\mathfrak{Q}_{Q^\vee})^W}(\mathfrak{Q}_{Q^\vee}),\ \mathbf{D}_{W_a} \to \mathrm{End}_{(\mathbf{D}_{Q^\vee})^W}(\mathbf{D}_{Q^\vee}),\ z \mapsto z \diamond -.$$

Proof (i) For a given $\xi = \sum_\lambda c_\lambda \eta_{t_\lambda}$, $c_\lambda \in \mathfrak{Q}$, we get

$$\begin{aligned}\eta_i \xi &= \sum_\lambda s_i(c_\lambda)\eta_i\eta_{t_\lambda} = \sum_\lambda s_i(c_\lambda)\eta_{t_{s_i(\lambda)}}\eta_i\\ &= \sum_{\lambda'} s_i(c_{s_i(\lambda')})\eta_{t_{\lambda'}}\eta_i, \text{ where } \lambda' = s_i(\lambda).\end{aligned}$$

On the other side, we have

$$\eta_i \diamond \xi = \psi(\eta_i \xi) = \psi(\eta_i \sum_\lambda c_\lambda \eta_{t_\lambda}) = \sum_\lambda s_i(c_{s_i(\lambda)})\eta_{t_\lambda}.$$

Therefore, $\eta_i \diamond \xi = \xi$ if and only if $s_i(c_{s_i(\lambda)}) = c_\lambda$ for any $\lambda \in Q^\vee$, which is equivalent to say that $\eta_i \xi = \xi \eta_i$.

Now if this condition is satisfied, then

$$c\eta_i \diamond (\xi\xi') = \psi(c\eta_i\xi\xi') = \psi(\xi c\eta_i\xi') \overset{(3.3)}{=} \xi\psi(c\eta_i\xi') \overset{(3.2)}{=} \xi(c\eta_i \diamond \xi').$$

(ii) Since $\mathfrak{Q}_{Q^\vee} = \imath(\mathfrak{Q}_{Q^\vee}) = C_{\mathfrak{Q}_{W_a}}(\mathfrak{Q})$, we have $Z(\mathfrak{Q}_{W_a}) \subset \mathfrak{Q}_{Q^\vee}$, and the first identity then follows. By part (i), we know that $\eta_i \diamond \xi = \xi$, $\forall \alpha_i \in I$ is equivalent to $\eta_w \xi = \xi \eta_w$, $\forall w \in W$ that is equivalent to $z\xi = \xi z$, $\forall z \in \mathfrak{Q}_W$. Since ξ already commutes with $\eta_{t_\lambda}, \lambda \in Q^\vee$, ξ belongs to the centre $Z(\mathfrak{Q}_{W_a})$. Conversely, if $\xi \in Z(\mathfrak{Q}_{W_a}) \cap \mathfrak{Q}_{Q^\vee}$, then it is invariant under all $\eta_i, \alpha_i \in I$. The description of the centre $Z(\mathbf{D}_{W_a})$ follows similarly.

(iii) Follows from parts (i) and (ii). ∎

4 Borel isomorphisms

In this section, we study Borel isomorphisms involving the FADA and the formal Peterson subalgebra. We assume $\mathbb{Q} \subset R$ throughout this section.

Consider the left S-linear dual $\mathbf{D}_W^*$ embedded into $\mathfrak{Q}_W^*$. The latter has a $\mathfrak{Q}$-basis $\{f_w\}_{w\in W}$. Following [4, Section 11] there is an (equivariant) characteristic map

$$\mathfrak{c}: S \to \mathbf{D}_W^*, \quad a \mapsto \sum_{w\in W} w(a) f_w \tag{4.1}$$

which induces the Borel isomorphism (see [4, Theorem 11.4])

$$\rho: S \otimes_{S^W} S \to \mathbf{D}_W^*, \quad a \otimes b \mapsto a\mathfrak{c}(b) = \sum_{w\in W} a w(b) f_w. \tag{4.2}$$

Recall that $\sigma = \sum_{w\in W} \eta_w \in \mathbf{D}_{W_\mathrm{a}}$. Denote $\mathbf{x} = \prod_{\alpha\in\Phi^+} x_{-\alpha}$ and $Y = \sigma\frac{1}{\mathbf{x}}$.

By [3, Lemma 10.12] ($Y = Y_\Pi$) we have $Y \in \mathbf{D}_W$. We denote by X_{I_u}, Y_{I_u} products corresponding to a reduced sequence I_u of $u \in W_\mathrm{a}$.

Lemma 4.1 *We have $\sigma\mathbf{D}_W = Y\mathbf{D}_W = YS$.*

Proof Observe that $Y = \frac{1}{|W|}\sigma Y$, so $YS \subset Y\mathbf{D}_W \subset \sigma Y\mathbf{D}_W \subset \sigma\mathbf{D}_W$. Conversely,

$$\sigma\mathbf{D}_W = Y\mathbf{x}\mathbf{D}_W \subset Y\mathbf{D}_W.$$

Note that $\mathbf{D}_W$ is also a right S module with basis $X_{I_v}, v \in W$, and $YX_{I_v} = \delta_{v,e}Y$. So given $X_{I_v}b \in \mathbf{D}_W$ with $v \in W$, $b \in S$, we have

$$YX_{I_v}b = \delta_{v,e}Yb \in YS.$$

So $\sigma\mathbf{D}_W \subset YS$, and the result follows. ■

Lemma 4.2 *We have $SYS = \mathbf{D}_W$. So $\mathbf{D}_W$ is a cyclic S-S-bimodule.*

Proof According to [3, Lemma 10.3] $\mathbf{x}f_e \in \mathbf{D}_W^*$. Let $\sum_i a_i \otimes b_i \in S \otimes_{S^W} S$ so that $\rho(\sum_i a_i \otimes b_i) = \mathbf{x}f_e$. Then,

$$\sum_i a_i w(b_i) = \begin{cases} \mathbf{x}, & w = e, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore,

$$\sum_i a_i Y b_i = \sum_{w\in W}\sum_i a_i w(b_i)\eta_w \tfrac{1}{\mathbf{x}} = 1.$$

Finally, by Lemma 4.1, for any $z \in \mathbf{D}_W$, we can write $z = \sum_i a_i Y b_i z = \sum_i a_i Y b_i'$ for some $b_i' \in S$. ■

Lemma 4.3 *We have $\psi(\sigma\mathbf{D}_{W_\mathrm{a}}) = Z(\mathbf{D}_{W_\mathrm{a}})$.*

Proof Let $z = c\eta_{t_\lambda u}$, where $c \in \mathfrak{Q}$, $u \in W$. We have

$$\psi(\sigma z) = \psi\Big(\sum_{w\in W} \eta_w c \eta_{t_\lambda u}\Big) = \psi\Big(\sum_{w\in W} w(c)\eta_{t_{w(\lambda)}}\eta_{wu}\Big)$$
$$= \sum_{w\in W} w(c)\eta_{t_{w(\lambda)}} = \sum_{w\in W} \eta_w c t_\lambda \eta_{w^{-1}},$$

so it obviously belongs to $Z(\mathbf{D}_{W_\mathrm{a}})$.

As for the opposite inclusion, take $z \in Z(\mathbf{D}_{W_a})$. Since $Z(\mathbf{D}_{W_a}) \subset C_{\mathbf{D}_{W_a}}(S) = \mathbf{D}_{Q^\vee}$, we get $\psi(z) = z$. Observe that $\mathrm{pr}(z'\sigma) = |W|\mathrm{pr}(z')$ for any $z' \in \mathbf{D}_{W_a}$. So we obtain

$$\psi\Big(\sigma z \tfrac{1}{|W|}\Big) = \psi\Big(z\sigma \tfrac{1}{|W|}\Big) = \psi(z) = z.$$

Thus, $z \in \psi(\sigma \mathbf{D}_{W_a})$, and the proof is finished. ∎

Consider two ring homomorphisms induced by the usual mutiplication:

$$\Theta\colon S \otimes_{S^W} Z(\mathbf{D}_{W_a}) \longrightarrow \mathbf{D}_{Q^\vee}. \tag{4.3}$$

$$\Xi\colon \mathbf{D}_W \otimes_{S^W} Z(\mathbf{D}_{W_a}) \longrightarrow \mathbf{D}_{W_a}. \tag{4.4}$$

Note that in the definition of Θ and Ξ, one can switch the tensor factors. Moreover, both homomorphisms are left S-linear. The following is our first main result.

