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A NEW RESULT ABOUT ALMOST UMBILICAL HYPERSURFACES OF REAL SPACE FORMS

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Abstract

In this short note, we prove that an almost umbilical compact hypersurface of a real space form with almost Codazzi umbilicity tensor is embedded, diffeomorphic and quasi-isometric to a round sphere. Then, we derive a new characterisation of geodesic spheres in space forms.

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1. Introduction

Let (M^n, g) be a connected and oriented compact Riemannian manifold isometrically immersed into the simply connected real space form $\mathbb{M}^{n+1}(\delta)$ of constant curvature δ. Let B be the second fundamental form of the hypersurface and H its mean curvature. Since we consider only hypersurfaces, we take B as the real-valued second fundamental form. We denote by $\tau = B - Hg$ the traceless part of the second fundamental form, also called the umbilicity tensor. We say that M is totally umbilical if $\tau = 0$.

It is a well-known fact that a compact (without boundary) totally umbilical hypersurface of a simply connected real space form is a geodesic sphere. In the present note, we will investigate the natural question of the stability of this rigidity result. In other words, if a compact hypersurface of a real space form is almost umbilical, is this hypersurface close to a sphere? In what sense?

Shiohama and Xu proved in [18, 19] that if $\|\tau\|_n$ is small enough, then M^n is homeomorphic to the sphere $\mathbb{S}^n$. Subsequently, we obtained quantitative results about the closeness of almost hypersurfaces to spheres in [6, 16]. For instance, we proved in [16], always for hypersurfaces of space forms, that if $\|B - kg\|_\infty$ is sufficiently small, for a positive constant k, then the hypersurface is quasi-isometric to a sphere of radius $1/k$. In [6] and also in [16], we obtained similar results with a less restrictive assumption replacing the L^∞-norm by the L^r-norm ($r > n$). In particular, the hypersurface is diffeomorphic to the sphere $\mathbb{S}^n$. The proximity to the sphere is stronger than for the results of Shiohama and Xu but, in counterpart, the assumption on the umbilicity is also stronger. Indeed, the hypothesis implies that the umbilicity tensor is close to zero and in addition that the mean curvature is close to a constant. We add that in a very recent paper [17], Scheuer obtained the following nice result. If M is embedded into $\mathbb{R}^{n+1}$ and mean convex, then $\|\tau\|_\infty$ sufficiently small implies that M is strictly convex and Hausdorff close to a geodesic sphere of appropriate radius $\sqrt{n/\lambda_1}$, where λ_1 is the first positive eigenvalue of the Laplacian on M.

The aim of the present note is to give a result comparable to those of [6, 16] with an alternative condition to the assumption that the mean curvature is almost constant. Starting from the remark that a hypersurface of a real space form has constant mean curvature if and only if its umbilicity tensor is Codazzi, we will prove that a hypersurface of a real space form which is almost umbilical with almost Codazzi umbilicity tensor is close to a sphere in the same sense as in [16] (see Theorem 3.1). Then, we prove a new rigidity result for geodesic spheres in real space forms (Corollary 4.4).

2. Preliminaries

Before stating the results, we will introduce some useful notation and recall some useful facts. First, we recall that the results obtained in [6, 16] are consequences of pinching results for the first eigenvalue of the Laplacian proved in [5, 6]. A key tool for these pinching results is the Michael–Simon extrinsic Sobolev inequality for submanifolds of the Euclidean space [10] and its generalisation by Hoffman and Spruck for any ambient manifold [8]. We begin by recalling the conditions under which these Sobolev inequalities are valid. Let $(N^{n+1}, \overline{g})$ be a Riemannian manifold with sectional curvature bounded above, say $K_N \leqslant b^2$, with b real or purely imaginary and $n \geqslant 2$. For $0 < \alpha < 1$, we denote by $\mathcal{H}_V(N, \alpha)$ the set of all connected, oriented and compact Riemannian manifolds without boundary (M^n, g), isometrically immersed into N and satisfying the two following conditions:

$$b^2(1-\alpha)^{-2/n}(\omega_n^{-1}\mathrm{Vol}(M))^{2/n} \leqslant 1, \tag{2.1}$$

$$2\rho_0 \leqslant \mathrm{inj}_M(N), \tag{2.2}$$

where $\mathrm{inj}_M(N)$ is the injectivity radius of N restricted to M and ρ_0 is given by

$$\rho_0 = \begin{cases} b^{-1}\sin^{-1}(b(1-\alpha)^{-1/n}(\omega_n^{-1}\mathrm{Vol}(M))^{1/n}) & \text{if } b \text{ is real,} \\ (1-\alpha)^{-1/n}(\omega_n^{-1}\mathrm{Vol}(M))^{1/n} & \text{if } b \text{ is imaginary.} \end{cases}$$

