Canad. Math. Bull. Vol. **47** (1), 2004 pp. 119–132

2-Primary Exponent Bounds for Lie Groups of Low Rank

Stephen D. Theriault

Abstract. Exponent information is proven about the Lie groups $SU(3)$, $SU(4)$, $Sp(2)$, and G_2 by showing some power of the H-space squaring map (on a suitably looped connected-cover) is null homotopic. The upper bounds obtained are 8, 32, 64, and 2^8 respectively. This null homotopy is best possible for $SU(3)$ given the number of loops, off by at most one power of 2 for $SU(4)$ and $Sp(2)$, and off by at most two powers of 2 for G_2.

1 Introduction

This paper computes upper bounds for the 2-primary exponents of the Lie groups $SU(3)$, $SU(4)$, $Sp(2)$, G_2, and $SO(3)$ through $SO(8)$. As we are concerned only with 2-primary information, assume all spaces and maps have been localized at 2.

Two types of exponents should be distinguised. The *homotopy exponent* of a space X, written $\exp(X)$, is 2^t if t is the minimal power of 2 which annihilates the 2-torsion in the homotopy groups of X. The *H-exponent* of an H-space Y, written $H\exp(Y)$, is 2^t if t is the minimal power of the H-space squaring map on Y which is null homotopic. Observe that if Y has H-exponent 2^t then its homotopy exponent is also bounded above by 2^t, but the reverse implication need not be true. In computing H-exponents, there is often interference from a few low dimensional integral homotopy classes. These misleadingly force the H-exponent to be infinite. To avoid this problem an appropriately connected cover is considered instead.

In what follows, a space X will often be looped beyond its connectivity. When this happens, let $\Omega_0^n X$ be the component of $\Omega^n X$ containing the basepoint. Equivalently, this is the n-th-loop space of the n-th-connected cover of X.

Theorem 1.1 *The following hold:*

(a) $H\exp\big(\Omega_0^6 SU(3)\big) = 8$,
(b) $H\exp\big(\Omega_0^8 SU(4)\big) \leq 32$,
(c) $H\exp\big(\Omega_0^8 Sp(2)\big) \leq 64$,
(d) $H\exp\big(\Omega_0^{12} G_2\big) \leq 2^8$.

The result for $SU(3)$ is best possible in the following sense. There are known elements of order 4 in the homotopy groups of $SU(3)$ but no known elements of order 8. It is conjectured that the homotopy exponent of $SU(3)$ is 4. On the other

Received by the editors October 5, 2001; revised May 2, 2003.
AMS subject classification: 55Q52.
Keywords: Lie group, exponent.

hand, Davis and Mahowald [DM2] show that 8 is a lower bound for the H-exponent of $\Omega_0^k SU(3)$ for $0 \leq k \leq 8$. So the result in Theorem 1.1 is best possible for the given number of loops. However, it may be the case that after looping $SU(3)$ more often the H-exponent drops to 4 to match the conjectured homotopy exponent. The potential discrepancy between the homotopy exponent and the H-exponent is, at this point, not understood. As for $SU(4)$, $Sp(2)$, and G_2, they have known elements in their homotopy groups of orders 16, 32 [MT] and 64 [DM1] respectively. So the results in Theorem 1.1 are off by at most one power of 2 for each of $SU(4)$ and $Sp(2)$, and by at most two powers of 2 for G_2.

As a corollary of Theorem 1.1 we are able to compute exponent bounds for the Lie groups $SO(3)$ through $SO(8)$. Recall that $\mathrm{Spin}(n)$ is the universal two-sheeted cover of $SO(n)$ if $n \geq 3$. As well, it follows from [C1, Section 5] that $H\exp(\Omega^2 S^3\langle 3\rangle) = 4$.

Corollary 1.2 *The following hold:*

(a) $\mathrm{Spin}(3) \simeq S^3$, *and so* $H\exp\big(\Omega^2 SO(3)\langle 3\rangle\big) = 4$,
(b) $\mathrm{Spin}(4) \simeq S^3 \times S^3$, *and so* $H\exp\big(\Omega^2 SO(4)\langle 3\rangle\big) = 4$,
(c) $\mathrm{Spin}(5) \simeq Sp(2)$, *and so* $H\exp\big(\Omega_0^8 SO(5)\big) \leq 64$,
(d) $\mathrm{Spin}(6) \simeq SU(4)$, *and so* $H\exp\big(\Omega_0^8 SO(6)\big) \leq 32$,
(e) $\mathrm{Spin}(7) \simeq G_2 \times S^7$, *and so* $H\exp\big(\Omega_0^{12} SO(7)\big) \leq 2^8$,
(f) $\mathrm{Spin}(8) \simeq \mathrm{Spin}(7) \times S^7$, *and so* $H\exp\big(\Omega_0^{12} SO(8)\big) \leq 2^8$. ■