Theorem 4.4 *Assume $\mathbb{Q} \subset R$. The maps Θ and Ξ are ring isomorphisms.*

Remark 4.5 The geometric interpretation of this theorem is well-known for equivariant homology and equivariant K-homology. As explained in [9, 12], we have the following isomorphisms of algebras

$$\mathbf{D}_W \simeq H_*^T(G/B), \qquad \mathbf{D}_{W_a} \simeq H_*^T(Fl_G), \qquad \mathbf{D}_{Q^\vee} \simeq H_*^T(Gr_G),$$

where G/B is the flag variety, Fl_G is the affine flag variety, and Gr_G is the affine Grassmannian. By (ii) of Lemma 3.5, we could identify

$$Z(\mathbf{D}_{W_a}) = (\mathbf{D}_{Q^\vee})^W = H_*^T(Gr_G)^W = H_*^G(Gr_G).$$

As a result, the morphisms Θ and Ξ can be viewed as the isomorphisms of algebras

$$\Theta\colon H_T^*(pt) \otimes_{H_G^*(pt)} H_*^G(Gr_G) \simeq H_*^T(Gr_G),$$
$$\Xi\colon H_*^T(G/B) \otimes_{H_G^*(pt)} H_*^G(Gr_G) \simeq H_*^T(Fl_G).$$

The isomorphism Ξ can be also rewritten in a more familiar form

$$H_*^T(G/B) \otimes_{H_T^*(pt)} H_*^T(Gr_G) \simeq H_*^T(Fl_G),$$

which is what Corollary 4.8 implies.

For a general equivariant oriented cohomology theory h, it follows from [5] that $\mathbf{D}_W^*$ is isomorphic to the equivariant oriented cohomology $h_T(G/B)$. However, the respective results for Fl_G and Gr_G are not known since they are not varieties of finite type. Therefore, our isomorphism in this case serves as the algebraic analogs of the potentially-correct geometric result. Observe also that in general, the isomorphisms in Corollary 4.8 are only isomorphisms of modules.

The proof of the theorem will occupy the rest of this section. We start proving the surjectivity first.

Lemma 4.6 *The map $\Theta\colon S \otimes_{S^W} Z(\mathbf{D}_{W_a}) \to \mathbf{D}_{Q^\vee}$ given in (4.3) is surjective.*

Proof Consider the following diagram

$$\begin{array}{ccc} S \otimes_{S^W} \sigma \mathbf{D}_{W_{\mathrm{a}}} & \longrightarrow & \mathbf{D}_{W_{\mathrm{a}}} \\ \Big\downarrow{\scriptstyle \mathrm{id}\otimes\psi} & & \Big\downarrow{\scriptstyle \psi} \\ S \otimes_{S^W} Z(\mathbf{D}_{W_{\mathrm{a}}}) & \xrightarrow{\Theta} & \mathbf{D}_{Q^\vee} \end{array}.$$

Since ψ is an S-module homomorphism, this diagram commutes.

By Lemma 4.2, we can write $1 = \sum_i a_i Y b_i$ for some $a_i, b_i \in S$. For any $z \in \mathbf{D}_{W_{\mathrm{a}}}$, we then have $z = \sum_i a_i Y b_i z$. This shows that elements of $Y\mathbf{D}_{W_{\mathrm{a}}}$ generate $\mathbf{D}_{W_{\mathrm{a}}}$ as a left S-module. Similarly to the proof of Lemma 4.1, we obtain that $Y\mathbf{D}_{W_{\mathrm{a}}} = \sigma\mathbf{D}_{W_{\mathrm{a}}}$. So the top horizontal map is surjective, and the result follows. ∎

Lemma 4.7 *The map* $\Xi\colon \mathbf{D}_W \otimes_{S^W} Z(\mathbf{D}_{W_{\mathrm{a}}}) \to \mathbf{D}_{W_{\mathrm{a}}}$ *given in (4.4) is surjective.*

Proof Since the elements of $\mathbf{D}_W$ and of $Z(\mathbf{D}_{W_{\mathrm{a}}})$ commute with each others, the image of Ξ is the subalgebra generated by $\mathbf{D}_W$ and $Z(\mathbf{D}_{W_{\mathrm{a}}})$. It contains S and X_i for $\alpha_i \in I$ by definition, so it suffices to show that it contains X_0 as well. Observe that

$$\begin{aligned} X_0 &= \tfrac{1}{x_\theta}(1 - \eta_{s_\theta}\eta_{t_{-\theta^\vee}}) \\ &= \tfrac{1}{x_\theta}(1-\eta_{s_\theta}) + \eta_{s_\theta}\tfrac{1}{x_{-\theta}}(1-\eta_{t_{-\theta^\vee}}) \\ &= X_\theta + \eta_{s_\theta}\tfrac{x_\theta}{x_{-\theta}} Z_\theta. \end{aligned}$$

Since $Z_\theta \in \mathbf{D}_{Q^\vee}$ by [18, Lemma 4.1], we have $X_\theta \in \mathbf{D}_W$ and $\eta_{s_\theta} \in \mathbf{D}_W$. So X_0 belongs to the subalgebra generated by $\mathbf{D}_W$ and $Z(\mathbf{D}_{W_{\mathrm{a}}})$. ∎

Corollary 4.8 *The maps* $\mathbf{D}_W \otimes_S \mathbf{D}_{Q^\vee} \to \mathbf{D}_{W_{\mathrm{a}}}$ *and* $\mathbf{D}_{Q^\vee} \otimes_S \mathbf{D}_W \to \mathbf{D}_{W_{\mathrm{a}}}$ *induced by the usual multiplication are isomorphisms of left S-modules (In the first map,* $\mathbf{D}_W$ *is viewed as an S-S-bimodule, and in the second map,* $\mathbf{D}_W$ *is viewed as a left S-module.)*

Proof Since $\mathbf{D}_{Q^\vee} \supset Z(\mathbf{D}_{W_{\mathrm{a}}})$, by Lemma 4.7, both maps are surjective. To prove the injectivity, we change the base to $\mathfrak{Q}$-modules by applying the exact functors $- \otimes_S \mathfrak{Q}$ and $\mathfrak{Q} \otimes -$. It then suffices to show that the induced maps

$$\mathbf{D}_W \otimes_S \mathbf{D}_{Q^\vee} \otimes_S \mathfrak{Q} = \mathfrak{Q}_W \otimes_{\mathfrak{Q}} \mathfrak{Q}_{Q^\vee} \longrightarrow \mathfrak{Q}_{W_{\mathrm{a}}} = \mathbf{D}_{W_{\mathrm{a}}} \otimes_S \mathfrak{Q}$$

$$\mathfrak{Q} \otimes_S \mathbf{D}_{Q^\vee} \otimes_S \mathbf{D}_W = \mathfrak{Q}_{Q^\vee} \otimes_{\mathfrak{Q}} \mathfrak{Q}_W \longrightarrow \mathfrak{Q}_{W_{\mathrm{a}}} = \mathfrak{Q} \otimes_S \mathbf{D}_{W_{\mathrm{a}}}$$

are injective. But these are even isomorphisms. So, the conclusion follows. ∎

We now discuss injectivity of the maps in the theorem.

For any parabolic subgroup W_P of W, we denote by W^P the subset of minimal length left coset representatives. Consider the $\mathfrak{Q}$-linear dual $\mathfrak{Q}^*_{W^P} = \mathrm{Hom}(W^P, \mathfrak{Q})$ with a basis $\{f_w\}_{w\in W^P}$. One can also identify it with the invariants $(\mathfrak{Q}^*_W)^{W_P}$ by identifying each f_w, $w \in W^P$ with $\sum_{v\in W_P} f_{wv} \in (\mathfrak{Q}^*_W)^{W_P}$ (see [3, Section 11] for more details).

Lemma 4.9 *The map* $\rho_{P,\mathfrak{Q}}\colon S \otimes_{S^W} \mathfrak{Q}^{W_P} \to \mathfrak{Q}^*_{W^P}$ *defined by*

$$\rho_{P,\mathfrak{Q}}(c_1 \otimes c_2) := \sum_{w\in W^P} c_1 w(c_2) f_w$$

is an isomorphism.

Proof Assume first that $P = B$ (the Borel case). Then, the map $\rho_{P,\mathfrak{Q}}$ is obtained from the isomorphism ρ by the base change with the functor $- \otimes_S \mathfrak{Q}$. So $\rho_{B,\mathfrak{Q}}$ is an isomorphism.

