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# THE SUBNORMAL SUBGROUP STRUCTURE OF THE INFINITE GENERAL LINEAR GROUP

*by* DAVID G. ARRELL

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## 1. Introduction and notation

Let $R$ be a ring with identity, let $\Omega$ be an infinite set and let $M$ be the free $R$-module $R^{(\Omega)}$. In [**1**] we investigated the problem of locating and classifying the normal subgroups of $GL(\Omega, R)$, the group of units of the endomorphism ring $\mathrm{End}_R\, M$, where $R$ was an arbitrary ring with identity. (This extended the work of [**3**] and [**8**] where it was necessary for $R$ to satisfy certain finiteness conditions.) When $R$ is a division ring, the complete classification of the normal subgroups of $GL(\Omega, R)$ is given in [**9**] and the corresponding results for a Hilbert space are given in [**6**] and [**7**]. The object of this paper is to extend the methods of [**1**] to yield a classification of the subnormal subgroups of $GL(\Omega, R)$ along the lines of that given by Wilson in [**10**] in the finite dimensional case.

For any two-sided ideal $\mathfrak{p}$ of $R$, we shall denote by $GL(\Omega, \mathfrak{p})$ the kernel of the natural group homomorphism induced by the projection $R \to R/\mathfrak{p}$, and by $GL'(\Omega, \mathfrak{p})$ the inverse image of the centre of $GL(\Omega, R/\mathfrak{p})$. Let $\{e_\lambda : \lambda \in \Omega\}$ denote the canonical basis of $M$. Suppose $\Lambda \subset \Omega$, $\mu \in \Omega - \Lambda$ and $f: \Lambda \to R$. (We shall adopt the convention that $f$ extends to a map $f: \Omega \to R$ by defining $f(\omega) = 0$ for all $\omega \in \Omega - \Lambda$ and we shall use $\subset$ to denote proper subset inclusion.) Define the $R$-automorphism of $M$ $t(\Lambda, f, \mu)$ by

$$t(\Lambda, f, \mu)e_\rho = e_\rho + e_\mu f(\rho), \quad \text{for all} \quad \rho \in \Omega.$$

We shall call the $t(\Lambda, f, \mu)$ *elementary matrices* since the $t(\Lambda, f, \mu)$ can be thought of as $\Omega \times \Omega$ matrices differing from the identity matrix in only the $\mu$th row with $\Lambda$ indexing the non-zero entries of that row. We identify with each $a \in R$ the map $a: \Lambda \to R$ with $a(\lambda) = a$, for all $\lambda \in \Lambda$. Define $E(\Omega, R)$ to be the subgroup of $GL(\Omega, R)$ generated by $\{t(\Lambda, f, \mu): \Lambda \subset \Omega,\ \mu \in \Omega - \Lambda,\ f: \Lambda \to R\}$. For any right ideal $\mathfrak{p}$ of $R$ we define $E(\Omega, \mathfrak{p})$ to be the normal closure of $\{t(\Lambda, f, \mu): \Lambda \subset \Omega,\ \mu \in \Omega - \Lambda,\ f: \Lambda \to \mathfrak{p}\}$ in $E(\Omega, R)$. Arguments similar to those of [**8**] show that $E(\Omega, R)$ and $E(\Omega, \mathfrak{p})$ are normal subgroups of $GL(\Omega, R)$. If $\Lambda = \{\lambda\}$ we shall abbreviate $t(\Lambda, f, \mu)$ to $t(\lambda, a, \mu)$, where $a = f(\lambda)$ and we shall define $EF(\Omega, R)$ to be the subgroup of $GL(\Omega, R)$ generated by $\{t(\lambda, a, \mu): \lambda, \mu \in \Omega,\ \lambda \neq \mu,\ a \in R\}$. Thus, if $\boldsymbol{N}$ denotes the set of natural numbers $\{1, 2, 3, \ldots\}$, then we see that $EF(\boldsymbol{N}, R)$ is just the subgroup $E(R)$ of the stable linear group of Bass [**4**]. For any right ideal $\mathfrak{p}$ of $R$, $EF(\Omega, \mathfrak{p})$ is defined to be the normal closure of $\{t(\lambda, a, \mu): \lambda, \mu \in \Omega,\ \lambda \neq \mu,\ a \in \mathfrak{p}\}$ in $EF(\Omega, R)$.

