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MOORE SPACES, SEMI-METRIC SPACES AND CONTINUOUS MAPPINGS CONNECTED WITH THEM

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1. Introduction. In [**1**] Arhangel'skiĭ announced that any σ-paracompact p-space could be mapped onto a Moore space by a perfect map. However Burke [**3**] recently showed that this is not true in general and he gave an example of a T_2, locally compact, σ-paracompact space which cannot be mapped onto a Moore space by a perfect map. Therefore the following question is suggested:

Question 1. How can the perfect preimages of Moore spaces be characterized?

In [**7**] Ponomarev established: "In order that a regular space X be paracompact, it is necessary and sufficient that for every open cover ω of the space X there exists an ω-mapping $f:X \rightarrow Y$ onto some metric space Y". This suggests further questions.

Question 2. How can the spaces which admit ω-mappings onto Moore spaces be characterized?

Question 3. How can the spaces which admit ω-mappings onto semi-metric spaces be characterized?

The main aim of this note is to answer the questions raised above.

Definitions of terms not given here can be found in [**1**]. A regular space in this paper is assumed to be T_1.

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2. ω-mappings and s-paracompact spaces.

Definition 2.1. Let ω be an open cover of a space X. A continuous mapping f from the space X onto some space Y is called an *ω-mapping* if for each point y in Y, there exists a neighbourhood O_y such that $f^{-1}O_y$ is contained in an element of the cover ω.

Definition 2.2. A topological space X is called s-paracompact if for each open cover $\mathscr{U}$ of X there exists a sequence $\{\mathscr{V}_i\}_{i=1}^{\infty}$ of open covers of X such that the following conditions are satisfied:

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(a) $\mathscr{V}_1 = \mathscr{U}$ and $\mathscr{V}_{i+1}$ refines $\mathscr{V}_i$ for each i;

(b) for each x in X there is i_x such that x is in exactly one member of $\mathscr{V}_i$ for all $i > i_x$;

(c) for each x in X, $A_x = \bigcap_{i=1}^{\infty} \mathrm{st}(x, \mathscr{V}_i)$ is closed in X.

Definition 2.3. A T_1-space X is a *semi-metric space* provided that there exists a distance function (or semi-metric) d such that for each (x, y) in $X \times X$, (1) $d(x, y) = d(y, x)$; (2) $d(x, y) \geqq 0$ and $d(x, y) = 0$ only if $x = y$; and (3) for every $M \subset X$, $\inf\{d(x, y)|y \in M\} = 0$ if and only if x is in the closure of M.

LEMMA 2.1. *If for every open cover ω of the space X there exists an ω-mapping $f:X \to Y$ onto a s-paracompact space Y, then X is a s-paracompact space.*

Proof. Let ω be an open cover of the space X. Then by the hypothesis there exists an ω-mapping f of the space X onto some s-paracompact space Y. For each y in Y, let O_y be an open neighbourhood of y such that $f^{-1}O_y$ is contained in some $U \in \omega$. Evidently, $\mathscr{O} = \{O_y|y \in Y\}$ is an open cover of Y. Since Y is s-paracompact, there exists a sequence $\{\mathscr{V}_i\}_{i=1}^{\infty}$ of open covers of Y such that (1) $\mathscr{V}_1 = \mathscr{O}$ and $\mathscr{V}_{i+1}$ refines $\mathscr{V}_i$ for each i; (2) given y in Y there is i_y such that y $i \geqq i_y$, y is exactly in one member of $\mathscr{V}_i$; and (3) for each y in Y, $A_y = \bigcap_{i=1}^{\infty} \mathrm{st}(y, \mathscr{V}_i)$ is closed in Y. Let

$$\mathscr{W}_{i+1} = f^{-1}\mathscr{V}_i = \{f^{-1}V|V \in \mathscr{V}_i\} \text{ for } i = 1, 2, \ldots,$$

and $\mathscr{W}_1 = \omega$.

Since $\mathscr{V}_{i+1}$ refines $\mathscr{V}_i$ for each i, $\mathscr{W}_{i+1}$ refines $\mathscr{W}_i$ for each. Given x in X, there is a unique y in Y for which x belongs to $f^{-1}y$. Also for y in Y there is an i_y such that for all $i \geqq i_y$, y is exactly one member of $\mathscr{V}_i$. By the construction of the sequence $\{\mathscr{W}_i\}_{i=1}^{\infty}$, we can conclude that x belongs to exactly one member of $\mathscr{W}_i$ for all $i \geqq i_y + 1$ and for any x in X, $A_x = f^{-1}A_{fx}$ is closed. Hence X is a s-paracompact space.