For a general parabolic W_P, there is a commutative diagram

$$\begin{array}{ccc} S \otimes_{S^W} \mathfrak{Q}^{W_P} & \xrightarrow{\rho_{P,\mathfrak{Q}}} & \mathfrak{Q}^*_{W^P} \\ \downarrow & & \downarrow \\ S \otimes_{S^W} \mathfrak{Q} & \xrightarrow[\sim]{\rho_{B,\mathfrak{Q}}} & \mathfrak{Q}^*_W \end{array}.$$

Both vertical maps identify the top with the W_P-invariant subsets of the bottom, so the top horizontal map is an isomorphism. ∎

Lemma 4.10 *The map $\Theta\colon S \otimes_{S^W} Z(\mathbf{D}_{W_a}) \to \mathbf{D}_{Q^\vee}$ defined in (4.3) is injective.*

Proof Let $z = \sum_{\lambda \in Q^\vee} c_\lambda \eta_{t_\lambda} \in Z(\mathbf{D}_{W_a}) \subset \mathbf{D}_{Q^\vee}$ with $c_\lambda \in \mathfrak{Q}$. Since $\eta_u z = z \eta_u$ for any $u \in W$, we have

$$(*) \qquad \forall u \in W, \quad u c_\lambda = c_{u(\lambda)}.$$

These properties give us an injective map:

$$\phi : Z(\mathbf{D}_{W_a}) \to \bigoplus_{\lambda \in Q^\vee_{\geq 0}} \mathfrak{Q}^{W_\lambda}, \qquad \sum_{\lambda \in Q^\vee} c_\lambda \eta_{t_\lambda} \longmapsto (c_\lambda)_{\lambda \in Q^\vee_{\geq 0}},$$

where $Q^\vee_{\geq 0}$ is the set of dominant coroots and W_λ is the stabilizer of λ, which is a parabolic subgroup of W.

Let W^λ denote the set of minimal length representatives of the cosets W/W_λ. Consider the following diagram

$$\begin{array}{ccccc} S \otimes_{S^W} Z(\mathbf{D}_{W_a}) & \xrightarrow{\qquad\Theta\qquad} & \mathbf{D}_{Q^\vee} & & \\ \Big\downarrow{\scriptstyle \mathrm{id} \otimes \phi} & & \Big\downarrow & & \\ \bigoplus_{\lambda \in Q^\vee_{\geq 0}} (S \otimes_{S^W} \mathfrak{Q}^{W_\lambda}) & \xrightarrow{\Theta'} & \bigoplus_{\lambda \in Q^\vee_{\geq 0}} \left(\bigoplus_{w \in W^\lambda} \mathfrak{Q} \eta_{t_{w(\lambda)}} \right) & = & \mathfrak{Q}_{Q^\vee}, \end{array}$$

where Θ' is the direct sum of maps

$$S \otimes_{S^W} \mathfrak{Q}^{W_\lambda} \longrightarrow \bigoplus_{w \in W^\lambda} \mathfrak{Q} \eta_{t_{w(\lambda)}}, \quad c_1 \otimes c_2 \longmapsto \sum_{w \in W^\lambda} c_1 w(c_2) \eta_{t_{w(\lambda)}},$$

for all $\lambda \in Q^\vee_{\geq 0}$. Since by Lemma 4.9, each such component map is injective, so is Θ'.

By direct computations and by the property (*), the diagram is commutative. Since both maps $\mathrm{id} \otimes \phi$ and Θ' are injective, so is Θ. ∎

Lemma 4.11 *The map $\Xi\colon \mathbf{D}_W \otimes_{S^W} Z(\mathbf{D}_{W_a}) \to \mathbf{D}_{W_a}$ defined in (4.4) is injective.*

Proof It follows from the combination of previous results:

$$\begin{aligned} \mathbf{D}_W \otimes_{S^W} Z(\mathbf{D}_{W_a}) &\simeq \mathbf{D}_W \otimes_S S \otimes_{S^W} Z(\mathbf{D}_{W_a}) \\ &\simeq \mathbf{D}_W \otimes_S \mathbf{D}_{Q^\vee} && (\Theta \text{ is an isomorphism}) \\ &\simeq \mathbf{D}_{W_a} && (\text{Corollary } 4.8). \end{aligned}$$ ∎

5 The $\hat{A}_1$-case

In this section, we discuss an example of the formal Peterson subalgebra for the affine root system of type $\hat{A}_1$. We show that it provides a natural model for "quantum" oriented cohomology of $\mathbb{P}^1$.

Recall that a root system of type $\hat{A}_1$ has two simple roots, $\alpha_1 = \theta = \alpha$ and $\alpha_0 = -\alpha + \delta$, and each $w \in W_{\mathrm{a}}$ has a unique reduced decomposition. We follow the notation of [11, Section 4.3] and define for $i \geq 1$:

$$\sigma_0 = e, \qquad \sigma_{2i} = (s_1 s_0)^i = t_{-i\alpha^\vee}, \qquad \sigma_{2i+1} = s_0 \sigma_{2i},$$
$$\sigma_{-2i} = (s_0 s_1)^i = t_{i\alpha^\vee}, \qquad \sigma_{-(2i+1)} = s_1 \sigma_{-2i}.$$

The set of minimal length coset representatives of W_{a}/W is then $W_{\mathrm{a}}^- = \{\sigma_i : i \geq 0\}$.

The root lattice is $Q = \mathbb{Z}\alpha$, and in the formal group algebra $S = R[[Q]]_F$, we have $x_n = x_{n\alpha} = n \cdot_F x_\alpha$, where $n \cdot_F x$, $n \in \mathbb{Z}$ is the n-fold formal sum (inverse) of x.

As for the Demazure elements, we have

$$X_j^2 = \kappa_{\alpha_j} X_j, \; j = 0, 1, \; \text{ and}$$
$$\kappa_{\alpha_j} = \kappa_\alpha = \tfrac{1}{x_\alpha} + \tfrac{1}{x_{-\alpha}} \text{ is } W_{\mathrm{a}}\text{-invariant.}$$

Set $\mu = -\frac{x_{-1}}{x_1}$. Observe that if F is of the form $F(x, y) = \frac{x+y-\beta xy}{g(x,y)}$ for some power series $g(x, y)$, we have $x_{-1} = \frac{x_1}{\beta x_1 - 1}$, hence, $\mu = \frac{1}{1-\beta x_1}$. Given a reduced expression $w = s_i s_j \ldots$, we will use the notation $Y_{ij\ldots}$ for $Y_w = Y_{I_w}$. Denote $\mathfrak{X}_w = \mathrm{pr}(X_w)$ and $\mathfrak{Y}_w = \mathrm{pr}(Y_w)$.

Example 5.1 Direct computations give:

$$\imath(\mathfrak{X}_0) = \imath(\mathfrak{X}_{\sigma_1}) = X_0 + X_1 - x_{-1} X_{01}$$
$$\imath(\mathfrak{X}_{10}) = \imath(\mathfrak{X}_{\sigma_2}) = X_{10} + \mu X_{01}$$
$$\imath(\mathfrak{X}_{010}) = \imath(\mathfrak{X}_{\sigma_3}) = X_{010} + X_{101} - x_{-1} X_{1010}$$

and

$$\mathfrak{Y}_0 = \mathfrak{Y}_{\sigma_1} = \tfrac{1}{x_1} + \tfrac{1}{x_{-1}} \eta_{t_{\alpha^\vee}}$$
$$\mathfrak{Y}_{10} = \mathfrak{Y}_{\sigma_2} = Y_1 \diamond \mathfrak{Y}_0 = \tfrac{2}{x_1 x_{-1}} + \tfrac{1}{x_{-1}^2} \eta_{t_{\alpha^\vee}} + \tfrac{1}{x_1^2} \eta_{t_{-\alpha^\vee}}$$
$$\mathfrak{Y}_{010} = \mathfrak{Y}_{\sigma_3} = \mathfrak{Y}_0 \mathfrak{Y}_{10} = \tfrac{3}{x_1^2 x_{-1}} + \tfrac{3}{x_1 x_{-1}^2} \eta_{t_{\alpha^\vee}} + \tfrac{1}{x_1^3} \eta_{t_{-\alpha^\vee}} + \tfrac{1}{x_{-1}^3} \eta_{t_{2\alpha^\vee}}.$$

These computations also show that $\mathfrak{X}_{\sigma_i}$, $i = 1, 2, 3$ satisfy identities similar to those of [10, Lemma 3].

We now look at various products of elements $\mathfrak{Y}_w \in \mathbf{D}_{Q^\vee}$.