For any two-sided ideal $\mathfrak{p}$ of $R$ we can write $\mathfrak{p}$ as a sum of finitely generated right ideals $\{\mathfrak{p}_\alpha : \alpha \in A\}$. It was shown in [1] that the normal subgroup $\prod_{\alpha \in A} E(\Omega, \mathfrak{p}_\alpha)$ is independent of the choice of the $\mathfrak{p}_\alpha$ and we shall denote this group by $E[\Omega, \mathfrak{p}]$. We also recall from [1] that a ring $R$ is said to be *d-finite* if every two-sided ideal of $R$ can be finitely generated as a right ideal. Thus, simple rings and Noetherian rings are $d$-finite.

We say that a subgroup $H$ of a group $G$ is a *subnormal* subgroup of $G$ if there exists a normal series of subgroups

$$H = H_d \lhd H_{d-1} \lhd \ldots \lhd H_0 = G.$$

We shall write $H \lhd^d G$. The least integer $d$ such that $H \lhd^d G$ is called the *defect* of $H$ in $G$. We define the terms $\gamma_i(G)$ of the lower central series of $G$ by $\gamma_1(G) = G$, $\gamma_i(G) = [G, \gamma_{i-1}(G)]$, $i = 2, 3, \ldots$. A group $G$ is called *nilpotent* if $\gamma_m(G) = 1$, for some integer $m$. If $c+1$ is the least value of $m$ satisfying this condition then $c$ is called the *class* of $G$.

For any $\Omega \times \Omega$ matrix $X$, we define the *level* of $X$ to be the two-sided ideal generated by the matrix entries $X_{\alpha\beta}$, $X_{\alpha\alpha} - X_{\beta\beta}$, for all $\alpha, \beta \in \Omega$, $\alpha \neq \beta$. For any subgroup $H$ of $GL(\Omega, R)$ we define the level of $H$ to be the two-sided ideal $J(H) = \sum_{X \in H} J(X)$, (c.f. [10]). We also define the ideal $K(H)$ to be the two-sided ideal $\sum J(X)$, where the summation is taken over all those $X \in H \cap E(\Omega, R)$ that have at least four trivial columns. (The $\varphi$th column of $X$ is said to be *trivial* if and only if $X(e_\varphi) = e_\varphi$.) Since matrices in $E(\Omega, R)$ differ from the $\Omega \times \Omega$ identity matrix in only finitely many rows we see that $K(H)$ is, in fact, the two-sided ideal generated by the matrix entries $X_{\alpha\beta}$, $X_{\alpha\alpha} - 1$, for all $\alpha, \beta \in \Omega$, $\alpha \neq \beta$, and all $X \in H \cap E(\Omega, R)$ that have at least four trivial columns. Clearly $K(H) \leqq J(H)$ and it was shown in [1] that whenever $H$ is a normal subgroup of $GL(\Omega, R)$, $K(H) = J(H)$.

## 2. Statement and discussion of results

We shall prove

**Theorem.** *Let $R$ be a ring with identity and let $\Omega$ be an infinite set. Let $G$ be a subgroup of $GL(\Omega, R)$ that contains $E(\Omega, R)$ and let $H$ be a subnormal subgroup of $G$, say $H \lhd^d G$. If we put $\mathfrak{p} = J(H)$ and $\mathfrak{q} = J(H^G)$ then*

$$E[\Omega, \mathfrak{p}^{f(d)}] \leqq H \leqq GL'(\Omega, \mathfrak{q})$$

*where $f(d) = (5^d - 1)/4$, for all integers $d \geqq 1$. Moreover, if $d = 1$, if $R$ is commutative or if $H \leqq E(\Omega, R)$ then $\mathfrak{p} = \mathfrak{q}$.*