Under these hypotheses, Hoffman and Spruck showed that for any C^1 function f on M, the following extrinsic Sobolev inequality holds:

$$\left(\int_M f^{n/(n-1)}\,dv_g\right)^{(n-1)/n} \leqslant K(n,\alpha)\int_M (|\nabla f| + |Hf|)\,dv_g,$$

where $K(n, \alpha)$ depends only on n and α (not on b). We remark that for Euclidean and hyperbolic spaces, that is, $\delta \leqslant 0$, (2.1) and (2.2) are trivially satisfied, whereas

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for spheres ($\delta > 0$), (2.1) and (2.2) amount to $\mathrm{Vol}(M) \leqslant (1-\alpha)\omega_n/\delta^{n/2}$, where ω_n is the volume of the n-dimensional unit sphere. An immediate consequence of this inequality is that $1 \leqslant K(n,\alpha)\|H\|_\infty \mathrm{Vol}(M)^{1/n}$ by taking $f = 1$. This extrinsic Sobolev inequality is of crucial importance to obtain pinching results for the first eigenvalue of the Laplacian (see [5, 6]). Another important fact under these assumptions is that the diameter of the hypersurface is bounded from above in terms of the mean curvature. Namely, Topping [20] (for the Euclidean space) and Wu and Zheng [21] (for any ambient manifold) proved that if (2.1) and (2.2) hold, then there exists a constant $C(n,\alpha)$ depending only on n and α so that

$$\mathrm{diam}(M) \leqslant C(n,\alpha) \int_M |H|^{n-1}\, dv_g. \tag{2.3}$$

For the statements of our results, we introduce the following subset of $\mathcal{H}_V(\mathbb{M}^{n+1}(\delta),\alpha)$. For $q > n$ and $A > 0$, we denote by $\mathcal{H}_V(n,\delta,\alpha,q,A)$ the subset of all manifolds in $\mathcal{H}_V(\mathbb{M}^{n+1}(\delta),\alpha)$ satisfying $\max_M\{\|H\|_\infty \mathrm{Vol}(M)^{1/n}, \|B\|_q \mathrm{Vol}(M)^{1/n}\} \leqslant A$ if $\delta \geqslant 0$ and $\max_M\{\|H\|_\infty \mathrm{Vol}(M)^{1/n}, \|B\|_q \mathrm{Vol}(M)^{1/n}, \|H\|_\infty/\sqrt{\|H\|_\infty^2 + \delta}\} \leqslant A$ if $\delta < 0$. Note that the quantity $\|H\|_\infty^2 + \delta$ is positive, even if δ is negative, and, hence, its square root has a sense. For instance, we can see this fact from the upper bound for the first positive eigenvalue of the Laplacian $0 < \lambda_1(\Delta) \leqslant n(\|H\|_\infty^2 + \delta)$, obtained by Heintze [7]. Note also that this last condition is invariant by any dilatation of the metric of the ambient space.

We also introduce the following useful functions. Let $r(x) = d(p,x)$ be the distance function to a base point p (in the sequel, p will be the centre of mass of M). We denote by Z the position vector defined by $Z = s_\delta(r)\overline{\nabla} r$, where $\overline{\nabla}$ is the connection of $\mathbb{M}^{n+1}(\delta)$. Moreover, the functions c_δ and s_δ are defined by

$$c_\delta(t) = \begin{cases} \cos(\sqrt{\delta}t) & \text{if } \delta > 0, \\ 1 & \text{if } \delta = 0, \\ \cosh(\sqrt{|\delta|}t) & \text{if } \delta < 0 \end{cases} \quad \text{and} \quad s_\delta(t) = \begin{cases} \dfrac{1}{\sqrt{\delta}}\sin(\sqrt{\delta}t) & \text{if } \delta > 0, \\ t & \text{if } \delta = 0, \\ \dfrac{1}{\sqrt{|\delta|}}\sinh(\sqrt{|\delta|}t) & \text{if } \delta < 0. \end{cases}$$