We prove Theorem 1.1 by manipulating the characteristic maps of the Lie groups. In Section 2 we develop a general method for computing exponent bounds which is applicable to spherically resolved spaces, based on the degrees of the characteristic maps. The method can also be applied in other circumstances, and as an example we compute exponent bounds on even dimensional spheres. In Section 3 we give a factorization of the second power map on $\Omega_0^4 S^3$ through two maps derived from the homotopy class η. In Section 4 we calculate bounds on the orders of the characteristic maps themselves. This involves factoring the 2-nd-power map on $\Omega^3 S^{4n+1}$ and the 4-th-power map on $\Omega^3 S^{4n-1}$ through the double suspension. Iterating gives a factorization through ΩS^3. After looping we then use the factorization from Section 3 mentioned above to establish null homotopies based on specific homotopy groups of the Lie groups. In Section 5 the results in Sections 2 and 4 are put together to prove the H-exponent bounds of the Lie groups.

2 A Method for Computing Upper Bounds on Exponents

Typically, an upper bound for the exponent of a space Y is estimated by identifying homotopy fibrations $X \to Y \to Z$ in which the exponents of both X and Z are known. Then $\exp(Y) \leq \exp(X) \cdot \exp(Z)$ (similarly for H-exponents). Often, though, this is a poor estimate. The purpose of this section is to show that a better estimate can be obtained in certain cases, in particular for spherically resolved spaces.

We will concentrate on the case of interest to us, that of $Z = S^{2n+1}$ and the degree 2^r map on S^{2n+1} factoring through Y for some r. The method generalizes to even dimensional spheres or odd primes, or to the case when Z is an H-space whose

p^r-power map factors through Y. It can be used inductively to obtain upper bounds on the exponent of spherically resolved spaces like $SU(n)$ which are better, but not significantly better, than those previously known. The one example we focus on will illustrate the method and allows us to address a couple nuances that arise for spheres at the prime 2 which do not occur at odd primes, or in the case of H-spaces and power maps.

We begin with the following lemma, which is a sort of Mayer-Vietoris sequence.

Lemma 2.1 *Suppose there is a homotopy pullback diagram*

$$\begin{tikzcd} Q \arrow[r, "f"] \arrow[d, "h"] & P \arrow[d, "i"] \\ M \arrow[r, "g"] & N \end{tikzcd}$$

where N is an H-space. Then there is a homotopy fibration

$$Q \xrightarrow{f\times h} P \times M \xrightarrow{i\cdot(-g)} N.$$

Proof Let F be the homotopy fiber of g, and equivalently, of f. Applying homotopy groups gives a commuting diagram of long exact sequences

$$\begin{tikzcd} \cdots \arrow[r] & \pi_i(F) \arrow[r] \arrow[d, equal] & \pi_i(Q) \arrow[r, "f_*"] \arrow[d, "h_*"] & \pi_i(P) \arrow[r] \arrow[d, "i_*"] & \pi_{i-1}(F) \arrow[r] \arrow[d, equal] & \cdots \\ \cdots \arrow[r] & \pi_i(F) \arrow[r] & \pi_i(M) \arrow[r, "g_*"] & \pi_i(N) \arrow[r] & \pi_{i-1}(F) \arrow[r] & \cdots . \end{tikzcd}$$

From this a diagram chase shows there is a long exact sequence

$$\cdots \longrightarrow \pi_i(Q) \xrightarrow{\alpha} \pi_i(P) \oplus \pi_i(M) \xrightarrow{\beta} \pi_i(N) \xrightarrow{\gamma} \pi_{i-1}(Q) \longrightarrow \cdots$$

where $\alpha = (f_*, h_*)$, $\beta = i_* + -g_*$, and γ is the connecting map given by the composite $\pi_i(N) \to \pi_{i-1}(F) \to \pi_{i-1}(Q)$.

Since N is an H-space, we can multiply i and $-g$ to obtain a map $P \times M \xrightarrow{i\cdot(-g)} N$. Now consider the homotopy fibration

$$T \xrightarrow{s} P \times M \xrightarrow{i\cdot(-g)} N.$$

On the level of homotopy groups, $\big(i\cdot(-g)\big)_* = \beta$, so $\pi_i(T) \cong \pi_i(Q)$ for all $i \geq 0$ and $s_* = \alpha$. The given homotopy $g\circ h \simeq i\circ f$ implies the composite $Q \xrightarrow{f\times h} P\times M \xrightarrow{i\cdot(-g)} N$ is null homotopic, so $f \times h$ factors as a composite $Q \xrightarrow{\lambda} T \xrightarrow{s} P \times M$ for some map λ. Since $(f \times h)_* = \alpha$, we must have that λ_* is an isomorphism. Thus λ is a homotopy equivalence, proving the lemma. ■

https://doi.org/10.4153/CMB-2004-013-x Published online by Cambridge University Press

Let $\underline{2}^r\colon S^{2n+1} \to S^{2n+1}$ be the map of degree 2^r. Let $S^{2n+1}\{\underline{2}^r\}$ be its homotopy fiber.

