12

Other renormalization schemes

In previous chapters, we have concentrated our discussions on the modified minimal subtraction $\overline{MS}$ scheme, which is the most convenient one for QCD. However, it is known that there is a freedom for choosing a renormalization scheme. Among different existing off-shell renormalization schemes discussed in the literature, we choose to discuss the following schemes which have been widely used in the 1980s. We shall also discuss their connections by comparing the renormalized QCD coupling in these different schemes.

12.1 The MS scheme

The MS scheme is the original minimal subtraction scheme proposed for dimensional renormalization. We have already discussed the difference between the MS and $\overline{MS}$ schemes, which one can illustrate by the comparison of the renormalized coupling in the two schemes:

$$\nu^{-\epsilon}\alpha_s^B = \alpha_s^R \left\{1 + \left(\frac{\alpha_s}{\pi}\right)\left(\frac{\beta_1}{\epsilon} + \delta\right) + \mathcal{O}\left(\frac{\alpha_s}{\pi}\right)^2\right\}, \tag{12.1}$$

where δ is an arbitrary constant characteristic of the scheme used. In the $\overline{MS}$ scheme:

$$\delta_{\overline{MS}} = \frac{\beta_1}{2}[\ln 4\pi - \gamma], \tag{12.2}$$

and the running couplings in the two schemes are related as:

$$\bar{\alpha}_s^{\overline{MS}} = \bar{\alpha}_s^{MS}\left[1 + \left(\frac{\bar{\alpha}_s}{\pi}\right)\delta + \mathcal{O}(\alpha_s^2)\right]. \tag{12.3}$$

This leading order relation can be translated by the relation between the scale Λ in the two schemes:

$$\Lambda_{\overline{MS}} \simeq \Lambda_{MS}\exp(\delta/\beta_1), \tag{12.4}$$

i.e., one obtains to this order:

$$\Lambda_{\overline{MS}} \simeq 2.66\ \Lambda_{MS}. \tag{12.5}$$

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Table 12.1. *Value of* $\delta(\alpha_G, n_f)$ *in the MS and MOM schemes*

Scheme		$\delta(\alpha_G, n_f)$
MS		0
$\overline{MS}$		$\delta_{\overline{MS}} \equiv (\beta_1/2)[\ln 4\pi - \gamma]$
MOM	Three-gluon	$\delta_{\overline{MS}} - \frac{11}{2} - \frac{23}{48}J - \alpha_G \frac{9}{16}(1-J) + \frac{\alpha_G^2}{8}(3-J) - \frac{\alpha_G^3}{16} + \frac{n_f}{3}\left(1 + \frac{2}{3}J\right)$
	Quark-gluon	$\delta_{\overline{MS}} - \frac{1}{16}\left(89 - \frac{85}{9}J\right) - \alpha_G \frac{25}{24}\left(1 - \frac{2}{3}J\right) + \frac{\alpha_G^2}{16}(3-J) + \frac{5n_f}{18}$
	Ghost-gluon	$\delta_{\overline{MS}} - \frac{5}{48}\left(41 + \frac{3}{2}J\right) - \frac{\alpha_G}{8}(9-2J) - \frac{\alpha_G^2}{16}\left(3 - \frac{J}{2}\right) + \frac{5n_f}{18}$

12.2 The momentum subtraction scheme

In the momentum subtraction scheme (MOM scheme), the renormalized two-point (or in general Green's) function is defined as [168–170]:

$$\Psi_5(q^2)_R = \Psi_5(q^2) - \Psi_5(q^2 = -\mu^2,\ m^2)\,, \tag{12.6}$$

where μ is the subtraction point in the Euclidean region. The choice of μ is arbitrary. It is often chosen at the symmetric point of the three-gluon vertex with which one defines the renormalized QCD coupling. However, the choice of the vertex is also arbitrary, as one can choose the quark-gluon-quark or ghost-gluon-ghost vertex for defining the renormalized coupling. In this scheme, the renormalization constants and universal parameters are mass-dependent, which is not convenient when one works with massive particles. However, due to the Appelquist–Carazzone decoupling theorem, one may ignore the effect of the heavy quarks having a mass larger than the momentum scale of the analysis. If one expresses the renormalized QCD coupling α_s in terms of the bare coupling α_s^B in $4-\epsilon$ dimensions, one has:

$$\nu^{-\epsilon}\alpha_s^B = \alpha_s\left[1 + \left(\delta(\alpha_G, n_f) + \frac{\beta_1}{\epsilon}\right)\alpha_s + \mathcal{O}\big(\alpha_s^2\big)\right]\,, \tag{12.7}$$

where $\delta(\alpha_G, n_f)$ is a finite term which depends on how α_s is renormalized, and are given in Table 12.1, where α_G is the gauge parameter; $\beta_1 = -(1/2)(11 - 2n_f/3)$ for $SU(3)_c \times SU(n)_f$, and:

$$J \equiv -2\int_0^1 dx\,\frac{\ln x}{x^2 - x + 1} = 2.3439072\ldots\,. \tag{12.8}$$

Therefore, one can derive the lowest order relation between the MOM and MS schemes in the case of massless quarks [170]:

$$\Lambda_{\rm mom} = \Lambda_{\rm MS}\exp\left\{\frac{\delta(\alpha_G, n_f)}{\beta_1}\right\}\,. \tag{12.9}$$

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In the usual case of three-gluon vertex, and for some particular values of the gauge parameter, one has:

$$\delta(0,3) = -8.46\,, \qquad \delta(1,3) = -7.68\,, \tag{12.10}$$

which leads to the numerically lowest order relation:

$$\Lambda_{\rm mom} \simeq \Lambda_{\rm MS} \begin{pmatrix} 6.55 & \text{for } \alpha_{\rm G} = 0: \text{ Landau gauge} \\ 5.51 & \text{for } \alpha_{\rm G} = 1: \text{ Feynman gauge} \end{pmatrix} \tag{12.11}$$

12.3 The Weinberg renormalization scheme

The Weinberg scheme [171] is variant of the MOM scheme. In this scheme the renormalized two-point function reads:

$$\Psi_5(q^2,\, m^2)_R = \Psi_5(q^2,\, m^2) - \Psi_5(q^2 = -\mu^2,\, m^2 = 0)\,, \tag{12.12}$$

and is renormalized at an off-shell space-like point $q^2 = -\mu^2$ and putting the particle masses to be zero. It coincides with the MOM scheme, in the case of massless theories. One can see that, in this scheme, the renormalization constants are also quark-mass dependent. It has been shown by [172] that the Weinberg scheme violates the Slavnov–Taylor identities due to the arbitrariness of the subtraction point at a specific vertex, the gauge dependence of the coupling and to the definition of the tensorial structure of the vertex at the subtraction point.

12.4 The BLM scheme

The BLM (Brodsky–Lepage–Mackenzie) scheme has been introduced in [173] and has been based on the analogy with QED where only the light fermion vacuum polarizations (VP) contribute to the renormalization of the strong coupling constant. In QED, the running effective charge can be defined as (see the next chapter on QED):

$$\alpha(Q) = \frac{\alpha}{1 + e^2 \Pi_{\rm em}(Q)}\,, \tag{12.13}$$

where to lowest order in α, and using an on-shell renormalization:

$$\Pi_{\rm em}(Q) = -\frac{1}{4\pi^2}\left(\frac{2}{3}\ln\frac{Q}{m_e} - \frac{5}{3}\right)\,. \tag{12.14}$$

The scheme states that an observable $\mathcal{O}$ which has the perturbative expansion:

$$\mathcal{O} = C_0 \alpha(Q)\left[1 + C_1 \frac{\alpha(Q)}{\pi} + \cdots\right] \tag{12.15}$$

can be replaced by:

$$\mathcal{O} = C_0 \alpha(Q_0^*)\left[1 + C_1^* \frac{\alpha(Q_1^*)}{\pi} + \cdots\right] \tag{12.16}$$