A last notion useful in the sequel is the extrinsic radius. We recall that the extrinsic radius $R(M)$ of M is defined by

$$R(M) = \inf\{\rho > 0 \mid \exists x \in \mathbb{M}^{n+1}(\delta) \text{ such that } \phi(M) \subset B(x,r)\},$$

where ϕ is the immersion of M into $\mathbb{M}^{n+1}(\delta)$. By a slight abuse of notation, we denote it by $R(M)$, although this radius depends not only on M but also on the immersion ϕ. Since, in this note, the considered immersion will be fixed, this notation does not lead to any ambiguity. Finally, even if this is not optimal, we remark that $R(M) \leqslant \mathrm{diam}(M)$.

3. Almost umbilical hypersurfaces

Now, we have all the ingredients to state the main result of this note, which gives a new result about the closeness to spheres for almost umbilical hypersurfaces.

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Theorem 3.1. *Let $(M^n, g) \in \mathcal{H}_V(n, \delta, \alpha, q, A)$. There exists $\varepsilon_0 \in (0,1)$, depending on n, q and A, so that if $\varepsilon \leqslant \varepsilon_0$ and*

$$\|\tau\|_\infty \leqslant \varepsilon\|H\|_\infty \quad \text{and} \quad \|d^\nabla \tau\|_\infty \leqslant \frac{\varepsilon\|H\|_\infty^2}{2nC(n,\alpha)A^n},$$

then M is embedded and (M, g) is ε-quasi-isometric to the round sphere $S\Big(p, s_\delta^{-1}\Big(1/\sqrt{\|H\|_\infty^2 + \delta}\Big)\Big)$, where p is the centre of mass of M. In particular, M is diffeomorphic to the sphere $\mathbb{S}^n$.

Before giving the proof of this theorem, we prove the following lemma.

Lemma 3.2. *Let (M^n, g) be a hypersurface of $\mathbb{M}^{n+1}(\delta)$. For any tangent vector fields X, Y,*

$$d^\nabla \tau(X, Y) = Y(H)X - X(H)Y.$$

Proof. We compute the curvature $\overline{R}(X, Y)\nu$. We have

$$\begin{aligned}
\overline{R}(X, Y)\nu &= \overline{\nabla}_X\overline{\nabla}_Y\nu - \overline{\nabla}_Y\overline{\nabla}_X\nu - \overline{\nabla}_{[X,Y]}\nu \\
&= -\overline{\nabla}_X A(Y) + \overline{\nabla}_Y A(X) + A([X, Y]) \\
&= -\overline{\nabla}_X \tau(Y) + \overline{\nabla}_Y \tau(X) + \tau([X, Y]) - X(H)Y + Y(H)X \\
&\qquad + H\overline{\nabla}_Y X - H\overline{\nabla}_X Y + H[X, Y] \\
&= -d^\nabla \tau(X, Y) - X(H)Y + Y(H)X,
\end{aligned}$$

where we have used the fact that $\overline{\nabla}_Y X - \overline{\nabla}_X Y + [X, Y] = 0$ since $\overline{\nabla}$ is torsion-free. Since M lies in a space of constant curvature, $\overline{R}(X, Y)\nu = 0$, which concludes the proof of the lemma. □

Remark 3.3. We deduce immediately from this lemma that M has constant mean curvature if and only if τ is Codazzi.

Proof of Theorem 3.1. Let $\varepsilon > 0$. We set $\eta = \varepsilon\|H\|_\infty^2/2nC(n,\alpha)A^n$. From the previous lemma and the assumption that $\|d^\nabla\tau\|_\infty \leqslant \eta$, we deduce that $|X(H)| \leqslant \eta$ for any unitary vector X. Thus, we have $\|\nabla H\| \leqslant n\eta$. Now, let $p \in M$ be a point where the maximum of $|H|$ is achieved. Then, for any $x \in M$, by the mean value inequality,

$$|H(p) - H(x)| \leqslant n\eta d(p, x) \leqslant n\eta \operatorname{diam}(M).$$

Since by assumption $M \in \mathcal{H}_V(n, \delta, \alpha, A)$, the diameter is bounded in terms of the mean curvature by (2.3). Namely,

$$\operatorname{diam}(M) \leqslant C(n,\alpha) \int_M |H|^{n-1}\, dv_g.$$

Hence,

$$|H(p) - H(x)| \leqslant n\eta C(n,\alpha)\|H\|_\infty^{n-1}\operatorname{Vol}(M)$$

and

$$|H^2 - \|H\|_\infty^2| \leqslant 2n\eta C(n,\alpha)\|H\|_\infty^n \operatorname{Vol}(M).$$

