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# SOME SUPERCUSPIDAL REPRESENTATIONS OF $Sp_4(\mathbf{k})$

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**1.** The purpose of this paper is to produce explicit realizations of supercuspidal representations of $Sp_4(\mathbf{k})$ where $\mathbf{k}$ is a $p$-adic field with odd residual characteristic. These representations will be constructed using the Weil representation of $Sp_4(\mathbf{k})$ associated with a certain 4-dimensional compact orthogonal group $O_Q$ over $\mathbf{k}$. The main problem addressed in this paper is the analysis of this representation; we need to find how the supercuspidal summands decompose into irreducible pieces.

The problem of decomposing Weil representations has been studied quite a bit already. The Weil representations of $SL_2(\mathbf{k})$ associated to 2-dimensional orthogonal groups were used by Casselman **[4]** and Shalika **[9]** to produce all supercuspidals of $SL_2(\mathbf{k})$. The explicit formulas for these representations were used by Sally and Shalika ( **[10]** ) to compute the characters and finally to write down a Plancherel formula for that group.

The ultimate object of this paper is to do the same for the group $Sp_4(\mathbf{k})$. To do this, we must consider the Weil representation of $Sp_4(\mathbf{k})$ associated to $O_Q$. (The 2-dimensional compact orthogonal groups produce Weil representations with only a single supercuspidal component each **[1]**.) The supercuspidals produced in the construction using $O_Q$ correspond naturally to characters of Cartan subgroups contained in an imbedding of $SL_2(\mathbf{k}) \times SL_2(\mathbf{k})$ in $Sp_4(\mathbf{k})$. It seems certain that there are classes of supercuspidals not included in this list, namely those that would correspond to characters of Cartan subgroups contained in $SL_2(E)$ imbedded in $Sp_4(\mathbf{k})$ where $E$ is a quadratic extension of $\mathbf{k}$.

The decomposition methods of **[9]** and **[4]** are based on computation of certain Gauss sums which arise as matrix coefficients in those Weil representations. This method appears to be hopeless in the case of $Sp_4(\mathbf{k})$. In this paper, we take a rather indirect route to the same destination; we use the decomposition of the Weil representation of $Sp_8(\mathbf{k})$ associated to $O_Q$ to provide information about $Sp_4(\mathbf{k})$. This method gets around the computations entirely and also works for the decomposition problem for $SL_2(\mathbf{k})$ associated to the compact 2-dimensional orthogonal groups; here one would use decomposition information about the corresponding Weil

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representations for $Sp_4(\mathbf{k})$. The main theorem of this paper can be found at the end of the second section following the introduction of notation and the statement of some known results. The third section uses the method mentioned above to show that the supercuspidal summands of the given Weil representation which correspond to representations of $O_Q$ are isotypic. The fourth and last section proves that the summands are irreducible. I would like to thank R. Howe for encouraging me to write up these results in the first place. I am particularly indebted to him for suggesting to me what the multiplicity of the isotypic summands should be.

**2.** In this section we present the explicit formulas for the Weil representation. These will be more or less the same as used in [**1**] which in turn were adapted from [**8**]. Let $\mathbf{k}$ be a $p$-adic field whose residual characteristic is odd. Let $\pi$ and $\epsilon$ be respectively a prime element and a non-square unit in the ring of integers of $\mathbf{k}$.

Now set

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\epsilon & 0 & 0 \\ 0 & 0 & -\pi & 0 \\ 0 & 0 & 0 & \epsilon\pi \end{bmatrix}$$

so that $Q$ is the matrix of the 4-dimensional anisotropic quadratic form over $k$. Let $O_Q$ be the corresponding orthogonal group. Let $W$ be an irreducible finite dimensional representation of $O_Q$ with space $V$. Then for $n = 1, 2, 3, \dots$, we have a smooth representation $T_n(W)$ of $Sp_{2n}(\mathbf{k})$. This representation is realized in the space $H_n(W)$ consisting of smooth $V$-valued functions on $M_{n,4}(\mathbf{k})$ satisfying the identity

$$f(X\alpha) = f(X)W(\alpha) \quad \text{for } \alpha \in O_Q.$$

We now give formulas for $T = T_n(W)$ on generators of $Sp_{2n}(\mathbf{k})$. Here the constant $\zeta$ is complex of modulus 1, and $dY$ is an appropriately normalized measure on the additive group of $M_{n,4}(\mathbf{k})$.