We see that when $H$ is a normal subgroup of $GL(\Omega, R)$ the theorem coincides with Theorem $A$ of [1]. We shall also prove

**Corollary.** *Let $R$ be a d-finite ring with identity, let $\Omega$ be an infinite set and let $H$ be a subgroup of $E(\Omega, R)$. The following assertions are equivalent.*

(i) *$H$ is a subnormal subgroup of $E(\Omega, R)$.*

(ii) *For some unique two-sided ideal* $\mathfrak{p}$ *of* $R$ *and some integer* $m$

$$E(\Omega, \mathfrak{p}^m) \leqq H \leqq GL'(\Omega, \mathfrak{p}).$$

*Moreover, if* (i) *holds then the least integer* $m$ *in* (ii) *satisfies* $d-1 \leqq m \leqq f(d)$. *If* (ii) *holds then the defect of* $H$ *in* $E(\Omega, R)$ *is at most* $m+1$. *(*$f$ *is as defined in the statement of the theorem.)*

We shall be interested in applying these results in two ways: first to investigate the simplicity of $E(\Omega, R)$ for arbitrary rings $R$ and secondly to look more closely at the structure of $E(\Omega, R)$ for specific rings $R$.

It is clear that simple rings with identity are $d$-finite and so the corollary shows that whenever $R$ is a simple ring with identity $E(\Omega, R)$ is a simple group. If we now consider rings that do not have an identity then $E(\Omega, R)$ need not necessarily be simple. In fact, we shall prove

**Proposition.** *Let* $R$ *be a simple ring without identity with* $R^2 \neq 0$. *If* $R$ *is* $d$*-finite then* $E(\Omega, R)$ *is simple. If* $R$ *is not* $d$*-finite then the derived group* $E(\Omega, R)'$ *of* $E(\Omega, R)$ *is a simple proper normal subgroup of* $E(\Omega, R)$; *indeed* $E(\Omega, R)'$ *is the unique minimal normal subgroup of* $E(\Omega, R)$.

It is clear that when $R$ is a simple ring with identity, $E(\Omega, R)$ is perfect and so we see from the proposition that when $R$ is a simple ring (with or without identity) $E(\Omega, R)$ is perfect if and only if $R$ is $d$-finite and $R^2 = R$. In fact, it is easy to extend the proposition to show that for any two-sided ideal $\mathfrak{p}$ of a ring $R$ with identity, $E(\Omega, \mathfrak{p})$ is perfect if and only if $\mathfrak{p}^2 = \mathfrak{p}$ and $\mathfrak{p}$ is finitely generated as a right ideal.

The proposition shows that the structure of $E(\Omega, R)/E(\Omega, R)'$ depends upon the way in which $R$ is generated as a right $R$-module. The following example shows just how far from trivial this factor group can be.

**Example.** For any ordinal $\alpha$ we can choose a ring $R$ and an infinite set $\Omega$ such that there are at least $\alpha$ normal subgroups between $E(\Omega, R)'$ and $E(\Omega, R)$.

We show first how to construct the ring $R$. For any ring $R$ define $d(R)$ to be the least cardinal $\mathfrak{u}$ amongst all those cardinals $\mathfrak{v}$ such that $R$ is generated as a right $R$-module by a set of cardinality $\mathfrak{v}$. For example, if $R$ is $d$-finite then $d(R) < \aleph_0$. We assert that, for any ordinal $\beta$ there exists a simple ring $R$ without identity such that $d(R) = \aleph_\beta$. Let $(\Lambda, \leqq)$ be a well-ordered set with card $\Lambda = \aleph_\beta$. Let $V$ be the free $\mathfrak{f}$-module ${}^{(\Lambda)}$, where $\mathfrak{f}$ is a field, and let $M_\beta(\mathfrak{f})$ denote the $\mathfrak{f}$-endomorphism ring of $V$. For each $X \in M_\beta(\mathfrak{f})$ define the rank of $X$, $\rho(X)$, as the $\mathfrak{f}$-dimension of the image space of $X$ and let $N_0 = \{X : X \in M_\beta(\mathfrak{f}),\ \rho(X) < \aleph_0\}$. Then $N_0$ is a simple ring without identity (for example, [**5**, page 109]) and since $\rho(X)$ is finite for each $X \in N_0$, it follows that $d(N_0) = \aleph_\beta$.