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where all VP corrections are absorbed into the effective coupling by an appropriate and unique choice of scales $Q_0^*,\ Q_1^*, \dots$. Since the number n_f of light flavour dependences usually enters the VP to this order, then, both Q_i^* and C_i^* are independent of n_f, while, in general, the scales Q_i^* can depend on the ratio of invariants. Taking the example of the anomalous magnetic moment of the leptons, which can be expressed as (see QED section):

$$a_e = \frac{\alpha}{2\pi}\left[1 - 0.657\frac{\alpha}{2\pi}\right], \tag{12.17}$$

and the VP contribution to the muon anomaly gives:

$$A_{VP}\frac{\alpha}{\pi}a_\mu^0 = \left[\frac{2}{3}\ln\frac{m_\mu}{m_e} - \frac{25}{18}\right]\frac{\alpha}{\pi}a_\mu^0 . \tag{12.18}$$

For the muon, one can expect that, at a scale $Q^* \sim m_\mu$, the exact result can be expressed as:

$$a_\mu = \frac{\alpha(Q^*)}{2\pi}, \tag{12.19}$$

where the running coupling is defined in Eq. (12.13), such that Eq. (12.18) and Eq. (12.19) must be equal. In this way, one obtains:

$$Q^* = m_\mu e^{5/12} . \tag{12.20}$$

In this procedure, the electron and the muon anomaly have the same expression to this order, as we replace:

$$a_\mu = \frac{\alpha}{2\pi}\left[1 + \frac{\alpha}{\pi}(A_{VP} + C_1) + \cdots\right], \tag{12.21}$$

by:

$$a_\mu = \frac{\alpha(Q^*)}{2\pi}\left[1 + \frac{\alpha(Q^*)}{\pi}C_1 + \cdots\right], \tag{12.22}$$

where:

$$\alpha(Q^*) \simeq \frac{\alpha}{1 - (\alpha/\pi)A_{VP}}, \tag{12.23}$$

and:

$$C_1 = -0.657 . \tag{12.24}$$

In the case of QCD, a similar approach can be made. The observable can be written as:

$$\mathcal{M} = C_0\alpha_{\overline{MS}}(Q)[1 + (\alpha_{\overline{MS}}(Q)/\pi)(n_f A_{VP} + B)] . \tag{12.25}$$

One can change the coupling by:

$$\alpha_{\overline{MS}}(Q^*) = \alpha_{\overline{MS}}(Q)\left[1 - \beta_1(\alpha_{\overline{MS}}(Q)/\pi)\ln\frac{Q^*}{Q} + \cdots\right]^{-1} . \tag{12.26}$$

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and express the observable as:

$$\mathcal{M} = C_0 \alpha_{\overline{MS}}(Q^*)[1 + (\alpha_{\overline{MS}}(Q^*)/\pi)C_1^* + \cdots] . \quad (12.27)$$

Then, one can deduce:

$$Q^* = Q \exp(3A_{VP}) ,$$
$$C_1^* = \frac{33}{2}A_{VP} + B , \quad (12.28)$$

where the term $\frac{33}{2}A_{VP}$ in C_1^* serves to remove the part of B which renormalizes the leading-order coupling.

The ratio of these gluonic corrections to the light quark ones is fixed by the β function. In some of the examples given by BLM, the value of Q^* appears to be lower than the original scale of the process, which might be inconvenient for the convergence of the QCD series. Moreover, the scheme dependence of the result in Eq. (12.28) has been pointed out in [174], while an extension of the BLM result beyond NLO shows an ambiguity in the prescription [175]. Recent interest in the resummation of perturbative QCD series using large value of β in the naïve Abelization of QCD (see Renormalons section) has revived the use of the BLM scheme despite these previous drawbacks of the procedure.