LEMMA 2.2. *Let X be a s-paracompact space. Then for each open cover ω of the space X there exists an ω-mapping $f:X \to Y$ onto some semi-metric space Y.*

Proof. Let $\omega = \{W_s|s \in S\}$ be an open cover of an s-paracompact space X. Then there exists a sequence $\{\mathscr{W}_i\}_{i=1}^{\infty}$ of open covers of X satisfying the condition that for each x in X, there is an i_x such that x is in exactly one member of $\mathscr{W}_i$ for $i \geqq i_x$, $\mathscr{W}_{i+1}$ refines $\mathscr{W}_i$ for each i and $A_x = \bigcap_{i=1}^{\infty} \mathrm{st}(x, \mathscr{W}_i)$ for each x in X is closed, where $\mathscr{W}_1 = \omega$.

For each i, define $U_i = \bigcup \{W \times W|W \in \mathscr{W}_i\}$. Clearly $\{U_i\}_{i=1}^{\infty}$ is a symmetric collection of subsets of $X \times X$ containing $\Delta = \{(x, x)|x \in X\}$ and $U_{i+1} \subset U_i$ for each i. Define on X an equivalence relation by setting $x \sim y$ if and only if $y \in \bigcap_{i=1}^{\infty} U_i[x]$ for any pair of points $x, y \in X$. It is easy to verify that $\sim$ is an equivalence relation on X. Note that $A_x = \bigcap_{i=1}^{\infty} U_i[x]$ for each x in X.

Define on X a new topology τ by setting a subset G of X to be open if and only if for each x in G, there is an i such that $U_i[x] \subset G$. Denote by X_τ the set X together with the new topology τ. Denote by $Y = X_\tau/\sim$ the quotient space of X_τ. Let $\psi: X \to X_\tau$ be the identity map and $\Phi: X_\tau \to Y$ be the quotient map. Now define $f = \Phi \circ \psi$. We claim that f is an ω-mapping of X onto Y and that Y is a semi-metric space.

f is an ω-mapping: Since τ is weaker than the original topology of X, ψ is continuous. Because Φ is the quotient map, it is continuous. Therefore $f = \Phi \circ \psi$ is continuous. That Φ is also an open map is seen as follows. Let U be an open subset of X_τ. To show that ΦU is open, it is enough to show that

$$U^{\#} = \Phi^{-1}\Phi U = \bigcup\{A_x | A_x \cap U \neq \emptyset, x \in X\}$$

is open in X_τ where $A_x = \bigcap_{i=1}^{\infty} U_i[x]$ for all x in X. If y belongs to $U^{\#}$, then $A_y \cap U \neq \emptyset$. Let $z \in A_y \cap U$. Then for some i we have $U_i[z] \subset U$, because U is an open subset of X_τ. But $A_y = A_z \subset U_i[z] \subset U$ implies that there is a j such that $y \in U_j[y] \subset U \subset U^{\#}$. Hence $U^{\#}$ is open and consequently, Φ is an open map. To each y in Y, assign $O_y = \Phi(\operatorname{int}_\tau U_{i_x}[x])$ where x is such that $fx = y$ and i_x is an index for which x is in exactly one member of $\mathscr{W}_{i_x}$; note that x is in the int $U_{i_x}[x]$. Now it is easy to show that $f^{-1}O_y$ is contained in some member of ω. Since y is arbitrary f is an ω-mapping of X onto Y.

Y is a semi-metric space: Define $\mathscr{W}_i' = f\mathscr{W}_i = \{fW | W \in \mathscr{W}_i\}$ for each i. It is easy to see that for each y in Y, $\{\operatorname{st}(y, \mathscr{W}_i')\}_{j=1}^{\infty}$ is a base for the open sets containing y. Also, $\mathscr{W}'_{i+1}$ refines $\mathscr{W}_i'$ for all i. Since $\mathscr{W}_{i+1}$ refines $\mathscr{W}_i$ for each i.

For $y_1, y_2 \in Y$, let $\alpha(y_1, y_2)$ denote the smallest integer n such that there is no element of $\mathscr{W}_n'$ containing both y_1 and y_2. If no such integer exists $\alpha(y_1, y_2) = \infty$. Now define $d: Y \times Y \to \mathbf{R}$ by setting $d(y_1, y_2) = 2^{-\alpha(y_1, y_2)}$ for (y_1, y_2) in $Y \times Y$. Then clearly, for each y, y_1, y_2 in Y, $d(y, y) = 0$ and $d(y_1, y_2) = d(y_2, y_1)$. Also, if $y_1 \neq y_2$, there is an open set U containing one of the points, say y_1, and not containing y_2; since Y is a T_1-space. Hence there is an n such that $y_1 \in \operatorname{st}(y_1, \mathscr{W}_n') \subset U$. Since $y_2 \notin U$ implies $y_2 \notin \operatorname{st}(y_1, \mathscr{W}_n')$, we have $d(y_1, y_2) \geqq 1/2^n > 0$.