Lemma 5.2 *For each $i \geq 1$ and $w \in W_{\mathrm{a}}^-$ we have $\mathfrak{Y}_{w\sigma_{2i}} = \mathfrak{Y}_w \mathfrak{Y}_{\sigma_{2i}}$.*
In particular, $\mathfrak{Y}_{\sigma_{2i}} = \mathfrak{Y}_{\sigma_2}^i = \mathfrak{Y}_{10}^i$ and, therefore, $\{\mathfrak{Y}_{\sigma_{2i}} : i \geq 1\}$ is a multiplicative set.

Proof Observe that $\sigma_{2i} = s_1 s_0 s_1 \cdots s_0$, so $Y_{\sigma_{2i}} = Y_1 Y_{\sigma_{2i-1}} = (1 + \eta_1) \frac{1}{x_{-\alpha}} Y_{\sigma_{2i-1}}$. By Lemma 3.1, we obtain

$$\mathfrak{Y}_{w\sigma_{2i}} = \mathrm{pr}(Y_w Y_{\sigma_{2i}}) = \mathrm{pr}(Y_w)\mathrm{pr}(Y_{\sigma_{2i}}) = \mathfrak{Y}_w \mathfrak{Y}_{\sigma_{2i}}$$

and the result follows. ∎

Lemma 5.3 *For each $i \geq 1$ we have $\mathfrak{Y}_{\sigma_{2i}} \in (\mathbf{D}_{Q^\vee})^W$, and for $j = 0, 1$*

$$Y_j \diamond \mathfrak{Y}_w = \begin{cases} \mathfrak{Y}_{s_i w}, & \text{if } s_j w > w, \\ \kappa_\alpha \mathfrak{Y}_w, & \text{if } s_j < s_j w. \end{cases}$$

In particular, we have

$$Y_{\sigma_i} \diamond \mathfrak{Y}_0 = Y_{\sigma_i} \diamond \mathfrak{Y}_{\sigma_1} = \begin{cases} \mathfrak{Y}_{\sigma_1}, & \text{if } i = 0, \\ \kappa_\alpha \mathfrak{Y}_{\sigma_i}, & \text{if } i > 0, \\ \mathfrak{Y}_{\sigma_{-i+1}}, & \text{if } i < 0. \end{cases} \tag{5.1}$$

Proof Since $\sigma_{2i} = s_1 \sigma_{2i-1}$, we get $Y_{\sigma_{2i}} = (1 + \eta_1) \frac{1}{x_{-\alpha}} Y_{\sigma_{2i-1}}$, which implies that $\eta_1 Y_{\sigma_{2i}} = Y_{\sigma_{2i}}$. By Lemma 3.1, we then obtain

$$\eta_1 \diamond \mathfrak{Y}_{\sigma_{2i}} = \eta_1 \diamond \mathrm{pr}(Y_{\sigma_{2i}}) = \mathrm{pr}(\eta_1 Y_{\sigma_{2i}}) = \mathrm{pr}(Y_{\sigma_{2i}}) = \mathfrak{Y}_{\sigma_{2i}},$$

therefore, $\mathfrak{Y}_{\sigma_{2i}} \in (\mathbf{D}_{Q^\vee})^W$. Similarly, we obtain $Y_j \diamond \mathfrak{Y}_w = \mathrm{pr}(Y_j Y_w)$, and the formula for the action follows. ∎

Corollary 5.4 *The set $\mathbf{D}_{Q^\vee}$ is a cyclic $\mathbf{D}_{W_a}$-module, generated by $\mathfrak{Y}_0 = \mathfrak{Y}_{\sigma_1}$.*
Moreover, the kernel of the map $\pi \colon \mathbf{D}_{W_a} \to \mathbf{D}_{Q^\vee}$ defined by $z \mapsto z \diamond \mathfrak{Y}_{\sigma_1}$ is $\mathbf{D}_{W_a} X_0$.

Proof The first part follows from (5.1). For the second part, we have

$$X_0 \diamond \mathfrak{Y}_{\sigma_1} = (\kappa_\alpha - Y_0) \diamond \mathfrak{Y}_{\sigma_1} = \kappa_\alpha \mathfrak{Y}_{\sigma_1} - \kappa_\alpha \mathfrak{Y}_{\sigma_1} = 0,$$

so $X_0 \in \ker \pi$.

Conversely, let $z = \sum_{i \geq 1} a_i Y_{\sigma_i} + \sum_{j \geq 0} b_j Y_{\sigma_{-j}} \in \ker \pi$. We then obtain

$$\begin{aligned} z \diamond \mathfrak{Y}_{\sigma_1} &= \Big(\sum_{i \geq 1} a_i Y_{\sigma_i} + \sum_{j \geq 0} b_j Y_{\sigma_{-j}}\Big) \diamond \mathfrak{Y}_{\sigma_1} \\ &= \sum_{i \geq 1} a_i \kappa_\alpha \mathfrak{Y}_{\sigma_i} + \sum_{j \geq 0} b_j \mathfrak{Y}_{\sigma_{j+1}} \\ &= \sum_{i \geq 1} a_i \kappa_\alpha \mathfrak{Y}_{\sigma_i} + \sum_{k \geq 1} b_{k-1} \mathfrak{Y}_{\sigma_k} \\ &= \sum_{i \geq 1} (a_i \kappa_\alpha + b_{i-1}) \mathfrak{Y}_{\sigma_i}. \end{aligned}$$

Therefore, if $z \diamond \mathfrak{Y}_{\sigma_1} = 0$, then $b_{i-1} = -a_i \kappa_\alpha$ for all $i \geq 1$, and we obtain

$$\begin{aligned} z &= \sum_{i \geq 1} a_i Y_{\sigma_i} - \sum_{j \geq 0} \kappa_\alpha a_{j+1} Y_{\sigma_{-j}} \\ &= \sum_{i \geq 1} a_i Y_{\sigma_{1-i}} Y_0 - \sum_{k \geq 1} \kappa_\alpha a_k Y_{\sigma_{1-k}} \\ &= \sum_{i \geq 1} a_i Y_{1-i} (Y_0 - \kappa_\alpha) \\ &= \Big(\sum_{i \geq 1} a_i Y_{1-i}\Big)(-X_0) \in \mathbf{D}_{W_a} X_0. \end{aligned}$$

∎

Remark 5.5 Observe that the map $\mathrm{pr} : \mathbf{D}_{W_a} \to \mathbf{D}_{W_a/W} \simeq \mathbf{D}_{Q^\vee}$ has the kernel $\mathbf{D}_{W_a} X_1 = \oplus_{i<0} S X_{\sigma_i}$, while the map π has the kernel $\mathbf{D}_{W_a} X_0 = \oplus_{i>0} S X_{\sigma_i}$.

For a general affine root system (for an extended Dynkin diagram), one can show that $\mathbf{D}_{Q^\vee}$ is a cyclic left $\mathbf{D}_{W_a}$-module, generated by $\mathrm{pr}(Y_0)$.

From the identities of Example 5.1 it follows that

$$\mathfrak{Y}_0^2 = x_{-1}\mathfrak{Y}_{010} + \mu\mathfrak{Y}_{10},$$

and we obtain the following presentation of the formal Peterson algebra in terms of generators and relations:

$$\mathbf{D}_{Q^\vee} \simeq S[\mathfrak{s}, \mathfrak{t}]/(\mathfrak{s}^2 - x_{-1}\mathfrak{s}\mathfrak{t} - \mu\mathfrak{t}),\ \mathfrak{Y}_0 \mapsto \mathfrak{s}, \mathfrak{Y}_{10} \mapsto \mathfrak{t}. \tag{5.2}$$

According to Lemma 5.2, we may define the localization

$$\mathbf{D}_{Q^\vee,\mathrm{loc}} = \mathbf{D}_{Q^\vee}[\tfrac{1}{\mathfrak{Y}_{2i}}, i \geq 1].$$

From (5.2), we then obtain our second main result.