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Since $\eta = (\varepsilon\|H\|_\infty^2)/(2nC(n,\alpha)A^n)$ and by assumption $\|H\|_\infty^n \mathrm{Vol}(M) \leqslant A^n$,

$$|H^2 - \|H\|_\infty^2| \leqslant \varepsilon \|H\|_\infty^2.$$

This, together with the other assumption that $\|\tau\|_\infty \leqslant \varepsilon\|H\|_\infty$, leads to the conclusion that if $\varepsilon \leqslant \varepsilon_0$, where ε_0 is a constant depending on n, q and A given by [6, Theorem 1.3], then M is ε-quasi-isometric to the the sphere $S\Big(p, s_\delta^{-1}\Big(1/\sqrt{\|H\|_\infty^2+\delta}\,\Big)\Big)$, where p is the centre of mass of M. In particular, M is diffeomorphic to $\mathbb{S}^n$. Moreover, in the proof of [6, Theorem 1.3], the diffeomorphism is explicitly given. Namely, this diffeomorphism is the map

$$\begin{aligned} F : M &\longrightarrow S(p,\rho) \\ x &\longmapsto \exp_p\Big(\rho\frac{X}{|X|}\Big), \end{aligned}$$

where $\rho = s_\delta^{-1}\Big(1/\sqrt{\|H\|_\infty^2+\delta}\,\Big)$ is the radius of the sphere, ϕ is the immersion of M into $\mathbb{M}^{n+1}(\delta)$ and $X = \exp_p^{-1}(\phi(x))$ is the position vector. Since F is of the form $F = G\circ\phi$ and is a diffeomorphism, ϕ is necessarily injective. Thus, the immersion ϕ is an embedding. This concludes the proof. □

Remark 3.4. The first condition $\|\tau\|_\infty \leqslant \varepsilon\|H\|_\infty$ is invariant by homothety. But, the second condition $\|d^\nabla\tau\|_\infty \leqslant \varepsilon\|H\|_\infty^2/(2nC(n,\alpha)A^n)$ is not invariant by homothety, because of the square on $\|H\|_\infty$. This square is required, since the diameter of M appears by the use of the mean value inequality.

4. A new characterisation of geodesic spheres

From Theorem 3.1, we can obtain a new characterisation of geodesic spheres in real space forms. We are motivated by the well-known Alexandrov theorem and the Yau conjecture [22]. Indeed, the Alexandrov theorem [2] states that a compact hypersurface of constant mean curvature embedded into the Euclidean space, the hyperbolic space or the half-sphere is a geodesic sphere. Many generalisations of this result have been proved. For instance, Ros proved that the same holds for higher order mean curvatures [12, 13]. In particular, in the Euclidean space, the second mean curvature $H_2 = \sigma_2(B)$ defined as the elementary symmetric homogeneous polynomial of the second fundamental form is (up to a multiplicative constant) the scalar curvature '*scal*'. Precisely, we have $scal = n(n-1)H_2$. More generally, in $\mathbb{M}^{n+1}(\delta)$, we have $scal = n(n-1)(H_2+\delta)$.

In the famous *Problem Section* of [22], Yau conjectured that the embedding is not necessary for the Alexandrov theorem for the scalar curvature. This conjecture is still open, although several partial answers have been given. We can cite, for instance, the cases where the hypersurface is convex [4], stable [1], of cohomogeneity two [11], locally conformally flat [3] or with pinched second fundamental form [9]. In [14, 15], we proved it with the assumption that the mean curvature is almost constant. In this section, we give another partial answer to this conjecture with a different additional assumption. Namely, we will show that hypersurfaces with constant scalar curvature and almost Codazzi umbilicity tensor are geodesic spheres. This characterisation is valid in the three ambient space forms (Euclidean space, hyperbolic space and half-sphere).

First, we prove a technical lemma.