We note here that

$$\{y | d(y_0, y) < 1/2^n\} = s(y_0; 1/2^n) = \operatorname{st}(y_0, \mathscr{W}_n')$$

for each y_0 in Y and each n. For y is in $s(y_0; 1/2^n)$ if and only if $d(y_0, y) < 1/2^n$ if and only if $\alpha(y_0, y) > n$ if and only if there exists W in $\mathscr{W}_n'$ such that $y_0, y \in W$, i.e., if and only if $y \in \operatorname{st}(y_0, \mathscr{W}_n')$. Now let $M \subset Y$. Then $y \in M^-$ if and only if $\operatorname{st}(y, \mathscr{W}_n') \cap M \neq \emptyset$ for each n if and only if $s(y; 1/2^n) \cap M \neq \emptyset$ for each n, i.e., if and only if $d(y, M) = 0$. Hence Y is a semi-metric space.

LEMMA 2.3. *Every semi-metric space X is s-paracompact.*

Proof. McAuley [**5**] pointed out that by using a proof analogous to that of Theorem 2 of [**2**], it follows that given any open cover $\mathscr{U} = \{U_s | s \in S\}$ of a semi-metric space X there exists a σ-discrete closed refinement $\mathscr{V} = \bigcup_{i=1}^{\infty} \mathscr{V}_i$, where $\mathscr{V}_i = \{\mathscr{V}_\alpha{}^i | \alpha \in \Lambda_i\}$ for each i, such that each member of $\mathscr{V}_i$ is contained in some member of $\mathscr{V}_{i+1}$ for each i.

Denote by $U_{\alpha,s}{}^1$ a member of $\mathscr{U}$ which contains $V_\alpha{}^i$ and let $O_\alpha{}^i = X - \bigcup\{V_\beta{}^i | \beta \in \Lambda_i, \alpha \neq \beta\}$. Define

$$\mathscr{W}_{i+1} = \{O_\alpha{}^i \cap U_{\alpha,s}{}^i | \alpha \in \Lambda_i\} \cup \{(X - \bigcup\{V_\alpha{}^i | \alpha \in \Lambda_i\}) \cap U_s | s \in S\}$$

for each $i = 1, 2, \ldots$, and $\mathscr{W}_1 = \mathscr{U}$. Now it is easy to see that the sequence $\{\mathscr{W}_i\}_{i=1}^\infty$ of open covers of X satisfies (a) $\mathscr{W}_1 = \mathscr{U}$ and $\mathscr{W}_{i+1}$ refines $\mathscr{W}_i$ for each i; (b) for each x in X, there is an i_x such that x is in exactly one member of $\mathscr{W}_i$ for all $i \geqq i_x$; and (c) for each x in X, $A_x = \bigcap_{i=1}^\infty \text{st}(x, \mathscr{W}_i)$ is closed in X. Hence X is s-paracompact.

THEOREM 2.4. *A topological space X is s-paracompact if and only if for each open cover ω of X there exists an ω-mapping $f: X \to Y$ onto some semi-metric space Y.*

The proof follows from Lemmas 2.1, 2.2, and 2.3.

3. ω-mappings and Moore spaces.

Definition 3.1. A topological space X is called *d-paracompact* if for each open cover $\mathscr{U}$ of X there exists a sequence $\{\mathscr{V}_i\}_{i=1}^\infty$ of open covers of X such that the following conditions are satisfied:

(i) $\mathscr{V}_i = \mathscr{U}$ and $\mathscr{V}_{i+1}$ refines $\mathscr{V}_i$ for each i;

(ii) given i and x in X, there is j (j depending on i and x) and some V in $\mathscr{V}_i$ such that $\text{st}(x, \mathscr{V}_j) \subset V$;

(iii) given i and x in X, there is j such that for any y in $\text{st}(x, \mathscr{V}_j)$ there is k_y such that $\text{st}(y, \mathscr{V}_{k_y}) \subset \text{st}(x, \mathscr{V}_i)$.

(One may note that for each x in X, $A_x = \bigcap_{i=1}^\infty \text{st}(x, \mathscr{V}_i)$ is closed in X.)