Theorem 5.6 *We have the following presentation*

$$\mathbf{D}_{Q^\vee,\mathrm{loc}} \simeq S[\mathfrak{t}, \mathfrak{t}^{-1}][\mathfrak{s}]/(\mathfrak{s}^2 - x_{-1}\mathfrak{s}\mathfrak{t} - \mu\mathfrak{t}).$$

In particular, the action of $\mathbf{D}_{W_a}$ on $\mathbf{D}_{Q^\vee}$ extends to an action on the localization $\mathbf{D}_{Q^\vee,\mathrm{loc}}$ by

$$z \diamond (\frac{\xi}{\mathfrak{Y}_{2i}}) = \frac{z \diamond \xi}{z \diamond \mathfrak{Y}_{2i}}, \quad i \geq 1,\ \xi \in \mathbf{D}_{Q^\vee}.$$

Observe that it is well-defined since for any $i, j \geq 1$, we have

$$z \diamond (\frac{\xi\mathfrak{Y}_{\sigma_{2j}}}{\mathfrak{Y}_{\sigma_{2(i+j)}}}) = \frac{z \diamond (\xi\mathfrak{Y}_{\sigma_{2j}})}{z \diamond (\mathfrak{Y}_{\sigma_{2i}}\mathfrak{Y}_{\sigma_{2j}})} \overset{5.3}{=} \frac{(z \diamond \xi)\mathfrak{Y}_{\sigma_{2j}}}{(z \diamond \mathfrak{Y}_{\sigma_{2i}})\mathfrak{Y}_{\sigma_{2j}}} = \frac{z \diamond \xi}{z \diamond \mathfrak{Y}_{\sigma_{2i}}} = z \diamond \frac{\xi}{\mathfrak{Y}_{\sigma_{2i}}}.$$

It then follows from Corollary 5.4 that

Corollary 5.7 *The localized algebra $\mathbf{D}_{Q^\vee,\mathrm{loc}}$ is a cyclic $\mathbf{D}_{W_a}$-module generated by $\mathfrak{Y}_0$.*

Remark 5.8 Let $F(x, y) = x + y - \beta xy$. Observe that for cohomology ($\beta = 0$) and K-theory ($\beta = 1$) the localization $\mathbf{D}_{Q^\vee,\mathrm{loc}}$ computes quantum cohomology and quantum K-theory of $\mathbb{P}^1$, respectively. For instance, for K-theory the presentation (5.2) recovers that of [10, Equation 17]. Therefore, it makes sense to think of $\mathbf{D}_{Q^\vee,\mathrm{loc}}$ as an algebraic model for "quantum" oriented cohomology of the projective line $\mathbb{P}^1$.

Finally, by the result of [18] $\mathbf{D}_{Q^\vee}$ is a Hopf algebra with coproduct defined by

$$\triangle(a\eta_{t_\lambda}) = a\eta_{t_\lambda} \otimes \eta_{t_\lambda} = \eta_{t_\lambda} \otimes a\eta_{t_\lambda}.$$

In our case, we obtain

$$\begin{aligned}
\triangle(\mathfrak{Y}_0) &= \tfrac{1}{x_1}(1-\mu) + \mu(\mathfrak{Y}_0 \otimes 1 + 1 \otimes \mathfrak{Y}_0) + x_{-1}\mathfrak{Y}_0 \otimes \mathfrak{Y}_0,\\
\triangle(\mathfrak{Y}_{10}) &= \kappa_\alpha^2 + (\tfrac{x_1}{x_{-1}^2} - \tfrac{1}{x_1})(1 \otimes \mathfrak{Y}_0 + \mathfrak{Y}_0 \otimes 1) + (1 + \tfrac{x_1^2}{x_{-1}^2})\mathfrak{Y}_0 \otimes \mathfrak{Y}_0\\
&\quad + \tfrac{1}{\mu}(\mathfrak{Y}_{10} \otimes 1 + 1 \otimes \mathfrak{Y}_{10}) - \tfrac{x_1^2}{x_{-1}}(\mathfrak{Y}_0 \otimes \mathfrak{Y}_{10} + \mathfrak{Y}_{10} \otimes \mathfrak{Y}_0) + x_1^2\mathfrak{Y}_{10} \otimes \mathfrak{Y}_{10}.
\end{aligned}$$

Example 5.9 In particular, for the cohomology we get

$$\begin{aligned}\triangle(\mathfrak{Y}_0) &= 1\otimes\mathfrak{Y}_0 + \mathfrak{Y}_0\otimes 1 - x_\alpha\mathfrak{Y}_0\otimes\mathfrak{Y}_0,\\ \triangle(\mathfrak{Y}_{10}) &= 2\mathfrak{Y}_0\otimes\mathfrak{Y}_0 + (1\otimes\mathfrak{Y}_{10} + \mathfrak{Y}_{10}\otimes 1)\\ &\quad + x_\alpha(\mathfrak{Y}_0\otimes\mathfrak{Y}_{10} + \mathfrak{Y}_{10}\otimes\mathfrak{Y}_0) + x_\alpha^2\mathfrak{Y}_{10}\otimes\mathfrak{Y}_{10},\end{aligned}$$

and for the K-theory (identifying $x_\alpha = 1 - e^{-\alpha}$) we get

$$\begin{aligned}\triangle(\mathfrak{Y}_0) &= -e^\alpha + e^\alpha(\mathfrak{Y}_0\otimes 1 + 1\otimes\mathfrak{Y}_0) + (1-e^\alpha)\mathfrak{Y}_0\otimes\mathfrak{Y}_0,\\ \triangle(\mathfrak{Y}_{10}) &= 1-(1+e^\alpha)(\mathfrak{Y}_0\otimes 1 + 1\otimes\mathfrak{Y}_0) + (1+e^{-2\alpha})\mathfrak{Y}_0\otimes\mathfrak{Y}_0\\ &\quad + e^{-\alpha}(\mathfrak{Y}_{10}\otimes 1 + 1\otimes\mathfrak{Y}_{10}) + (e^{-\alpha}-e^{-2\alpha})(\mathfrak{Y}_0\otimes\mathfrak{Y}_{10} + \mathfrak{Y}_{10}\otimes\mathfrak{Y}_0)\\ &\quad + (1-e^{-\alpha})^2\mathfrak{Y}_{10}\otimes\mathfrak{Y}_{10}.\end{aligned}$$

6 The dual of the formal Peterson subalgebra

In this section, we study the dual of the formal Peterson subalgebra.

Consider the $\mathfrak{Q}$-linear dual of the twisted group algebra $\mathfrak{Q}_{W_a}^* = \mathrm{Hom}_{\mathfrak{Q}}(\mathfrak{Q}_{W_a}, \mathfrak{Q})$. It is generated by f_w, $w \in W_a$. Following [14] there are two actions of $\mathfrak{Q}_{W_a}$ on the dual $\mathfrak{Q}_{W_a}^*$ defined as follows:

$$a\eta_w \bullet bf_v = bvw^{-1}(a)f_{vw^{-1}} \text{ and } a\eta_w \odot bf_v = aw(b)f_{wv}.$$

Indeed, the $\odot$-action comes from left multiplication in $\mathfrak{Q}_{W_a}$, and the $\bullet$-action comes from right multiplication. Observe that these two actions commute, which makes $\mathfrak{Q}_{W_a}^*$ into a $\mathfrak{Q}$-$\mathfrak{Q}$-bimodule. Moreover, $(z\bullet f)(z') = f(zz')$, $z, z' \in \mathfrak{Q}_{W_a}$, $f \in \mathfrak{Q}_{W_a}^*$.

We now define two tensor products.

The first one is the tensor product $\mathfrak{Q}_{W_a}\otimes\mathfrak{Q}_{W_a}$ of left $\mathfrak{Q}$-modules that is

$$az\otimes z' = z\otimes az',\ a\in\mathfrak{Q},\ z, z'\in\mathfrak{Q}_{W_a}.$$

There is a canonical map $\Delta\colon \mathfrak{Q}_{W_a}\to\mathfrak{Q}_{W_a}\otimes\mathfrak{Q}_{W_a}$ given by $a\eta_w \mapsto a\eta_w\otimes\eta_w$. This map defines a co-commutative coproduct structure on $\mathfrak{Q}_{W_a}$ with the co-unit $\mathfrak{Q}\to\mathfrak{Q}_{W_a}$, $a\mapsto a\eta_e$.

The second tensor product $\hat{\otimes}$ was introduced in [13]. Here, we provide a different but equivalent definition:

$$\mathfrak{Q}_{W_a}\hat{\otimes}\mathfrak{Q}_{W_a} := \mathfrak{Q}_{W_a}\times\mathfrak{Q}_{W_a}/\langle(a\eta_w, b\eta_v) - (aw(b)\eta_w, \eta_v)\rangle.$$

Observe that $\mathfrak{Q}_{W_a}\hat{\otimes}\mathfrak{Q}_{W_a}$ is also a left $\mathfrak{Q}$-module.