Lemma 4.1. *Let* $(M^n, g) \in \mathcal{H}_V(n, \delta, \alpha, q, A)$ *and* s *be a positive constant. Let* $\varepsilon > 0$ *and assume that*

$$|H - \|H\|_\infty| \leqslant \varepsilon \|H\|_\infty \quad \text{and} \quad \|scal - s\|_\infty \leqslant \varepsilon \|H\|_\infty^2.$$

Then

$$\|\tau\|_\infty \leqslant D \|H\|_\infty \varepsilon,$$

where $D > 1$ *is an explicit constant depending on* $n, \|H\|_\infty, \delta$ *and* A.

Remark 4.2.

(1) The constant D does not depend on s. Moreover, we will see in the proof that s is then close to $n(n-1)(\|H\|_\infty^2 + \delta)$.

(2) If $\delta \geqslant 0$, then D does not depend on $\|H\|_\infty$.

Proof. The proof of this lemma comes directly from the Hsiung–Minkowski formulas. We recall that the Hsiung–Minkowksi formulas are integral formulas involving two consecutive higher order mean curvatures. In particular, we have the first two:

$$\int_M (H\langle Z, \nu\rangle + c_\delta(r))\, dv_g = 0, \tag{4.1}$$

$$\int_M (H_2\langle Z, \nu\rangle + c_\delta(r)H)\, dv_g = 0. \tag{4.2}$$

Since we assume that $|scal - s| < \varepsilon$ and $scal = n(n-1)(H_2 + \delta)$,

$$\left| H_2 - \left(\frac{s}{n(n-1)} - \delta \right) \right| < \frac{1}{n(n-1)} \varepsilon \|H\|_\infty^2. \tag{4.3}$$

For more convenience, we will denote $h_2 = s/(n(n-1)) - \delta$ and $\|H\|_\infty = h$. Then, from (4.2),

$$\begin{aligned}
0 &= \int_M (H_2\langle Z, \nu\rangle + c_\delta(r)H)\, dv_g \\
&= \int_M (h_2\langle Z, \nu\rangle + c_\delta(r)H)\, dv_g + \int_M (H_2 - h_2)\langle Z, \nu\rangle\, dv_g \\
&= \frac{h_2}{h} \int_M h\langle Z, \nu\rangle\, dv_g + \int_M c_\delta(r)H\, dv_g + \int_M (H_2 - h_2)\langle Z, \nu\rangle\, dv_g \\
&= \frac{h_2}{h} \int_M H\langle Z, \nu\rangle\, dv_g + \frac{h_2}{h} \int_M (h - H)\langle Z, \nu\rangle\, dv_g + \int_M c_\delta(r)h\, dv_g \\
&\quad + \int_M c_\delta(r)(H - h)\, dv_g + \int_M (H_2 - h_2)\langle Z, \nu\rangle\, dv_g.
\end{aligned}$$

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Now, we use the other Hsiung–Minkowski formula (4.1) to get

$$\begin{aligned}
0 &= -\frac{h_2}{h}\int_M c_\delta(r)\,dv_g + \frac{h_2}{h}\int_M (h-H)\langle Z,\nu\rangle\,dv_g \\
&\quad + \int_M c_\delta(r)h\,dv_g + \int_M c_\delta(r)(H-h)\,dv_g + \int_M (H_2-h_2)\langle Z,\nu\rangle\,dv_g \\
&= \Big(h-\frac{h_2}{h}\Big)\int_M c_\delta(r)\,dv_g + \frac{h_2}{h}\int_M (h-H)\langle Z,\nu\rangle\,dv_g + \int_M c_\delta(r)(H-h)\,dv_g \\
&\quad + \int_M (H_2-h_2)\langle Z,\nu\rangle\,dv_g.
\end{aligned}$$

Then, since $|Z| = s_\delta(r)$ and using the assumption $|H-h| \leqslant h\varepsilon$ and (4.3),

$$\Big|h-\frac{h_2}{h}\Big|\int_M c_\delta(r)\,dv_g \leqslant h_2\varepsilon\int_M s_\delta(r)\,dv_g + \varepsilon h\int_M c_\delta(r)\,dv_g + \frac{\varepsilon h^2}{n(n-1)}\int_M s_\delta(r)\,dv_g.$$

Using the fact that $|H_2| \leqslant H^2$, we deduce from the assumptions on h and h_2 that

$$\begin{aligned}
h_2 &\leqslant |H_2| + \frac{1}{n(n-1)}\varepsilon h^2 \\
&\leqslant H^2 + \frac{1}{n(n-1)}\varepsilon h^2 \\
&\leqslant h^2 + 2\varepsilon h^2 + \frac{1}{n(n-1)}\varepsilon h^2 \leqslant h^2(1+3\varepsilon).
\end{aligned}$$