Definition 3.2. A topological space X is called *developable* if there exists a sequence $\{\mathscr{V}_i\}_{i=1}^\infty$ of open covers of X satisfying the condition that for each x in X and any open set U containing x, there is i such that $\text{st}(x, \mathscr{V}_i) \subset U$.

A regular developable space is called a Moore space.

LEMMA 3.1. *Every developable space X is a d-paracompact space.*

The proof follows immediately from the definition of developable spaces.

LEMMA 3.2. *If for each open cover ω of the space X there exists an ω-mapping $f: X \to Y$ onto a d-paracompact space Y, then X is a d-paracompact space.*

The proof is similar to Lemma 2.1.

LEMMA 3.3. *Let X be a d-paracompact space. Then for each open cover ω of the space X there exists an ω-mapping $f: X \to Y$ onto some T_1 developable space Y.*

Proof. Let X be a d-paracompact space. Then for each open cover ω of X, there exists a sequence $\{\mathscr{V}_i\}_{i=1}^\infty$ of open covers of X satisfying

(i) $\mathscr{V}_i = \omega$ and $\mathscr{V}_{i+1}$ refines $\mathscr{V}_i$ for each i;

(ii) given i and x in X there is a $j > i$ (depending on x and i) and some V in $\mathscr{V}_i$ such that $\mathrm{st}(x, \mathscr{V}_j) \subset V$; and

(iii) given i and x in X, there is a j such that for any y in $\mathrm{st}(x, \mathscr{V}_j)$, there is a k_y such that $\mathrm{st}(y, \mathscr{V}_{k_y}) \subset \mathrm{st}(x, \mathscr{V}_i)$.

Define a new topology τ on X by taking $\{\mathrm{st}(x, \mathscr{V}_i)\}_{i=1}^{\infty}$ as a base for the neighbourhood system at x in X_τ. Denote by X_τ the set X with the new topology τ. Define an equivalence relation $\sim$ on X by setting $x \sim y$ if and only if $y \in \bigcap_{i=1}^{\infty} \mathrm{st}(x, \mathscr{V}_i)$. Let $Y = X_\tau/\sim$ be the quotient space of X_τ, let $\Phi: X_\tau \to Y$ be the quotient map, and let $\psi: X \to X$ be the identity map. Define $f = \Phi \circ \psi$. Then we claim that f is an ω-mapping of X onto Y and Y is a T_1 developable space.

Using a proof analogous to Lemma 2.3 one can show that f is an ω-mapping of X onto Y and that Φ is open.

Y is a developable space: First note that $\Phi^{-1}\Phi\, \mathrm{int}_\tau A = \mathrm{int}_\tau A$ where A is any subset of X. Define $\mathscr{W}_1 = \{\Phi\, \mathrm{int}_\tau V | V \in \mathscr{V}_i\}$ for each i. We claim that $\{\mathscr{W}_i\}_{i=1}^{\infty}$ is a development for Y. Let $y \in Y$ and let U be an open set in Y containing y. Then $f^{-1}y \subset f^{-1}U$, i.e.,

$$(\Phi \circ \psi)^{-1}y \subset (\Phi \circ \psi)^{-1}U \text{ implies } \Phi^{-1}y \subset \Phi^{-1}U.$$

For some x in X, $\Phi^{-1}y = \bigcap_{i=1}^{\infty} \mathrm{st}(x, \mathscr{V}_i)$. Since $\Phi^{-1}U$ is open and contains $\bigcap_{i=1}^{\infty} \mathrm{st}(x, \mathscr{V}_i)$ we have $\mathrm{st}(\Phi^{-1}y, \mathscr{V}_i) \subset \Phi^{-1}U$ for some i. Also, it follows from Note [**2**] that Y is T_1. Hence the lemma is proved.

THEOREM 3.4. *A topological space X is a d-paracompact space if and only if for each open cover ω of X there exists an ω-mapping of X onto some T_1 developable space.*

The proof follows from Lemmas 3.1, 3.2 and 3.3.

Remark. In view of Theorem 9 of Bing [**2**] and Theorem 3.4, it is easy to show that a space X is d-paracompact if and only if for each open cover $\mathscr{U}$ of X there exists a sequence $\{\mathscr{V}_i\}_{i=1}^{\infty}$ of open covers of X satisfying the following properties:

(a′) $\mathscr{V}_1 = \mathscr{U}$ and $\mathscr{V}_{i+1}$ refines $\mathscr{V}_i$ for each i;

(b′) for each x in X there is an i_x such that x is exactly in one member of $\mathscr{V}_i$ for all $i \geqq i_x$;

(c′) given i and x in X, there is j such that for any y in $\mathrm{st}(x, \mathscr{V}_j)$ there is a k_y such that $\mathrm{st}(y, \mathscr{V}_{k_y}) \subset \mathrm{st}(x, \mathscr{V}_i)$.