(1) For $A \in GL_n(\mathbf{k})$ $\quad T\begin{bmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{bmatrix} f(X) = |\det A|^2 f({}^tAX)$

(2) For $B$ $n \times n$ symmetric $T\begin{bmatrix} I & B \\ 0 & I \end{bmatrix} f(X) = \Phi(\mathrm{tr}\, BXQ{}^tX) f(X)$

(3) $T\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} f(X) = \zeta \int_{M_{n4}} f(Y)\Phi(\mathrm{tr}\, YQ{}^tX)dY.$

From [**1**] we know that all $H_4(W)$ are non-zero irreducible distinct

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representations of $Sp_8(\mathbf{k})$. We also have:

(a) Let $P$ be a parabolic subgroup of $Sp_8(\mathbf{k})$ whose reductive part $M$ consists of elements of the form

$$\begin{bmatrix} R & 0 & 0 & 0 \\ 0 & A & 0 & B \\ 0 & 0 & S & 0 \\ 0 & C & 0 & D \end{bmatrix}$$

where $R$ and $S$ are $2 \times 2$ diagonal matrices such that $RS = 1$ and

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in Sp_4(\mathbf{k}).$$

Then there exists an extension of $T_2(W)$ to $P$ such that $T_4(W)$ is a subrepresentation of $\mathrm{Ind}_P^G\, T_2(W)$ where $G = Sp_8(\mathbf{k})$. For $f \in H_4(W)$, $g \in G$, and $X \in M_{2,4}(\mathbf{k})$ the embedding $A$ is given by

$$[A(f)(g)](X) = [T_4(g)f]\begin{bmatrix} 0 \\ X \end{bmatrix}.$$

(b) For any irreducible $W$, $T_2(W)$ is supercuspidal if and only if both $H_1(W) = \{0\}$ and $H_2(W) \neq \{0\}$. This amounts to saying that a summand $H_2(W)$ is supercuspidal if and only if it is non-zero and any function $f$ therein is supported only on elements of rank 2.

The behaviour of representations $T_n(W)$ is illustrated by the orbits of the action of $O_Q$ on $M_{n,4}(\mathbf{k})$. Two elements $X$ and $Y$ are in the same orbit if and only if $XQ^tX = YQ^tY$. This means that the spectrum of a function $f$ in $H_n(W)$ under the operators in formula (2) above depends only on which orbits support $f$. An eigencharacter of this unipotent group is given by a choice of a symmetric $n \times n$ matrix, namely some $XQ^tX$. Operators of formula (1) permute these eigencharacters but not transitively. Two eigencharacters are in the same orbit under this action if their corresponding symmetric matrices are equivalent. The irreducibility arguments for representations of $SL_2(\mathbf{k})$ in **[9]** and **[4]** use this structure to produce two inequivalent irreducible representations of the parabolic subgroup, each corresponding to a different class of 1-dimensional symmetric matrices (i.e., an element of $\mathbf{k}^x/(\mathbf{k}^x)^2$). Then an explicit computation shows that the Weyl element fuses these two into an irreducible representation of $SL_2(\mathbf{k})$ itself. In **[1]** the cases where $n \geqq 4$ are considered. Here the operators of formulas (1) and (2) act irreducibly already and no calculation involving the Weyl elements was needed. There was only one supercuspidal summand in these cases (when $n = 4$) so the interesting case is when $n = 2$.

We now state the main theorem of the paper.

THEOREM 2.1. *Let an irreducible representation $W$ of $O_Q$ be chosen so that $T_2(W)$ is supercuspidal. Then $T_2(W)$ is irreducible; if $T_2(W)$ and $T_2(W')$ are two such supercuspidal representations with $W$ and $W'$ not equivalent then $T_2(W)$ and $T_2(W')$ are also not equivalent.*

This theorem suggests that this particular construction produces all or most of the supercuspidal representations which correspond to tori which are imbeddable in $SL_2 \times SL_2$. One might look in Weil representations constructed with non-compact orthogonal groups for other types.

**3.** In this section we prove that $T_2(W)$ is isotypic. We apply the following fact taken from the main theorem of [**3**]. Let $\tau_1$ and $\tau_2$ be supercuspidal representations of $M$. Then the representations $\rho_i = \mathrm{Ind}_P^G \tau_i$ of $G = Sp_8(\mathbf{k})$ have a common subrepresentation if and only if $\tau_1$ and $\tau_2$ are conjugate by some element $y = G$. It is reasonably easy to see that in the present application we may take $y$ to be of the form

$$\begin{bmatrix} R & 0 & S & 0 \\ 0 & I & 0 & 0 \\ T & 0 & U & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$