Similarly, we define:

$$\mathfrak{Q}_{W_a}^*\hat{\otimes}\mathfrak{Q}_{W_a}^* = \mathfrak{Q}_{W_a}^*\times\mathfrak{Q}_{W_a}^*/\langle(af_w, bf_v) - (aw(b)f_w, f_v)\rangle.$$

By definition, there is an isomorphism of $\mathfrak{Q}$-modules:

$$\mathfrak{Q}_{W_a}^*\hat{\otimes}\mathfrak{Q}_{W_a}^* \simeq (\mathfrak{Q}_{W_a}\hat{\otimes}\mathfrak{Q}_{W_a})^*, \quad (af_w\hat{\otimes}bf_v)(c\eta_x\hat{\otimes}d\eta_y) := acw(bd)\delta_{w,x}\delta_{v,y}.$$

There is a left $\mathfrak{Q}$-module homomorphism defined by the product structure of $\mathfrak{Q}_{W_a}$:

$$m\colon \mathfrak{Q}_{W_a}\hat{\otimes}\mathfrak{Q}_{W_a} \longrightarrow \mathfrak{Q}_{W_a}, \quad z_1\hat{\otimes}z_2 \mapsto z_1z_2,$$

whose dual is given by

$$m^*\colon \mathfrak{Q}^*_{W_a} \longrightarrow (\mathfrak{Q}_{W_a} \hat{\otimes} \mathfrak{Q}_{W_a})^* \simeq \mathfrak{Q}^*_{W_a} \hat{\otimes} \mathfrak{Q}^*_{W_a}, \quad m^*(cf_w) = \prod_u cf_u \hat{\otimes} f_{u^{-1}w}.$$

Indeed, given any element $a\eta_u \hat{\otimes} b\eta_v = au(b)\eta_u \hat{\otimes} \eta_v$, we have

$$\begin{aligned} m^*(cf_w)(a\eta_u \hat{\otimes} b\eta_v) &= cf_w(au(b)\eta_u\eta_v) = cf_w(au(b)\eta_{uv}) \\ &= \eta_{uv,w} cau(b) = (\prod_u cf_u \hat{\otimes} f_{u^{-1}w})(a\eta_u \hat{\otimes} b\eta_v). \end{aligned}$$

Recall (see also [18, Section 1.7]) that there is the Borel map defined via the characteristic map

$$\rho\colon \mathfrak{Q} \otimes_{\mathfrak{Q}^{W_a}} \mathfrak{Q} \longrightarrow \mathfrak{Q}^*_{W_a}, \quad a \otimes b \mapsto a\mathbf{c}(b) = \prod_{w \in W_a} aw(b)f_w.$$

Similar to [13] one obtains the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{Q} \otimes_{\mathfrak{Q}^{W_a}} \mathfrak{Q} & \xrightarrow{a\otimes b \mapsto a \otimes 1 \hat{\otimes} 1 \otimes b} & (\mathfrak{Q} \otimes_{\mathfrak{Q}^{W_a}} \mathfrak{Q}) \hat{\otimes} (\mathfrak{Q} \otimes_{\mathfrak{Q}^{W_a}} \mathfrak{Q}) \\ \downarrow \rho & & \downarrow \rho \hat{\otimes} \rho \\ \mathfrak{Q}^*_{W_a} & \longrightarrow & \mathfrak{Q}^*_{W_a} \hat{\otimes} \mathfrak{Q}^*_{W_a} \end{array}$$

Definition 6.1 We define the 0-th Hochschild homology of the bimodule $\mathfrak{Q}^*_{W_a}$ to be the quotient

$$\mathrm{HH}_0(\mathfrak{Q}^*_{W_a}) := \mathfrak{Q}^*_{W_a} / \langle a \bullet f - a \odot f : a \in \mathfrak{Q}, f \in \mathfrak{Q}_{W_a} \rangle.$$

Consider the dual of the map $\imath\colon \mathfrak{Q}_{Q^\vee} \to \mathfrak{Q}_{W_a}$, $\eta_{t_\lambda} \mapsto \eta_{t_\lambda}$. It gives a surjection

$$\imath^*\colon \mathfrak{Q}^*_{W_a} \twoheadrightarrow \mathfrak{Q}^*_{Q^\vee}, \quad af_{t_\lambda w} \mapsto a\delta_{w,e} f_{t_\lambda}.$$

We have

$$\imath^*(a \bullet bf_{t_\lambda v} - a \odot bf_{t_\lambda v}) = \imath^*(v(a)bf_{t_\lambda v} - abf_{t_{\lambda v}}) = (v(a)b - ab)\delta_{v,e} f_{t_\lambda v} = 0.$$

So it induces a surjection

$$\imath^*\colon \mathrm{HH}_0(\mathfrak{Q}^*_{W_a}) \twoheadrightarrow \mathfrak{Q}^*_{Q^\vee}.$$

On the other hand, $\ker \imath^* = \prod_{w \neq e, \lambda \in Q^\vee} \mathfrak{Q} f_{t_\lambda w}$. Now for any w, let $x_\mu \in S$ so that $w(\mu) \neq \mu$, then we obtain

$$f_{t_\lambda w} = \tfrac{1}{x_\mu - w(x_\mu)}(x_\mu \odot f_{t_\lambda w} - x_\mu \bullet f_{t_\lambda w}) \in \ker \imath^*.$$

Therefore, we have proven the following lemma.

Lemma 6.1 *There is an isomorphism $HH_0(\mathfrak{Q}^*_{W_a}) \simeq \mathfrak{Q}^*_{Q^\vee}$ which fits into the following commutative diagram*

$$\begin{array}{ccc} \mathfrak{Q}^*_{W_a} & \twoheadrightarrow & \mathrm{HH}_0(\mathfrak{Q}^*_{W_a}) \\ & \searrow^{\imath^*} & \downarrow \simeq \\ & & \mathfrak{Q}^*_{Q^\vee} \end{array}$$

By definition, we have the commutative diagram of left S-modules:

$$\begin{array}{ccccc} \mathfrak{Q}^*_{W_{\mathrm{a}}} & \xrightarrow{\imath^*} & \mathfrak{Q}^*_{Q^\vee} & \xhookrightarrow{\mathrm{pr}^*} & \mathfrak{Q}_{W_{\mathrm{a}}} \\ \uparrow & & \uparrow & & \uparrow \\ \mathbf{D}^*_{W_{\mathrm{a}}} & \xrightarrow{\imath'^*} & \mathbf{D}^*_{Q^\vee} & \xhookrightarrow{\mathrm{pr}^*} & \mathbf{D}_{W_{\mathrm{a}}} \end{array} .$$

It is also easy to see that the surjection $\imath^* \colon \mathfrak{Q}^*_{W_{\mathrm{a}}} \twoheadrightarrow \mathfrak{Q}^*_{Q^\vee}$ induces a surjective map

$$\imath^{*\prime} \colon \mathrm{HH}_0(\mathbf{D}^*_{W_{\mathrm{a}}}) \twoheadrightarrow \mathbf{D}^*_{Q^\vee}.$$

Our goal is to show that the isomorphism and the diagram of Lemma 6.1 can be restricted to the formal Peterson subalgebra $\mathbf{D}_{Q^\vee}$. Namely, we want to prove the following.

Theorem 6.2 *The map $\imath^{*\prime}$ gives an isomorphism of Hopf algebras $HH_0(\mathbf{D}^*_{W_{\mathrm{a}}}) \simeq \mathbf{D}^*_{Q^\vee}$.*

Since the product structure on both the domain and the codomain is induced by the coproduct structure

$$\mathfrak{Q}_{W_{\mathrm{a}}} \longrightarrow \mathfrak{Q}_{W_{\mathrm{a}}} \otimes \mathfrak{Q}_{W_{\mathrm{a}}}, \quad a\eta_w \mapsto a\eta_w \otimes \eta_w,$$

where the codomain is the tensor product of left $\mathfrak{Q}$-modules, the map $\imath^{*\prime}$ is a ring homomorphism.

Moreover, since the coproduct structure on both the domain and the codomain is induced by the product structure in $\mathfrak{Q}_{W_{\mathrm{a}}}$ and $\mathfrak{Q}_{Q^\vee}$, the map $\imath^{*\prime}$ is a coalgebra homomorphism. Therefore, it only suffices to prove the injectivity of $\imath^{*\prime}$.

To prove the latter, we introduce the following filtration on the dual $\mathbf{D}^*_{W_{\mathrm{a}}}$ of the FADA.