LEMMA 3.5. *A topological space X is a regular d-paracompact space if and only if for each open cover $\mathscr{U}$ of X there exists a sequence $\{\mathscr{V}_i\}_{i=1}^{\infty}$ of open covers of X such that the following conditions are satisfied:*

(1) $\mathscr{V}_1 = \mathscr{U}$ and $\mathscr{V}_{i+1}$ refines $\mathscr{V}_i$ for each i;

(2) given i and x in X, there is j (j depending on i and x) and some V in $\mathscr{V}_i$ such that $\mathrm{st}(x, \mathscr{V}_j)^- \subset V$;

(3) *given i and x in X, there is j such that for any y in* $\mathrm{st}(x,\mathscr{V}_j)$ *there is a k_y such that* $\mathrm{st}(y,\mathscr{V}_{k_y}) \subset \mathrm{st}(x,\mathscr{V}_i)$.

Proof. Let $\mathscr{U}$ be an open cover of a regular d-paracompact space X. Then there exists a sequence $\{\mathscr{W}_i\}_{i=1}^{\infty}$ of open covers of X satisfying the following properties:

(1′) $\mathscr{W}_1 = \mathscr{U}$, for each i the closure of each member of $\mathscr{W}_{i+1}$ is contained in some member of U, and $\mathscr{W}_{i+1}$ refines $\mathscr{W}_i$ for each i;

(2′) given i and x in X, there is j (j depending on i and x) and some W in $\mathscr{W}_i$ such that $\mathrm{st}(x,\mathscr{W}_j) \subset W$; and

(3′) given i and x in X, there is a j such that for any y in $\mathrm{st}(x,\mathscr{W}_j)$ there is a k_y such that $\mathrm{st}(y,\mathscr{W}_{k_y}) \subset \mathrm{st}(x,\mathscr{W}_i)$.

Now given an integer i_0 assume we can construct a sequence $\{\mathscr{V}_i\}_{i=1}^{\infty}$ of open covers of X satisfying (i) $\mathscr{V}_1 = \mathscr{W}_1 = \mathscr{U}$ and $\mathscr{V}_{j+1}$ refines $\mathscr{V}_j$ for each j; (ii) given $i < i_0$ and x in X, there is a j (depending on i and x) and some V in $\mathscr{V}_i$ such that $\mathrm{st}(x,\mathscr{V}_j)^- \subset V$; (iii) given j and x in X, there is a l such that for any y in $\mathrm{st}(x,\mathscr{V}_l)$ there is a k_y such that $\mathrm{st}(y,\mathscr{V}_{k_y}) \subset \mathrm{st}(x,\mathscr{V}_j)$. Now define $\{\mathscr{V}_i'\}_{i=1}^{\infty}$, a sequence of open covers of X such that $\mathscr{V}_i' = \mathscr{V}_i$ for $i = 1, \ldots, i_0$ and $\mathscr{V}'_{i_0+j} = \mathscr{W}_{i_0 j+1}$ for $j = 1, 2, \ldots$ where $\{\mathscr{W}_{i_0 j}\}_{j=1}^{\infty}$ is a sequence of open covers of X satisfying (a′) $\mathscr{W}_{i_0 1} = \mathscr{V}_{i_0+1}$ for each i, the closure of members of $\mathscr{W}_{i_0 j+1}$ is contained in some member of $\mathscr{V}_{i_0+1}$, and $\mathscr{W}_{i_0 j+1}$ refines $\mathscr{W}_{i_0 j}$; (b′) given i and x in X, there is j (j depending on i and x) and some V in $\mathscr{V}_i'$ such that $\mathrm{st}(x,\mathscr{V}_j) \subset V$; (c′) given i and $x \in X$, there is a j such that for any y in $\mathrm{st}(x,\mathscr{V}_j')$ there is k_y such that $\mathrm{st}(y,\mathscr{V}_{k_y}') \subset \mathrm{st}(x,\mathscr{V}_i')$. Hence by the induction there exists a sequence $\{\mathscr{W}_i\}_{i=1}^{\infty}$ of open covers of X satisfying conditions 1, 2 and 3. This proves the lemma.

The converse is trivial.