where $\begin{bmatrix} R & S \\ T & U \end{bmatrix}$ is a Weyl group element in $Sp_4(\mathbf{k})$. Let $H$ be the subgroup of $M$ isomorphic to $Sp_4(\mathbf{k})$. Then we can say more specifically that if the representations $\rho_i$ share a summand, then $\tau_1$ and $\tau_2$ restricted to $H$ are equivalent. Let $V$ be any $H$-subspace of $H_2(W)$. Let $\mathrm{Ind}_H^G V$ be the corresponding induced $G$-subspace of the representation space $H_2(W)^G$ of $\mathrm{Ind}_H^G T_2(W)$. Let $h \in V$ and assume $h(X_0) = v \neq 0 (v \in C^j)$. Then one can construct by elementary means an arbitrarily small neighborhood $U$ of $X_0$ and an element $h_U \in V$ such that $h_U$ is supported in $UL$ and $h_U = v$ on $U$.

Let $p_V: H_2(W)^G \to \mathrm{Ind}_H^G V$ be given by

$$(p_v(f))(g) = (\pi_v \cdot f)(g)$$

where $\pi: H_2(W) \to V$ is the natural orthogonal projection. It is easily shown that $p_v$ is a $G$-homomorphism. To prove the first part of the main theorem, it now suffices to show that

$$p_v(A(H_4)(W)) \neq \{0\}.$$

This would mean that all $H$-subspaces of $H_2(W)$ induce to contain a common irreducible summand, namely $H_4(W)$, and would therefore be equivalent by the above remarks. To show this, pick $h' \in H_4(W)$ so that

$$h'\begin{bmatrix} 0 \\ x_0 \end{bmatrix} = v = h(X_0).$$

Thus $[A(h')(I)](X_0) = v$. We can certainly choose a neighborhood $U$ of $X_0$ sufficiently small so that

$$\langle A((h')I), h_U \rangle \neq 0.$$

Therefore $p_v(A(h'))$ is non-zero and the result follows.

**4.** To determine the decomposition of the isotypic space $H_2(W)$, we need to look more closely at the action of the orthogonal group $O_Q$. We will make use of some properties of the representations of $O_Q$. These will be based on results in **[2]** which were in turn derived from [7] and [**5**]. We use $D$, the quaternions over $\mathbf{k}$ to construct $O_Q$. Let $D$ be considered as a 4-dimensional quadratic space over $\mathbf{k}$ with the quadratic form $Q$ being just the reduced norm $\nu$. We let $D^x \times D^x$ act on $D$ by $(r, s):z \to r^{-1}zs$. When $\nu(r) = \nu(s)$ this transformation is orthogonal. Let $SO_Q$ be the quotient

$$\{(r, s):\nu(r) = \nu(s)\}/\{(x, x):x \in \mathbf{k}^x \subseteq D^x\}.$$

Let $D$ consist of elements $a + bi + cj + dk$ where $a$, $b$, $c$, and $d$ are in $k$, $i^2 = \epsilon$, $j^2 = \pi$ and $ij = -ji = k$. We generate the rest of $L$ by including the element $\sigma$ where

$$(a + bi + cj + dk)\sigma = a - bi - cj - dk.$$

Representations of $SO_Q$ are thus subrepresentations of tensor products $I \otimes J$ of $D^x \times D^x$. In **[2]** we develop a system of labels for representations of $SO_Q$ of the form $(R, S)$ where $R$ and $S$ are representations of various subgroups of $D^x$. It is known that if a representation $W$ of $O_Q$ of degree $> 1$ contains $(R, S)$ in its restriction to $SO_Q$, then $H_1(W)$ is non-zero if and only if $R = S$. Since $\sigma$ sends $(R, S)$ to $(S, R)$, we may conclude that for such $W$ that $H_2(W)$ is supercuspidal if and only if it is irreducibly induced from $SO_Q$. Now let $U$ be a representation of $SO_Q$ with representation space $V_0$. Let

$$H_2^0(U) = \{f:M_{2,4}(\mathbf{k}) \to V_0| \text{ for any } \gamma \in SO_Q,$$
$$f(X\gamma) = f(X)U(\gamma)\}.$$

LEMMA 4.1. *If $U$ induces irreducibly to a representation $W$ of $O_Q$ then $H_2^0(U)$ and $H_2(W)$ are isomorphic as $Sp_4$-spaces.*

*Proof.* The representation space $Y$ of $W$ may be written as $Y_0 \oplus V_0W(\sigma)$. Let $p:Y \to Y_0$ be the natural projection. The map $f \to p \cdot f$ is clearly an $Sp_4$-homomorphism from $H_2(W)$ to $H_2^0(U)$. The action of $W(\sigma)$ forces the kernel to be zero and the map $f_0 \to f$ given by

$$f(x) = (f_0(X), f_0(X)W(\sigma))$$

is the inverse.