12.5 The PMS optimization scheme

The principle of minimal sensitivity (PMS) scheme has been introduced by Stevenson [176] in QCD. It consists on the fact that physical quantities should be insensitive to a small variation of unphysical parameters, and is based on variational approach. It is more instructive to illustrate the method by the classical example of the $e^+e^- \to$ hadrons total cross-section, which is known to high-accuracy in perturbative QCD. To order α_s^2, the corresponding Adler D-function reads:

$$D(q^2) \equiv -q^2 \int_0^\infty \frac{dt}{(t-q^2)^2} R(t) \simeq \sum_i Q_i^2\{1 + [D_2 \equiv a_s(1 + a_s F_3)] + \cdots\} , \quad (12.29)$$

where:

$$R \equiv \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} . \quad (12.30)$$

Q_i is the quark charge in units of e; F_3 is renormalization scheme dependent; $a_s \equiv \bar{\alpha}_s/\pi$ is the QCD running coupling. The ν (subtraction scale) dependence of the dimensional renormalization scheme can be introduced via:

$$\tau \equiv -\beta_1 \ln(\nu/\Lambda) . \quad (12.31)$$

Using the differential equation obeyed by the running coupling:

$$-\beta_1 \frac{\partial a_s}{\partial \tau} = a_s \beta(a_s) = \beta_1 a_s^2 \left(1 + \frac{\beta_2}{\beta_1} a_s\right) , \quad (12.32)$$

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one obtains:

$$\frac{\partial D_2}{\partial \tau} = -a_s^2 \left(1 + \frac{\beta_2}{\beta_1} a_s\right)(1 + 2F_3 a_s) + a_s^2 \frac{\partial F_3}{\partial \tau} \ . \tag{12.33}$$

Using the fact that D_2 is independent of τ, the a_s^2 term in Eq. (12.33) must vanish, which leads to:

$$F_3(\tau) = \tau - \tau_0 + F_3(\tau_0) \ . \tag{12.34}$$

The optimization criterion imposes that the remainder term of $\partial D_2/\partial \tau$ also vanishes at a critical value $\tau \equiv \tau_c$. The optimal value of F_3 corresponds to:

$$\frac{\beta_2}{\beta_1} + 2F_3^{\rm opt}\left(1 + \frac{\beta_2}{\beta_1} a_s(\tau_c)\right) = 0 \ , \tag{12.35}$$

where the rôle of $a_s \beta_2$ can be increased by computing the next order terms. From this result, one can deduce the optimal value of D_2:

$$D_2^{\rm opt} = a_s(\tau_c)\left[1 - \frac{(\beta_2/\beta_1)a_s(\tau_c)}{2[1 + (\beta_2/\beta_1)a_s(\tau_c)]}\right] . \tag{12.36}$$

The last step of the analysis is to find $a_s(\tau_c)$. This can be done by integrating Eq. (12.32). One obtains to two loops:

$$\hat{K}_2(a_s) \equiv \tau = \int_{a_s}^{\infty} \frac{dx}{x^2[1 + (\beta_2/\beta_1)a_s(x)]} = \frac{1}{a_s} + \frac{\beta_2}{\beta_1} \ln\left(\frac{(\beta_2/\beta_1)a_s}{1 + (\beta_2/\beta_1)a_s}\right) , \tag{12.37}$$

where the upper limit of integration is equivalent to the choice of Λ in $\tau = -\beta_1 \ln(\nu/\Lambda)$. Using Eq. (11.53) by including next leading corrections, one can derive the relation:

$$\Lambda_{\rm opt} = \Lambda_{MS}\left(-2\frac{\beta_2}{\beta_1}\right)^{\beta_2/\beta_1{}^2} \ . \tag{12.38}$$