Now let $\Omega$ be a set of cardinality $\aleph_\alpha$ and let $R$ be a simple ring without identity with $d(R) = \aleph_{\alpha+1}$. Let $X \in E(\Omega, R)$ and let $\mathfrak{c}$ be the cardinality of a minimal generating set for the right ideal generated by $X_{\alpha\beta}$, $X_{\alpha\alpha} - X_{\beta\beta}$, for all $\alpha, \beta \in \Omega$, $\alpha \neq \beta$; let $\aleph_\beta$ be the first infinite cardinal greater than $\mathfrak{c}$. We shall say that $X$ has $\aleph_\beta$-support. Let

$$E(\beta) = \langle X : X \in E(\Omega, R), X \text{ has } \aleph_\gamma\text{-support}, \gamma \leqq \beta \rangle^{E(\Omega, R)}.$$

Methods similar to those used in the proof of Theorem B of [**1**] show that $E(\beta)$ is a proper normal subgroup of $E(\Omega, R)$ whenever $\beta \leqq \alpha$ and hence $\{E(\mu): 0 \leqq \mu \leqq \alpha+1\}$ is a tower of normal subgroups of $E(\Omega, R)$ with $E(0)=E(\Omega, R)'$ and $E(\alpha+1)=E(\Omega, R)$. We deduce that there are at least $\alpha$ distinct normal subgroups between $E(\Omega, R)$ and $E(\Omega, R)'$.

## 3. Basic lemmas

We first remark that the familiar commutator relations for elementary matrices, namely

$$[t(\Lambda_1, f, \mu), t(\Lambda_2, g, \rho)] = \begin{cases} t(\Lambda_2, f(\rho)g, \mu), & \mu \notin \Lambda_2, \\ t(\Lambda_1, -g(\mu)f, \rho), & \rho \notin \Lambda_1, \end{cases}$$

hold for the generators of $E(\Omega, R)$. Next notice that the proof of Lemma B of [**1**] essentially yields

**Lemma 1.** *Let $H$ be a subgroup of $GL(\Omega, R)$ that is normalised by $EF(\Omega, \mathfrak{p})$, for some two-sided ideal $\mathfrak{p}$ of $R$; then $J(H)\mathfrak{p} \leqq K(H)$.*

Also notice that the methods of Lemma C and Corollary A of [**1**] may be extended to give

**Lemma 2.** *Let $H$ be a subgroup of $E(\Omega, R)$ that is normalised by $EF(\Omega, \mathfrak{p})$, for some two-sided ideal $\mathfrak{p}$ of $R$; then $H$ contains $EF(\Omega, \mathfrak{p}^2K(H)\mathfrak{p}^2)$.*

We complete this section by quoting from [**1**]

**Lemma 3.** *If $H$ is a subgroup of $GL(\Omega, R)$ that is normalised by $E(\Omega, R)$ and if $H$ contains $EF(\Omega, \mathfrak{q})$, for some two-sided ideal $\mathfrak{q}$ of $R$, then $H$ contains $E(\Omega, \mathfrak{p})$, for any finitely generated right ideal $\mathfrak{p}$ of $R$ contained in $\mathfrak{q}$.*