Lemma 2.2 *Suppose there is a homotopy fibration*

$$X \xrightarrow{f} Y \xrightarrow{q} S^{2n+1}$$

where Y is an H-space and there is a map $S^{2n+1} \xrightarrow{i} Y$ such that $q \circ i \simeq \underline{2}^r$. Then there is a homotopy fibration

$$\Omega X \times \Omega S^{2n+1} \xrightarrow{\Omega f\cdot(-\Omega i)} \Omega Y \longrightarrow S^{2n+1}\{\underline{2}^r\}.$$

Proof The homotopy $q \circ i \simeq \underline{2}^r$ results in a homotopy pullback diagram

$$\begin{tikzcd} S^{2n+1}\{\underline{2}^r\} \arrow[r] \arrow[d] & S^{2n+1} \arrow[r, "\underline{2}^r"] \arrow[d, "i"] & S^{2n+1} \arrow[d, equal] \\ X \arrow[r, "f"] & Y \arrow[r, "q"] & S^{2n+1}. \end{tikzcd}$$

Apply Lemma 2.1 to get a homotopy fibration $S^{2n+1}\{\underline{2}^r\} \to X \times S^{2n+1} \xrightarrow{f\cdot(-i)} Y$. Continuing the fibration sequence to the left two steps gives the desired fibration. ∎

Before stating the exponent information coming out of Lemma 2.2 we need to elaborate on the nuances at the prime 2 mentioned earlier which do not occur at odd primes or in the case of H-spaces and power maps. Let $2\colon \Omega S^{2n+1} \to \Omega S^{2n+1}$ be the H-space squaring map. Let $\Omega S^{2n+1}\{2\}$ be its homotopy fiber. Except in cases involving Hopf invariant one, 2 is not homotopic to $\Omega\underline{2}$. But if we loop then by [C1, Section 4] we do have $2^r \simeq 2\Omega^2\underline{2}^{r-1} \simeq \Omega^2\underline{2}^r$ if $r \geq 2$. In particular $\Omega^2 S^{2n+1}\{\underline{2}^r\} \simeq \Omega^2 S^{2n+1}\{2^r\}$ if $r \geq 2$. Looping one more time gives an H-equivalence, and since the 2^r-power map on $\Omega^3 S^{2n+1}\{2^r\}$ is null homotopic [N, 5.2], the same is true for $\Omega^3 S^{2n+1}\{\underline{2}^r\}$.

Keeping all this in mind, Lemma 2.2 has the following corollary.

Corollary 2.3 *Suppose $r \geq 2$. Let t be such that $2^t = \max\bigl(\exp(X), \exp(S^{2n+1})\bigr)$. Then $\exp(Y) \leq 2^{t+r}$.* ∎

The $r = 1$ case is a bit different. Here, the homotopy exponent of $S^{2n+1}\{\underline{2}\}$ is bounded above by 8, is conjectured to be 4, and in either case does not match the 2 within the braces. But the discrepancy in powers of 2 can sometimes be recovered by modifying the homotopy fibration in Lemma 2.2 by 'exchanging' a factor of 2 from the fiber to the base.

Lemma 2.4 *Given the homotopy fibration*

$$\Omega X \times \Omega S^{2n+1} \xrightarrow{\Omega f \cdot (-\Omega i)} \Omega Y \longrightarrow S^{2n+1}\{\underline{2}^r\}$$

of Lemma 2.2, then there is a homotopy fibration

$$\Omega^3 X \times \Omega^3 S^{2n+1} \xrightarrow{\Omega^3 f \cdot (-2\Omega^3 i)} \Omega^3 Y \longrightarrow \Omega^2 S^{2n+1}\{\underline{2}^{r+1}\}.$$