We now finish the proof of the main theorem. If $\deg(W) = 1$ we are done since the eigenspaces of the subgroup

$$N = \left\{ \begin{bmatrix} I & B \\ 0 & I \end{bmatrix} : B \text{ symmetric} \right\} \subseteq Sp_4(\mathbf{k})$$

would be one-dimensional. This is because functions transforming according to $W$ are determined over an entire orbit by their scalar value at one point in that orbit. Since $T_2(W)$ is isotypic this means it has multiplicity one. Let $X$ be of rank two in $M_{2,4}(\mathbf{k})$. Let $E_1$ be the span of $w$ and $z$ in $\mathbf{k}^4$ and $E_2$ the orthogonal complement of $E_1$. Let $\mathscr{R}(X)$ and $\mathscr{S}(X)$ be respectively the maximal abelian subgroups fixing $E_2$ and $E_1$ pointwise. Since $E_1$ is thus $\mathscr{R}(X)$-invariant, we have an imbedding $A:\mathscr{R}(X) \to GL_2(\mathbf{k})$ having the property that for $r \in \mathscr{R}(X)$ we have $Xr = A(r)X$. Now let $\mathscr{H}(X) \subseteq Sp_4(\mathbf{k})$ consist of elements of the form

$$\mathscr{B}(r) = \begin{bmatrix} {}^tA(r) & 0 \\ 0 & A(r)^{-1} \end{bmatrix}.$$

Let $[X] \subseteq M_{2,4}(\mathbf{k})$ be the orbit of $X$ under $SO_Q$. For $f \in H_2(W)$ let $\rho f = f| [X]$ so that $\rho$ is an $\mathscr{H}(X)$-map from $H_2^0$ to $C_c^\infty([X], Y_0)$. Thus if $V_1$ and $V_2$ are isomorphic $Sp_4(\mathbf{k})$-spaces of $H_2(U)$, then $\rho V_1$ and $\rho V_2$ are $\mathscr{H}(X)$-isomorphic spaces of $C_c^\infty([X], Y_0)$. Let $f \in H_2(U)$ and let $f(X) = v$ in $Y_0$. Then necessarily $v$ is fixed by $U(S(X))$. Suppose, also, that $v$ is an eigenvector of $R(X)$ with eigencharacter $\psi$. Then for $r \in R(X)$ we have

$$[T(\mathscr{B}(r)f)](X\gamma) = \psi(r)f(X\gamma).$$

Thus $\rho f$ is in the $\psi$-space of $\mathscr{H}(X)$. The argument now comes down to two facts. First, if $V$ is an irreducible $Sp_4(\mathbf{k})$-space of $H_2^0(U)$, the decomposition of $\rho V$ into $\mathscr{H}(X)$-eigenspaces depends only on the isomorphism class of $V$. This can be seen by looking at the basic operators of the Weil representation itself. Second, the multiplicity of a $\psi$-eigenspace can never exceed the multiplicity of $\psi \otimes 1$ in $U$ restricted to $R(X) \times S(X)$. A key fact here is that this restriction applies for any $\psi \otimes 1$ occurring in the restriction of $U$. Given any $W$ satisfying the conditions of 2.1, we may take $X$ to be $\begin{bmatrix} 1 \\ z \end{bmatrix}$ for some $z$ in $D^x$. Then

$$R(X) = \{ (\alpha, \alpha) \in SO_Q : \alpha \in \mathbf{k}(z)^x \}$$

$$S(X) = \{ (\alpha^{-1}, \alpha) \in SO_Q : \alpha \in \mathbf{k}(z)^x \}.$$

Let $U = (R, S)$ as described above. We know from work of Howe and Corwin (see **[5]**, **[6]**, and **[7]** ) that representations of various subgroups of

$D^x$ can be classified corresponding to quadratic extensions of **k**. Let $z$ generate a quadratic extension of **k** different from those corresponding to $R$ or $S$. Then the character of $R$ when restricted to the torus corresponding to $z$ is zero outside the kernel of $R$. The same is true for $S$. This implies that there is a choice of $\psi$ such that the multiplicity in question is exactly one. Since all summands of $H_2(W)$ must share equally the various $\mathscr{H}(X)$-eigenspaces we see that there must be exactly one such summand. Finally, a survey of these same formulas gives us that for $W_1$ and $W_2$ distinct, the characters occurring in the representations of $\mathscr{H}(X)$ in $C_c^\infty([X], Y_0)$ are different. This means that $H_2(W_1)$ and $H_2(W_2)$ since $C_c^\infty([X], Y_0)$ is an entire eigenspace of $N$ (defined above) the character of which depends only on $X$ and not on $W$.

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