LEMMA 3.6. *Let X be a regular d-paracompact space. Then for each open cover ω of the space X there exists an ω-mapping $f:X \to Y$ onto some Moore space Y.*

Proof. Let X be a regular d-paracompact space. Then by Lemma 3.5 for each open cover ω of X, there exists a sequence $\{\mathscr{V}_i\}_{i=1}^{\infty}$ of open covers of X satisfying (i) $\mathscr{V}_1 = \omega$ and $\mathscr{V}_{i+1}$ refines $\mathscr{V}_i$ for each i; (ii) given i and x in X there is $j > i$ (j depending on x and i) and some V in $\mathscr{V}_i$ such that $\mathrm{st}(x,\mathscr{V}_j)^- \subset V$; and (iii) given i and x in X, there is a j such that for any y in $\mathrm{st}(x,\mathscr{V}_j)$, there is k_y such that $\mathrm{st}(y,\mathscr{V}_{k_y}) \subset \mathrm{st}(x,\mathscr{V}_i)$.

Define a new topology τ on X by setting $\{\mathrm{st}(x,\mathscr{V}_i)\}_{i=1}^{\infty}$ as a base for the neighbourhood system at x in X. Denote by X_τ the set X with the new topology τ. Define an equivalence relation $\sim$ on X by setting $x \sim y$ if and only if $y \in \bigcap_{i=1}^{\infty} \mathrm{st}(x,\mathscr{V}_i)$. Let $Y = X_\tau/\sim$ be the quotient space of X_τ, $\Phi:X_\tau \to Y$ be the quotient map, and $\psi:X \to X_\tau$ be the identity map. Now define $f = \Phi \circ \psi$. Then as in Lemma 3.3 we can show that f is an ω-mapping of X onto Y and Y is a developable space. It remains to show that Y is regular.

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For this, it is enough to show that $\{\text{st}(x, \mathscr{V}_i)^-\}_{i=1}^{\infty}$ is a base for the neighbourhood system at x in X in the topology τ.

Let U be an open subset of X relative to τ and let x be in U. Then for some i we have $x \in \text{st}(x, \mathscr{V}_i) \subset U$. By condition (ii) on the sequence of covers $\{\mathscr{V}_i\}_{i=1}^{\infty}$ we have

$$x \in \text{st}(x, \mathscr{V}_k)^- \subset V \subset \text{st}(x, \mathscr{V}_i) \subset U$$

for some k and V in $\mathscr{V}_i$. Hence $\{\text{st}(x, \mathscr{V}_i)^-\}_{i=1}^{\infty}$ is a base at x in X in the topology τ. Y is obviously T_1 (by the remark in Definition 3.1).

LEMMA 3.7. *If for each open cover ω of the space X there exists an ω-mapping $f:X \to Y$ onto a regular d-paracompact space Y, then X is a regular d-paracompact space.*

We leave the proof of Lemma 3.7 to the reader.

THEOREM 3.8. *A topological space X is regular and d-paracompact if and only if for each open cover ω of X there exists an ω-mapping $f:X \to Y$ onto some Moore space.*

The proof follows from Lemmas 3.5, 3.6 and 3.7.

4. Perfect mappings and Moore spaces.

Definition 4.1. Let X be a topological space. A *decomposition* of X is a collection $\mathscr{A}$ of nonempty subsets of X such that $X = \bigcup\{A | A \in \mathscr{A}\}$. A *compact decomposition* of X is a decomposition $\mathscr{A}$ of X such that every member of $\mathscr{A}$ is a compact subset of X.

Definition 4.2. Let X be a topological space, let $\mathscr{A}$ be a decomposition of X and let $\mathscr{U}$ be an open cover of X. Then $\mathscr{U}$ is called *cover modulo the decomposition* $\mathscr{A}$ of X if for A in $\mathscr{A}$ and U in $\mathscr{U}$, $A \cap U \neq \emptyset$ implies $A \subset U$.

Definition 4.3. Let X be a topological space and let $\mathscr{A}$ be a decomposition of X. Then X is said to have *development modulo a decomposition* provided that there exists a sequence $\{\mathscr{V}_i\}_{i=1}^{\infty}$ of open covers of X satisfying the following properties:

(a) $\mathscr{V}_{i+1}$ refines $\mathscr{V}_i$ for each i;

(b) $\mathscr{V}_i$ is a cover modulo the decomposition $\mathscr{A}$ of X for each i;

(c) for each A in $\mathscr{A}$ and any open set U of X containing A there is an i such that $\text{st}(A, \mathscr{V}_i) \subset U$.

Definition 4.4. A topological space X is said to be *developable modulo a decomposition* provided that for some decomposition $\mathscr{A}$ of X there exists a development modulo a decomposition $\mathscr{A}$.

