NIL RINGS SATISFYING CERTAIN CHAIN CONDITIONS: AN ADDENDUM

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We prove here a result closely related both in spirit and technique to those obtained in our paper **(1)**. We also take this opportunity to present a counterexample, due to E. Sąsiada, to the conjecture we made there. Finally, we extend a result of Wedderburn **(3)** from algebras over fields to algebras over commutative rings; the result follows easily from the paper by Posner **(2)** although it does not appear there explicitly.

Let R be an algebra over a commutative integral domain Ω. We say that R satisfies a polynomial identity over Ω if there exists an element $p[x_1, \ldots, x_n]$ in the n non-commuting variables such that $p(a_1, \ldots, a_n) = 0$ for all $a_1, \ldots, a_n \in R$.

It is both well known and easy to show that if R satisfies a polynomial identity over Ω it satisfies a multilinear such identity. We can write this identity as

$$p(x_1, \ldots, x_n) = \sum_{\pi \in S_n} \alpha_\sigma \, x_{\sigma(1)} \ldots x_{\sigma(n)},$$

where $\alpha_\sigma \in \Omega$ and σ varies over the symmetric group, S_n, of degree n. In order to avoid counterexamples of a trivial kind we assume that some α_σ, hence without loss of generality α_1, is a unit in Ω.

We prove the following theorems.

THEOREM 1. *Let R be a nil ring satisfying a polynomial identity (of the type described above) over Ω. If R satisfies the ascending chain condition on left annihilators, then it must be nilpotent.*

Proof. By the ascending chain condition on left annihilators there is an integer k such that $xR^m = (0)$ implies that $xR^k = (0)$. Let

$$U = \{x \in R \mid xR^k = 0\}$$

and consider $\bar{R} = R/U$. If $\bar{x}\bar{R} = (0)$, then $xR \subset U$; hence $xR^{k+1} = (0)$, resulting in $xR^k = (0)$, that is $x \in U$ and so $\bar{x} = 0$. If $U = R$, then R is nilpotent; if $U \neq R$, since ΩU clearly is in U, $\bar{R}$ is a nil ring in which $\bar{x}\bar{R} \neq (0)$ for $\bar{x} \neq 0$ and which satisfies the same identity as does R. Since U is a left annihilator, by **(1**, Lemma 3**)** $\bar{R}$ satisfies the ascending chain condition on left annihilators. All in all we have shown (by passing to $\bar{R}$) that if R is not nilpotent

Received November 19, 1964. The work of the first-named author was supported by a grant from the Army Research Office (AROD) at the University of Chicago.

we may assume that $xR = (0)$ only if $x = 0$. We proceed to show that this situation is impossible.

Let $A = \{x \in R \mid Rx = (0)\}$; by (**1**, Lemma 2) $A \neq (0)$. If $x_1 = a \in A$ and $x_2, \ldots, x_n$ are arbitrary in R, then

$$0 = p(a, x_2, \ldots, x_n) = \sum \alpha_\sigma x_{\sigma(1)} \ldots x_{\sigma(n)}$$

gives us, since $Ra = (0)$, that

$$a \sum_{\sigma \in S_{n-1}} \alpha_\sigma x_{\sigma(2)} \ldots x_{\alpha(n)} = 0,$$

that is

$$A\Big(\sum_{\sigma \in S_{n-1}} \alpha_\sigma x_{\sigma(2)} \ldots x_{\sigma(n)} \Big) = (0).$$

Since $\Omega r(A) \subset r(A)$ (where $r(A) = \{x \in R \mid Ax = (0)\}$), $R' = R/r(A)$ satisfies the identity

$$\sum_{\sigma \in S_{n-1}} \alpha_\sigma x_{\sigma(2)} \ldots x_{\sigma(n)} = 0,$$

where α_1 is a unit in Ω, of degree $n - 1$. Since R' is nil and satisfies a polynomial identity of degree $n - 1$ and the ascending chain condition on left annihilators (**1**, Lemma 3), by induction we have that R' is nilpotent, say $(R')^m = (0)$. Therefore, $R^m \subset r(A)$, leading to $AR^m = (0)$. Since in R we know that $xR \neq (0)$ for $x \neq 0$, we conclude that $A = (0)$, a contradiction. The theorem is thus proved.

Although merely a special case of the theorem, because it has independent interest we cite the following corollary.

COROLLARY. *If R is a ring satisfying the ascending chain condition on left annihilators and if $x^n = 0$ for all $x \in R$, n a fixed integer, then R is nilpotent.*

Proof. R is an algebra over Z, the ring of integers, and satisfies the polynomial

$$\sum_{\sigma \in S_n} x_{\sigma(1)} \ldots x_{\sigma(n)}$$

over Z; thus by the theorem it is nilpotent.

We had conjectured that nil rings satisfying the ascending chain condition on left annihilators were nilpotent. This is false, as the following example of E. Sąsiada shows. Let R be the free ring over the integers in $x_0, x_1, \ldots, x_n, \ldots$ reduced modulo the relations $x_i x_j = 0$ for $i \geqslant j$. This is a non-nilpotent nil ring satisfying the ascending chain condition on left annihilators, as can be readily verified.

We conclude this brief note with an extension of a beautiful result due to Wedderburn.

THEOREM 2. *Let A be a commutative ring and suppose that R is an algebra over A. If $R = Ax_1 + \ldots + Ax_n$, where the x_i are all nilpotent, then R is nilpotent.*

Proof. By assumption, $R = Ax_1 + \ldots + Ax_n$; hence $x_i x_j = \sum \alpha_{ijk} x_k$, where $\alpha_{ijk} \in A$. Let $A_0 = Z[\alpha_{ijk}]$ be the ring obtained by adjoining all the α_{ijk} to the ring of integers Z. A_0 is a commutative Noetherian ring. The ring $R_0 = A_0 x_1 + \ldots + A_0 x_n$ is thus an algebra over A_0 finitely generated as an A_0-module. Since A_0 is Noetherian, R_0 is left Noetherian. Moreover, just as in the case of fields, R_0 satisfies the standard identity

$$\sum_{\sigma \in S_{n+1}} (-1)^{\sigma} y_{\sigma(1)} \ldots y_{\sigma(n+1)}.$$

By a result of Posner **(2)**, since every element in R_0 is a sum of nilpotent elements, R_0 is nilpotent. Thus for some integer T all words in $x_1, \ldots, x_n$ of length larger than or equal to T vanish. But then $R^T = (0)$ follows and the theorem is established.

References

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3. J. H. M. Wedderburn. *Note on algebras*, Ann. of Math., *38* (1937), 854–856.

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