Rewriting Eq. (12.34) as:

$$F_3 = \hat{K}_2(a_s) - \rho_1(Q) \ , \tag{12.39}$$

where:

$$\rho_1(Q) \equiv \tau_0 - F_3(\tau_0) \ , \tag{12.40}$$

is a constant term independent of the unphysical variable τ at fixed Q, where at $Q^2 \equiv -q^2 = \nu^2$, it reads:

$$\rho_1(Q^2 = \nu^2) = -\beta_1 \ln(Q/\Lambda) - F_3 \ . \tag{12.41}$$

It is also a renormalization scheme invariant quantity, as the scheme dependence of Λ cancels the one of F_3. Substituting the value of F_3 from Eq. (12.39) into Eq. (12.35), one

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gets the following transcendental equation for $a_s(\tau_c)$:

$$\hat{K}_2(a_s(\tau_c)) + \frac{1}{2}\frac{\beta_2}{\beta_1}\left(1 + \frac{\beta_2}{\beta_1}(a_s(\tau_c)\right)^{-1} = \rho_1(Q)\ , \tag{12.42}$$

where the solution $a_s(\tau_c)$ is the one to be used in $D_2^{\rm opt}$. As ρ_1 behaves like $1/a_s$, it needs to be large for a good description of the process. The PMS scheme has been quite popular in the period of 1980–1990.

At present, the interest in the method has decreased. This is probably related to the fact that it does not yet incorporate the power corrections which plays a non-negligible rôle in the extraction of the QCD coupling from different processes. However, an extension of the method including these non-perturbative corrections, although small, should be more attractive.

12.6 The effective charge scheme

Like the PMS, this scheme is also conceptually based on the construction of scheme-invariant quantities from combinations of scheme-dependent coefficients [177]. In order to illustrate the discussion, let's start from the D function defined in Eq. (12.29), which we rewrite as:[1]

$$D_n \simeq \sum_i Q_i^2 \left\{1 + a_s d_0 \left(1 + \sum_{i=1}^{n-1} d_i a_s^i\right) + \cdots\right\}\ , \tag{12.43}$$

where all higher order corrections and scheme dependence of the process are absorbed into the definition of the coupling constant. The ECH approach imposes the condition that all coefficients $d_i = 0$ for all $i \geq 2$. Writing the β function as:

$$\beta(\alpha_s) = -\beta_1 a_s \left(1 + \sum_{i=1}^{n-1} c_i a_s^i\right)\ , \tag{12.44}$$

and:

$$D_n^{\rm ECH} = D_n(a_s) + \delta D_n^{\rm ECH}\ , \tag{12.45}$$

these conditions imply for the remaining corrections to the physical quantities [178]:

$$\begin{aligned}\delta D_2^{\rm ECH} &= d_0 d_1 (c_1 + d_2)\\ \delta D_3^{\rm ECH} &= d_0 d_1 \left(c_2 - \frac{1}{2}c_1 d_1 - 2d_1^2 + 3d_2 + d_2\right)\ .\end{aligned} \tag{12.46}$$

These conditions are realized provided that the expansion of the β function in terms of a_s makes sense, which translates into the renormalization scheme-independent constraint:

$$c_1 a_s \equiv \frac{\beta_2}{\beta_1} a_s < 1\ , \tag{12.47}$$

[1] We neglect in this discussion the small contribution due to the light by the light-scattering diagram (see next chapter).

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which for four flavours corresponds to $Q > 1.62\lambda$. However, it is interesting to see the modification of this constraint when non-perturbative terms are included in the QCD series. In [178], relations between the corrections to D_n in the PMS and ECH scheme have been also derived with the result:

$$\begin{aligned}\delta D_2^{\mathrm{PMS}} &= \delta D_2^{\mathrm{ECH}} + \frac{d_0 c_1^2}{4} \\ \delta D_3^{\mathrm{PMS}} &= \delta D_3^{\mathrm{ECH}} \,,\end{aligned} \tag{12.48}$$

as well as an extension of the analysis to $n = 4$.

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