Definition 4.5. A mapping $f:X \to Y$ of a space X onto Y is called *perfect* if f is closed, continuous and $f^{-1}y$ is compact for y in Y.

LEMMA 4.1. *If a topological space X is a perfect preimage of a developable space Y, then X is developable modulo a compact decomposition.*

Proof. Let f be a perfect mapping of a space X onto a developable space Y. Since Y is developable there exists a sequence $\{\mathscr{V}_i\}_{i=1}^{\infty}$ of open covers of Y such that for each y in Y, $\{\mathrm{st}(y,\mathscr{V}_i)\}_{i=1}^{\infty}$ is a base for the neighbourhood system at y. Without loss of generality, assume that $\mathscr{V}_{i+1}$ refines $\mathscr{V}_i$ for each i. Then it is easy to see that $\mathscr{A} = \{f^{-1}y|y \in Y\}$ is a compact decomposition of X and $\{\mathscr{W}_i\}_{i=1}^{\infty}$ is a development modulo the compact decomposition $\mathscr{A}$ for X, where $\mathscr{W}_i = \{f^{-1}V|V \in \mathscr{V}_i\}$ for each i. Hence X is developable modulo a compact decomposition.

LEMMA 4.2. *If a regular space X is developable modulo compact decomposition, then there exists a perfect mapping of X onto some Moore space Y.*

Proof. Suppose X is a regular developable modulo compact decomposition space. Then there exists a compact decomposition $\mathscr{A} = \{A_\alpha|\alpha \in \Lambda\}$ of X and there exists a sequence $\{\mathscr{V}_i\}_{i=1}^{\infty}$ of open covers of X such that it is a development modulo compact decomposition $\mathscr{A}$. Define on X an equivalence relation by setting $x \sim y$ if and only if $x, y \in A_\alpha$ for some α in Λ. Let $Y = X/\sim$ be the quotient space of X with the quotient topology and let $f:X \to Y$ be the quotient map. It is easy to see that f is continuous and compact; i.e., $f^{-1}y$ is compact for each y in Y. Let C be a closed subset of X. Then

$$C^{\#} = \bigcup\{A_\alpha|A_\alpha \cap C \neq \emptyset; \alpha \in \Lambda\}$$

is a subset of X. We shall show that $C^{\#}$ is closed. Suppose $y \notin C^{\#}$. Then there is an A_{α_y} for some $\alpha_y \in \Lambda$ such that $y \in A_{\alpha_y}$ and $A_{\alpha_y} \cap C = \emptyset$. Therefore for some i we have $C \cap \mathrm{st}(y,\mathscr{V}_i) = \emptyset$. But then $\mathrm{st}(y,\mathscr{V}_i) \cap C^{\#} = \emptyset$, for otherwise some A_α will be contained in $\mathrm{st}(y,\mathscr{V}_i)$ and will intersect C. Hence $y \notin C^{\#}$ implies y is not a limit point of $C^{\#}$. Consequently, $C^{\#}$ is a closed subset of X. Now $f^{-1}fC = C^{\#}$ and Y carries the quotient topology, so that fC is closed. Hence f is a perfect mapping of X onto Y. Now we show that Y is a Moore space.

For each i define $\mathscr{W}_i = \{\mathrm{int} fV|V \in \mathscr{V}_i\}$. We claim that $\{\mathscr{W}_i\}_{i=1}^{\infty}$ is a development for Y. Let $y \in Y$ and U be any open set containing y. Since f is continuous $f^{-1}U$ is open. Now there is i such that $\mathrm{st}(f^{-1}y,\mathscr{V}_i) \subset f^{-1}U$ and therefore $\mathrm{st}(y,\mathscr{W}_i) \subset U$. By the fact that V in $\mathscr{V}_i$ which intersects $f^{-1}y$ contains $f^{-1}y$ and the fact that f is continuous and closed, $\mathrm{int} fV \neq \emptyset$ for V in $\mathscr{V}_i$ and each i. That Y is regular follows trivially. Hence Y is a Moore space and the lemma is proved.

THEOREM 4.3. *A regular space X is developable modulo compact decomposition if and only if there exists a perfect mapping of X onto some Moore space Y.*

The proof follows from Lemmas 4.1 and 4.2.