Hence,

$$\Big|h-\frac{h_2}{h}\Big|\int_M c_\delta(r)\,dv_g \leqslant \varepsilon h\int_M c_\delta(r)\,dv_g + \Big(1+3\varepsilon+\frac{1}{n(n-1)}\Big)\varepsilon h^2\int_M s_\delta(r)\,dv_g.$$

Therefore, using the fact that $\varepsilon < 1$,

$$\begin{aligned}
|h^2-h_2| &\leqslant \bigg[1 + h\Big(1+3\varepsilon+\frac{1}{n(n-1)}\Big)\frac{\int_M s_\delta(r)\,dv_g}{\int_M c_\delta(r)\,dv_g}\bigg]\varepsilon h^2 \\
&\leqslant \bigg[1 + 5h\frac{\int_M s_\delta(r)\,dv_g}{\int_M c_\delta(r)\,dv_g}\bigg]\varepsilon h^2.
\end{aligned}$$

From now on, we will discuss the three cases $\delta = 0$, $\delta > 0$ and $\delta < 0$.

First case: $\delta = 0$. In this case, we have $s_\delta(r) = r$ and $c_\delta(r) = 1$, so

$$\frac{\int_M s_\delta(r)\,dv_g}{\int_M c_\delta(r)\,dv_g} \leqslant R.$$

Since R is the extrinsic radius, we have the obvious relation with the interior diameter $R \leqslant \mathrm{diam}(M)$ and, therefore, by the result of Topping,

$$\frac{\int_M s_\delta(r)\,dv_g}{\int_M c_\delta(r)\,dv_g} \leqslant R \leqslant \mathrm{diam}(M) \leqslant C(n,\alpha)\int_M |H|^{n-1}\,dv_g \leqslant C(n,\alpha)\frac{A^n}{h}.$$

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Second case: $\delta > 0$. In this case, $c_\delta(r) = \cos(\sqrt{\delta} r)$ and $s_\delta(r) = (1/\sqrt{\delta}) \sin(\sqrt{\delta} r)$ and

$$\frac{\int_M s_\delta(r)\, dv_g}{\int_M c_\delta(r)\, dv_g} \leqslant t_\delta(R) \leqslant R,$$

since we have assumed that M is contained in a sphere a radius $\pi/(4\sqrt{\delta})$. Thus, as in the case $\delta = 0$,

$$\frac{\int_M s_\delta(r)\, dv_g}{\int_M c_\delta(r)\, dv_g} \leqslant C(n, \alpha) \frac{A^n}{h}.$$

Third case: $\delta < 0$. In this case, $c_\delta(r) = \cosh(\sqrt{|\delta|})$ and $s_\delta(r) = (1/\sqrt{|\delta|}) \sinh(\sqrt{|\delta|})$. Hence,

$$\frac{\int_M s_\delta(r)\, dv_g}{\int_M c_\delta(r)\, dv_g} \leqslant \frac{1}{\sqrt{|\delta|}}.$$

Thus, in all three cases,

$$|h^2 - h_2| \leqslant E\varepsilon h^2,$$

where

$$E = \begin{cases} 1 + 5C(n, \alpha) A^n & \text{if } \delta \geqslant 0, \\ 1 + \dfrac{5h}{\sqrt{|\delta|}} & \text{if } \delta < 0 \end{cases}$$

is a constant depending on n, α, δ, h and A.

Now, we recall the Gauss formula. For $X, Y, Z, W \in \Gamma(TM)$,

$$R(X, Y, Z, W) = \overline{R}(X, Y, Z, W) + \langle SX, Z\rangle\langle SY, W\rangle - \langle SY, Z\rangle\langle SX, W\rangle,$$

where R and $\overline{R}$ are respectively the curvature tensors of M and $\mathbb{M}^{n+1}(\delta)$. By taking the trace in X and Z and for $W = Y$,

$$Ric(Y) = \overline{Ric}(Y) - \overline{R}(\nu, Y, \nu, Y) + nH\langle SY, Y\rangle - \langle S^2 Y, Y\rangle,$$

where S is the shape operator. Since the ambient space is of constant sectional curvature δ, by taking the trace a second time,

$$scal = n(n-1)\delta + n^2 H^2 - |S|^2$$

or, equivalently,

$$scal = n(n-1)(H^2 + \delta) - |\tau|^2.$$

Hence,

$$\begin{aligned} \|\tau\|^2 &= n(n-1)(H^2 - H_2) \\ &\leqslant n(n-1)(|H^2 - h^2| + |h^2 - h_2|) \\ &\leqslant n(n-1)(2h^2\varepsilon + Eh^2\varepsilon) \\ &\leqslant Dh^2\varepsilon, \end{aligned}$$

where we have set $D = n(n-1)(2 + E)$. This concludes the proof of the lemma. □

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Now, from this lemma, we can prove the following result.