## 4. The main proofs

We begin with the proof of the theorem. Let $H, G, d, f(d), \mathfrak{p}$ and $\mathfrak{q}$ be as in the statement of the theorem; we shall argue by induction on $d$. If $d=1$ then $H$ is normalised by $E(\Omega, R)$ and Theorem A of [**1**] shows that $E[\Omega, J(H)] \leqq H \leqq GL'(\Omega, J(H))$. This establishes an inductive basis. Now take as inductive hypothesis that the inclusions hold for all subnormal subgroups with normal chains of length less than $d$. If we write $H = H_d \lhd H_{d-1} \lhd^{d-1} G$ then $H_{d-1}$ contains $E[\Omega, J_0]$, where $J_0 = J(H_{d-1})^{f(d-1)}$. It follows that $H$ is normalised by $EF(\Omega, J_0)$ so that Lemma 2 shows that $H$ contains $EF(\Omega, J_0^2K(H)J_0^2)$. However, for any $Y \in E(\Omega, R)$, $K(H^Y) = K(H)$ and $H^Y \lhd^d G$ so that

$$EF(\Omega, J_0^2K(H)J_0^2) \leqq \bigcap_{Y \in E(\Omega, R)} H^Y \leqq H.$$

It follows from Lemma 3 that $H$ contains $E[\Omega, J_0^2K(H)J_0^2]$. But $J(H) \leqq J(H_{d-1})$ and this shows that $J(H)^{5f(d-1)+1} \leqq J_0^2K(H)J_0^2$, since $J(H)J_0 \leqq K(H)$ from Lemma 1. We deduce that $H$ contains $E[\Omega, J(H)^{f(d)}]$. We next remark that $H \leqq H^G \lhd G$ and by the inductive basis $H \leqq GL'(\Omega, J(H^G))$. We complete the proof by observing that if $R$ is commutative or if $H \leqq E(\Omega, R)$ then $J(H^G) = J(H)$.

We continue this section with the proof of the corollary. If (i) holds then (ii) follows from the theorem since, for $d$-finite rings $R$, $E[\Omega, \mathfrak{p}] = E(\Omega, \mathfrak{p})$, for any two-sided ideal $\mathfrak{p}$ of $R$. Now suppose that (ii) holds. Since $(E(\Omega, R) \cap GL'(\Omega, \mathfrak{p}))/E(\Omega, \mathfrak{p}^m)$ is nilpotent

$$H \lhd \gamma_m H \lhd \ldots \gamma_1 H \lhd E(\Omega, R)$$

is a normal series from $H$ to $E(\Omega, R)$ where $\gamma_i = \gamma_i(E(\Omega, R) \cap GL'(\Omega, p))$, $i = 1, \ldots, m$; it is clear that the defect of $H$ in $E(\Omega, R)$ is at most $m+1$. Moreover, if (i) holds then from the theorem we can always take $m = f(d)$, although a lesser value may suffice.

We complete this section with the proof of the proposition. The first assertion of the proposition follows immediately from the corollary if we embed $R$ in $R^* = \mathbf{Z} \times R$ in the usual way, for then normal subgroups of $E(\Omega, R)$ become subnormal subgroups of $E(\Omega, R^*)$ and the simplicity of $R$ shows that $J(H) = R$. Suppose now that $R$ is not $d$-finite but is simple and let $H$ be a non-trivial normal subgroup of $E(\Omega, R)$. Lemma 2 shows that $H$ contains $EF(\Omega, R)$ and from the commutator relations we deduce that $H$ contains $E(\Omega, R)'$. It follows that $E(\Omega, R)'$ is the unique minimal normal subgroup of $E(\Omega, R)$ and, in particular, $E(\Omega, R)' = E[\Omega, R]$. If we now let $H$ be a non-trivial normal subgroup of $E(\Omega, R)'$ then the theorem shows that $E[\Omega, R] \leqq H$ since $J(H) = R$ and we deduce that $E(\Omega, R)'$ is simple. The proof of Theorem B of [**1**] shows that every $X \in E(\Omega, R)'$ has $\aleph_0$-support and since there exist $X \in E(\Omega, R)$ that do not have $\aleph_0$-support ($R$ is not $d$-finite) we see that $E(\Omega, R)'$ is a proper subgroup of $E(\Omega, R)$. This completes the proof of the proposition.

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School of Mathematics and Computing
Leeds Polytechnic
Leeds LS1 3HE