In view of Theorem 3.1 in [**6**] we take the definition of p-space as follows:

Definition 4.6. A topological space X is called a *p-space* if there exists a sequence $\{\mathscr{V}_i\}_{i=1}$ of open covers of X satisfying the properties: if $\{F_s | s \in S\}$ is a family of closed sets with finite intersection property and there is x in X such that for each i there is V in $\mathscr{V}_i$ containing x and F_{s_i} for some $s_i \in S$, then $\bigcap\{F_s | s \in S\} \neq \emptyset$.

THEOREM 4.4. *A topological space X is a regular d-paracompact p-space if and only if for each open cover ω of X there exists a perfect ω-mapping $f:X \to Y$ onto some Moore space Y.*

Proof. If X is a regular d-paracompact p-space, by using the techniques of Lemma 3.5, for each open cover ω of X there exists a sequence $\{\mathscr{V}_i\}_{i=1}^{\infty}$ of open covers of X satisfying (a) $\mathscr{V}_1 = \omega$ and $\mathscr{V}_{i+1}$ refines $\mathscr{V}_i$ for each i; (b) given i and x in X, there is j (j depending on i and x) and some V in $\mathscr{V}_i$ such that $\mathrm{st}(x, \mathscr{V}_j)^- \subset V$; (c) given i and x in X, there is a j such that for any y in $\mathrm{st}(x, \mathscr{V}_j)$ there is k_y such that $\mathrm{st}(y, \mathscr{V}_{k_y}) \subset \mathrm{st}(x, \mathscr{V}_i)$; and (d) if $\{F | F \in \mathscr{F}\}$ is a family of closed sets with finite intersection property and there is x in X such that for each i some F in $\mathscr{F}$ is contained in $\mathrm{st}(x, \mathscr{V}_i)$ then $\bigcap\{F | F \in \mathscr{F}\} \neq \emptyset$. Now using a proof analogous to Theorem 3.4, one can show that there exists a perfect mapping of X onto a Moore space Y.

The converse follows from Lemma 3.7 and [**1**].

Remark. Theorem 4.4 in view of Theorem 4.3 suggests that a regular d-paracompact p-space is developable modulo compact decomposition. We conjecture that a regular developable modulo compact decomposition space is a d-paracompact space.

5. One-to-one continuous mappings.

PROPOSITION 5.1. *Let X be a d-paracompact space with the diagonal a G_δ-set in $X \times X$. Then there exists a T_1 developable space Y and a one-to-one continuous map f from X onto Y.*

Proof. If X is a d-paracompact space with the diagonal a G_δ-set in $X \times X$, then in view of [**4**, p. 112, Lemma 5.4], there exists a sequence $\{\mathscr{V}_i\}_{i=1}^{\infty}$ of open covers of X satisfying the conditions: (a) $\mathscr{V}_{i+1}$ refines $\mathscr{V}_i$ for each i and for any x in X we have $\{x\} = \bigcap_{i=1}^{\infty} \mathrm{st}(x, \mathscr{V}_i)$; (b) given i and x in X, there is j (j depending on i and x) and some V in $\mathscr{V}_i$ such that $\mathrm{st}(x, \mathscr{V}_j) \subset V$; and (c) given i and x in X, there is a j such that for any y in $\mathrm{st}(x, \mathscr{V}_j)$ there is a k_y such that $\mathrm{st}(y, \mathscr{V}_{k_y}) \subset \mathrm{st}(x\ \mathscr{V}_i)$.

Let $Y = X$ and define a topology on Y by taking $\{\mathrm{st}(x, \mathscr{V}_i)\}_{i=1}^{\infty}$ as a base for the neighbourhood system at x in X. Denote by τ the topology defined above. Now it is easy to see that the sequence $\{\mathscr{W}_i\}_{i=1}^{\infty}$, where

$$\mathscr{W}_i = \{\mathrm{int}_\tau V | V \in \mathscr{V}_i\}$$

for each i, is a development for Y. Also, for each y in Y, $\{y\} = \bigcap_{i=1}^{\infty} \mathrm{st}(y, \mathscr{W}_i)$

implies Y is a T_1-space. Now define $f:X \to Y$ by setting $fx = x$. Clearly f is a one-to-one continuous map. Hence the proposition is proved.

PROPOSITION 5.2. *Let X be a s-paracompact space with the diagonal a G_δ-set in $X \times X$. Then there exists a semi-metric space Y and a one-to-one continuous map f from X onto Y.*

We leave the proof of the above proposition to the reader.

6. Problems.

6.1. Is it true that a normal d-paracompact p-space always admits a perfect mapping onto a normal Moore space?

6.2. Is a normal d-paracompact space metacompact?

6.3. Is a normal metacompact space d-paracompact?

6.4. Find necessary and sufficient conditions for an s-paracompact space to be d-paracompact.

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