THEOREM 4.3. *Let* $(M^n, g) \in \mathcal{H}_V(n, \delta, \alpha, q, A)$. *There exists* $D > 1$ *depending on* n, α, δ, $\|H\|_\infty$ *and* A *and there exists* $\eta_0 \in (0, 1)$ *depending on* n, q *and* A *and so that if* $\eta \leqslant \eta_0$ *and*

$$\|scal - s\|_\infty \leqslant \frac{\eta\|H\|_\infty^2}{D} \quad \textit{and} \quad \|d^\nabla \tau\|_\infty \leqslant \frac{\eta\|H\|_\infty^2}{2nC(n, \alpha)AD},$$

then M *is embedded and* (M, g) *is* η*-quasi-isometric to the round sphere* $S\Big(p, s_\delta^{-1}\Big(\sqrt{s/n(n-1)}\Big)\Big)$, *where* p *is the centre of mass of* M. *In particular,* M *is diffeomorphic to the sphere* $\mathbb{S}^n$.

PROOF. First, the constant D of the theorem is the one computed in Lemma 4.1. The second assumption that $\|d^\nabla \tau\|_\infty \leqslant \eta\|H\|_\infty^2/(2nC(n, \alpha)AD)$ implies by the computations of the proof of Theorem 3.1 that $|H - \|H\|_\infty| \leqslant \eta\|H\|_\infty/D$. Thus, we can apply Lemma 4.1 with $\varepsilon = \eta/D$ to get

$$\|\tau\|_\infty \leqslant D\|H\|_\infty \varepsilon = \eta\|H\|_\infty. \tag{4.4}$$

Moreover, since $D > 1$,

$$\|d^\nabla \tau\|_\infty \leqslant \frac{\eta\|H\|_\infty^2}{2nC(n, \alpha)AD} \leqslant \frac{\eta\|H\|_\infty^2}{2nC(n, \alpha)A}. \tag{4.5}$$

Thus, (4.4) and (4.5) are exactly the hypotheses of Theorem 3.1. Then, we conclude that (M, g) is η-quasi-isometric to the round sphere $S\Big(p, s_\delta^{-1}\Big(\sqrt{s/n(n-1)}\Big)\Big)$, where p is the centre of mass of M. In particular, M is diffeomorphic to $\mathbb{S}^n$. In addition, Theorem 3.1 ensures that M is embedded into $\mathbb{M}^{n+1}(\delta)$. □

We deduce easily the following corollary, which is a new characterisation of geodesic spheres and gives a new partial answer to the Yau conjecture.

COROLLARY 4.4. *Let* $(M^n, g) \in \mathcal{H}_V(n, \delta, \alpha, q, A)$ *and* s *be a positive constant. There exists* $D > 1$ *depending on* n, α, δ, $\|H\|_\infty$ *and* A *and there exists* $\eta_0 \in (0, 1)$ *depending on* n, q *and* A *so that if* $\eta \leqslant \eta_0$ *and*

$$scal = s \quad \textit{and} \quad \|d^\nabla \tau\|_\infty \leqslant \frac{\eta\|H\|_\infty^2}{2nC(n, \alpha)A},$$

then M *is a geodesic sphere of radius* $s_\delta^{-1}\Big(\sqrt{s/n(n-1)}\Big)$.

PROOF. This corollary is a direct consequence of Theorem 4.3 together with the Alexandrov theorem for the scalar curvature proved by Ros [13]. Indeed, from the assumptions, we can apply Theorem 4.3 and get that M is embedded. Since M is assumed to have constant scalar curvature, by [13], M is a geodesic sphere. Moreover, the radius of this sphere is determined by the scalar curvature and is $s_\delta^{-1}\Big(\sqrt{s/n(n-1)}\Big)$. □

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