Definition 6.2 Let w_λ be as defined in the appendix. Set $F_i = \bigcup_{\ell(w_\lambda)\geq i} t_\lambda W$. For any $f \in \mathfrak{Q}^*_{W_{\mathrm{a}}} = \mathrm{Hom}(W_{\mathrm{a}}, \mathfrak{Q})$ set $\mathrm{supp} f = \{w \in W_{\mathrm{a}} \colon f(w) \neq 0\}$. Define the i-th stratum of the filtration to be

$$\mathcal{Z}_i := \{f \in \mathbf{D}^*_{W_{\mathrm{a}}} \colon \mathrm{supp} f \subseteq F_i\}.$$

We have $\mathcal{Z}_i = \prod_{w\in F_i} S \cdot Y^*_{I_w}$. Each $\mathcal{Z}_i$ is a S-bimodule, since $x \bullet af_u = u(x)af_u$ and $x \odot af_u = xaf_u$, $x, a \in S$, $u \in W_{\mathrm{a}}$.

Consider the following two conditions:

$$f(\eta_{t_\lambda u}) \in x_\alpha^{\ell_\alpha(w_\lambda)} S \quad \text{for any } u \in W, \tag{6.1}$$

$$f(\eta_{t_\lambda u} - \eta_{t_\lambda s_\alpha u}) \in x_\alpha^{\ell_\alpha(w_\lambda)+1} S \quad \text{for any } u \in W \text{ and root } \alpha \in \Phi. \tag{6.2}$$

Lemma 6.3 *Elements of $\mathcal{Z}_i$ satisfy the conditions (6.1) and (6.2).*

Proof Let $f \in \mathcal{Z}_i$ and $\ell(w_\lambda) = i$. Assume $\langle \lambda, \alpha^\vee \rangle \leq 0$. For $k \in [1, \ell_\alpha(w_\lambda)]$ by Lemma 7.2 of the appendix, we conclude that $w_{\lambda+k\alpha^\vee} < w_\lambda$ and, in particular, $\ell(w_{\lambda+k\alpha^\vee}) < \ell(w_\lambda)$. So $f(\eta_{\lambda+k\alpha^\vee}) = 0$, and we have for any $u \in W$,

$$\begin{aligned}
(Z_\alpha^{\ell_\alpha(w_\lambda)}\eta_{t_\lambda}\odot f)(\eta_u) &= \frac{1}{x_{-\alpha}^{\ell_\alpha(w_\lambda)}} f\big((1-\eta_{t_{\alpha^\vee}})^{\ell_\alpha(w_\lambda)}\eta_{t_\lambda u}\big)\\
&= \frac{1}{x_{-\alpha}^{\ell_\alpha(w_\lambda)}} f(\eta_{t_\lambda u}) \in S, \text{ and}\\
(Z_\alpha^{\ell_\alpha(w_\lambda)}\eta_{t_\lambda}X_\alpha\odot f)(\eta_u) &= \frac{1}{x_{-\alpha}^{\ell_\alpha(w_\lambda)}x_\alpha} f\big((1-\eta_{t_{\alpha^\vee}})^{\ell_\alpha(w_\lambda)}(\eta_{t_\lambda u}-\eta_{t_\lambda s_\alpha u})\big)\\
&= \frac{1}{x_{-\alpha}^{\ell_\alpha(w_\lambda)}x_\alpha} f(\eta_{t_\lambda u}-\eta_{t_\lambda s_\alpha u}) \in S.
\end{aligned}$$

Note that $\frac{x_\alpha}{x_{-\alpha}}$ is invertible in S, so we can replace x_α by $x_{-\alpha}$ whenever needed.

Similarly, if $\langle\lambda,\alpha^\vee\rangle<0$, then $\ell(w_{\lambda-k\delta})<\ell(w_\lambda)$ for $k\in[1,\ell_\alpha(w_\lambda)]$. Thus,

$$\begin{aligned}
(Z_{-\alpha}^{\ell_\alpha(w_\lambda)}\odot f)(\eta_u) &= \frac{1}{x_\alpha^{\ell_\alpha(w_\lambda)}} f\big((1-\eta_{t_{-\alpha^\vee}})^{\ell_\alpha(w_\lambda)}\eta_{t_\lambda u}\big)\\
&= \frac{1}{x_\alpha^{\ell_\alpha(w_\lambda)}} f(\eta_{t_\lambda w}) \in S, \text{ and}\\
(Z_{-\alpha}^{\ell_\alpha(w_\lambda)}\eta_{t_\lambda}X_\alpha\odot f)(\eta_u) &= \frac{1}{x_\alpha^{\ell_\alpha(w_\lambda)+1}} f\big((1-\eta_{t_{-\alpha^\vee}})^{\ell_\alpha(w_\lambda)}(\eta_{t_\lambda w}-\eta_{t_\lambda s_\alpha u})\big)\\
&= \frac{1}{x_\alpha^{\ell_\alpha(w_\lambda)+1}} f(\eta_{t_\lambda u}-\eta_{t_\lambda s_\alpha u}) \in S.
\end{aligned}$$

The result then follows. ■

Define

$$\mathcal{Y}_{(i)} = \{f\in \mathrm{Hom}(F_i\backslash F_{i+1}, S) \colon f \text{ satisfies the conditions (6.1) and (6.2)}\}.$$

Here, $f(\eta_{v_1}-\eta_{v_2}) := f(v_1)-f(v_2)$. Note that $\mathcal{Y}_{(i)}$ is a S-bimodule in the usual sense, that is $(a\odot f)(v) = af(v)$ and $(a\bullet f)(v) = v(a)f(v)$. By Lemma 6.3, we have a natural projection $\mathcal{Z}_i\to\mathcal{Y}_{(i)}$, which induces an injective S-bimodule map

$$\mathrm{res}\colon \mathcal{Z}_i/\mathcal{Z}_{i+1} \longrightarrow \mathcal{Y}_{(i)}.$$

Lemma 6.4 *The map* res *is an isomorphism of S-bimodules. In particular, $\mathcal{Z}_i/\mathcal{Z}_{i+1}$ is free of rank $|F_i\backslash F_{i+1}|$.*

Proof We only need to prove that res is surjective. Let $f\in\mathcal{Y}_{(i)}$. We pick a minimal element $w\in\mathrm{supp}(f)$. We first show that

$$f(\eta_w)\in\prod_{\alpha\in\Phi^+} x_\alpha^{\ell_\alpha(w)}S.$$

As x_α's are relatively prime, it reduces to show

$$f(\eta_w)\in x_\alpha^{\ell_\alpha(w)}S \tag{6.3}$$

for each root α.

Let $w = t_\lambda u$ for $\lambda\in Q^\vee$, $u\in W$ and $\ell_\alpha(w_\lambda)=i$. By Lemma 7.1 of the appendix, $\ell_\alpha(w)\in\{\ell_\alpha(w_\lambda),\ell_\alpha(w_\lambda)+1\}$.

If $\ell_\alpha(w) = \ell_\alpha(w_\lambda)$, then (6.3) follows from (6.1) directly. If $\ell_\alpha(w) = \ell_\alpha(w_\lambda) + 1$, by (7.2) and (7.3), we have

$$\ell_\alpha(t_\lambda s_\alpha u) = \ell_\alpha(w_\lambda) < \ell_\alpha(w).$$

It implies $t_\lambda s_\alpha u < w$ (note that $t_\lambda s_\alpha u$ and w are always comparable under the Bruhat order). By (6.2), we have

$$f(\eta_{t_\lambda u} - \eta_{t_\lambda s_\alpha u}) = f(\eta_w) \in x_\alpha^{\ell_\alpha(w_\lambda)+1} S.$$

Observe that the images of $Y^*_{I_w}$ for $w \in Z_i \backslash Z_{i+1}$ form a basis of $\mathcal{Z}_i/\mathcal{Z}_{i+1}$, and we have $Y^*_{I_w}(\eta_w) = \prod_{\alpha>0} x_\alpha^{\ell_\alpha(w)}$.

The conclusion then follows after replacing f by $f - \frac{f(\eta_w)}{\prod_{\alpha\in\Phi^+} x_\alpha^{\ell_\alpha(w_\lambda)}} Y^*_{I_w}$. ■

Denote for each $\lambda \in Q^\vee$, $\Delta_\lambda = \prod_{\alpha>0} x_\alpha^{\ell_\alpha(w_\lambda)} \in S$. It is clear that we have an S-bimodule isomorphism

$$\mathcal{Y}_{(i)} \simeq \bigoplus_{\ell(w_\lambda)=i} \Delta_\lambda \cdot \mathbf{D}_W.$$

To finish the proof of Theorem 6.2, we define a filtration on $\mathbf{D}^*_{Q^\vee}$ by

$$\mathcal{X}_i = \{f \in \mathbf{D}^*_{Q^\vee} : \mathrm{supp}(f) \subset F_i\}.$$

Then, $Y^*_{I_w}$ with $\ell(w_\lambda) = i$ is a S-basis of $\mathcal{X}_i/\mathcal{X}_{i+1}$. So the rank of $\mathcal{X}_i/\mathcal{X}_{i+1}$ is $|F_i \backslash F_{i+1}|$. Moreover, by definition, we know that $\imath^{*\prime}$ induces a map on each associated graded piece:

$$\imath^{*\prime} : \mathcal{Z}_i/\mathcal{Z}_{i+1} \to \mathcal{X}_i/\mathcal{X}_{i+1}.$$

From Lemma 6.4, the rank of $\mathcal{Z}_i/\mathcal{Z}_{i+1}$ is $|F_i \backslash F_{i+1}|$, therefore, $\imath^{*\prime}$ is an isomorphism.

7 Appendix

Here, we prove several combinatorial properties of the affine Weyl group that are used in the proof of Theorem 6.2.

For $w \in W_{\mathrm{a}}$, denote

$$\ell_\alpha(w) = |\{\beta = \pm\alpha + k\delta > 0 : w^{-1}(\beta) < 0\}|.$$

It is clear that $\ell(w) = \sum_{\alpha>0} \ell_\alpha(w)$. Also denote by $w_\lambda \in W_a^-$ the minimal representative of $t_\lambda W$. Then, $w_\lambda \le w_\mu$ if and only if there exists $w \in t_\lambda W$ and $y \in t_\mu W$ such that $w \le y$. Note that $w \le y$ implies $\ell_\alpha(w) \le \ell_\alpha(y)$ for all $\alpha \in \Phi_+$, so after fixing α $\ell_\alpha(w_\lambda)$ becomes minimal for elements w from $t_\lambda W$.

Lemma 7.1 *We have the following property:*

$$\ell_\alpha(w_\lambda) = \begin{cases} -\langle\lambda,\alpha\rangle, & \text{if } \langle\lambda,\alpha\rangle \le 0, \\ \langle\lambda,\alpha\rangle - 1, & \text{if } \langle\lambda,\alpha\rangle > 0. \end{cases}$$

Proof For $w = t_\lambda u \in t_\lambda W$, $\beta = \pm\alpha + k\delta > 0$ (so $k \ge 0$), we have

$$w^{-1}(\beta) = w^{-1}(\pm\alpha + k\delta) = \pm u^{-1}\alpha + (k \pm \langle\lambda,\alpha\rangle)\delta. \tag{7.1}$$

If $\langle\lambda,\alpha\rangle \le 0$, then $w^{-1}(\beta) < 0$ implies $\beta = \alpha + k\delta$, and moreover, $k \in [0, \ell_\alpha(w) - 1]$ (since $\ell_\alpha(w) = |Inv_\alpha(w)|$). So we have

$$\ell_\alpha(w) = -\langle\lambda,\alpha\rangle + \begin{cases} 1, & u^{-1}(\alpha) < 0, \\ 0, & u^{-1}(\alpha) > 0, \end{cases} \tag{7.2}$$

and the minimal value is $-\langle\lambda,\alpha\rangle$.

If $\langle\lambda,\alpha\rangle > 0$, then $w^{-1}(\beta) < 0$ if and only if $\beta = -\alpha + k\delta$ and $k \in [1, \ell_\alpha(w)]$, in which case we have

$$\ell_\alpha(w) = \langle\lambda,\alpha\rangle - \begin{cases} 1, & u^{-1}(\alpha) < 0, \\ 0, & u^{-1}(\alpha) > 0. \end{cases} \tag{7.3}$$

The minimal value is $\langle\lambda,\alpha\rangle - 1$. ■

Lemma 7.2 *Let $\alpha \in \Phi^+$, $\lambda \in Q^\vee$, and $k \in [0, \ell_\alpha(w_\lambda)]$.*

If $\langle\lambda,\alpha\rangle \le 0$, then $w_\lambda > w_{\lambda+k\alpha^\vee}$. If $\langle\lambda,\alpha\rangle > 0$, then $w_\lambda > w_{\lambda-k\alpha^\vee}$.

Proof If $\langle\lambda,\alpha\rangle \le 0$, consider $w \in t_\lambda W$ such that $w = t_\lambda u$ with $u^{-1}(\alpha) < 0$. We have

$$w^{-1}(\alpha) = u^{-1}(\alpha) + \langle\lambda,\alpha\rangle\delta < 0,$$

so $w > s_\alpha w$. From (7.2), we get $\ell_\alpha(w) = \ell_\alpha(w_\lambda) + 1 = -\langle\lambda,\alpha\rangle + 1$.

Since $1 \le k \le \ell_\alpha(w_\lambda) = -\langle\lambda,\alpha\rangle$, we get

$$(s_\alpha w)^{-1}(-\alpha + k\delta) = u^{-1}\alpha + (k + \langle\lambda,\alpha\rangle)\delta < 0,$$

which implies

$$s_\alpha w > s_{-\alpha+k\delta} s_\alpha w = t_{k\alpha^\vee} w.$$

Therefore, $w > t_{k\alpha^\vee} w$.

Since $w \in t_\lambda W$ and $t_{k\alpha^\vee} w \in t_{\lambda+k\alpha^\vee} W$, we get $w_\lambda > w_{\lambda+k\alpha^\vee}$.

If $\langle\lambda,\alpha\rangle > 0$, consider $w = t_\lambda u$ with $u^{-1}(\alpha) > 0$. We have

$$w^{-1}(-\alpha + \delta) = u^{-1}(\alpha) + (1 - \langle\lambda,\alpha\rangle)\delta < 0,$$

so $w > s_{-\alpha+\delta} w$. From (7.3), we have $\ell_\alpha(w) = \langle\lambda,\alpha\rangle = \ell_\alpha(w_\lambda) + 1$.

Since $1 \le k \le \ell_\alpha(w) - 1 = \ell_\alpha(w_\lambda) = \langle\lambda,\alpha\rangle - 1$, we get

$$s_{-\alpha+\delta} w > s_{\alpha+(k-1)\delta} s_{-\alpha+\delta} x = t_{-k\alpha^\vee} w.$$

So $w > t_{-k\alpha^\vee} w$. Since $w \in t_\lambda W$ and $t_{k\alpha^\vee} w \in t_{\lambda+k\alpha^\vee} W$, we have $w_\lambda > w_{\lambda+k\alpha^\vee}$. ■

Lemma 7.3 *If $\langle\lambda,\alpha\rangle = 0$, then we have the following sequence:*

$$w_\lambda < w_{\lambda+\alpha^\vee} < w_{\lambda-\alpha^\vee} < w_{\lambda+2\alpha^\vee} < w_{\lambda-2\alpha^\vee} < \cdots.$$

If $\langle\lambda,\alpha\rangle = 1$, then we have the following sequence:

$$w_\lambda < w_{\lambda-\alpha^\vee} < w_{\lambda+\alpha^\vee} < w_{\lambda-2\alpha^\vee} < w_{\lambda+2\alpha^\vee} < \cdots.$$

The lengths ℓ_α are given by $(0, 1, 2, 3, 4, \ldots)$.

Proof We only prove the case when $\langle \lambda, \alpha \rangle = 0$. Consider $\lambda + k\alpha^\vee$ and $\lambda - k\alpha^\vee$ with $k > 0$, then $\langle \lambda - k\alpha^\vee, \alpha \rangle = -2k < 0$, and $2k = \ell_\alpha(w_{\lambda - k\alpha^\vee})$, so by Lemma 7.2,

$$w_{\lambda - k\alpha^\vee} > w_{\lambda - k\alpha^\vee + 2k\alpha^\vee} = w_{\lambda + k\alpha^\vee}.$$

Finally, consider $\lambda - k\alpha^\vee$ and $\lambda + (k+1)\alpha^\vee$, $k \geq 0$, then $\langle \lambda + (k+1)\alpha^\vee, \alpha \rangle = 2(k + 1) \geq 2$, and $\ell_\alpha(w_{\lambda+(k+1)\alpha^\vee}) = 2k + 1$, so by Lemma 7.2,

$$w_{\lambda+(k+1)\alpha^\vee} \geq w_{\lambda+(k+1)\alpha^\vee-(2k+1)\alpha^\vee} = w_{\lambda - k\alpha^\vee}.$$ ∎

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