Proof As $q \circ i \simeq \underline{2}^r$, we have $\Omega^2 q \circ (2\Omega^2 i) \simeq 2\Omega^2 \underline{2}^r \simeq \Omega^2 \underline{2}^{r+1}$. Now, in place of the pullback used in the proof of Lemma 2.2, use the pullback

$$\begin{tikzcd}
\Omega^2 S^{2n+1}\{\underline{2}^{r+1}\} \arrow[r] \arrow[d] & \Omega^2 S^{2n+1} \arrow[r, "\Omega^2 \underline{2}^{r+1}"] \arrow[d, "2\Omega^2 i"] & \Omega^2 S^{2n+1} \arrow[d, equal] \\
\Omega^2 X \arrow[r, "\Omega^2 f"] & \Omega^2 Y \arrow[r, "\Omega^2 q"] & \Omega^2 S^{2n+1}.
\end{tikzcd}$$

∎

Remark 2.5 The exponent bounds resulting from Lemmas 2.2 and 2.4 can be further improved if the order of the map $S^{2n+1} \xrightarrow{i} Y$ is less than the exponent of S^{2n+1}. This is what we do in Section 4 when Y is a Lie group and we bound the order of the characteristic map.

As an example of Lemma 2.4 in action, consider the even dimensional sphere S^{2n}, localized at the prime 2. As input we need some information about the homotopy exponents of odd dimensional spheres. Mark Mahowald has conjectured the following exponent pattern:

$$\exp(S^{2n+1}) = \begin{cases} 2^n & \text{if } n \equiv 0, 3 \bmod 4 \\ 2^{n+1} & \text{if } n \equiv 1, 2 \bmod 4. \end{cases}$$

The homotopy of even dimensional spheres is intertwined with that of odd dimensional spheres by the Hopf fibration $S^{2n-1} \to \Omega S^{2n} \to \Omega S^{4n-1}$. The conjectured exponent pattern for even dimensional spheres is:

$$\exp(S^{2n}) = \begin{cases} 2 \cdot \exp(S^{4n-1}) & \text{if } n \equiv 0 \bmod 2 \\ \frac{1}{2} \cdot \exp(S^{4n-1}) & \text{if } n \equiv 1 \bmod 2. \end{cases}$$

Note that the above estimates are realized as lower bounds on the exponent, that is, in each case there are known elements in the homotopy groups whose order matches the conjectured order.

Proposition 2.6 *We have* $\exp(S^{2n}) \leq 2 \cdot \exp(S^{4n-1})$. *In particular, if* $n \equiv 0$ *mod* 2 *and* $\exp(S^{4n-1})$ *is as conjectured then* $\exp(S^{2n})$ *is also as conjectured.*

Proof Consider the composition $\Omega S^{4n-1} \xrightarrow{\Omega[1,1]} \Omega S^{2n} \xrightarrow{H} \Omega S^{4n-1}$, where $[1,1]$ is the Whitehead product and H is the second James-Hopf invariant. By [C2] this composition is homotopic to $\Omega\underline{2}$. This gives a homotopy pullback

$$\begin{tikzcd}
\Omega S^{4n-1}\{\underline{2}\} \arrow[r] \arrow[d] & \Omega S^{4n-1} \arrow[r, "\Omega\underline{2}"] \arrow[d, "\Omega[1,1]"] & \Omega S^{4n-1} \arrow[d, equal] \\
S^{2n-1} \arrow[r] & \Omega S^{2n} \arrow[r, "H"] & \Omega S^{4n-1}.
\end{tikzcd}$$

By Lemma 2.4, there is a homotopy fibration

$$\Omega^2 S^{2n-1} \times \Omega^3 S^{4n-1} \xrightarrow{\theta} \Omega^3 S^{2n} \xrightarrow{f} \Omega^2 S^{4n-1}\{\underline{4}\},$$

where $\theta = \Omega^2 E \cdot (-2\Omega^3[1,1])$.

Now consider the map induced by θ on homotopy groups. When restricted to the 2-torsion of the domain, $\pi_*(\theta)$ has order $\frac{1}{2} \cdot \exp(S^{4n-1})$. On the other hand, $\pi_*(f)$ has order 4. Thus $\exp(S^{2n}) \leq 4 \cdot \left(\frac{1}{2} \cdot \exp(S^{4n-1})\right) = 2 \cdot \exp(S^{4n-1})$. ■

3 A Factorization of the Second Power Map on $\Omega^4_0 S^3$

The better-known factorization of the second power map on $\Omega^2 S^3\langle 3\rangle$ [C1, *cf.* Section 5] is as a composite $\Omega^2 S^3\langle 3\rangle \xrightarrow{\Omega\overline{H}} \Omega^2 S^5 \xrightarrow{\lambda} \Omega^2 S^3\langle 3\rangle$, where $\overline{H}$ is the connected cover of the James-Hopf invariant $\Omega S^3 \xrightarrow{H} \Omega S^5$, and λ is some map. Our factorization can be roughly thought of as describing λ; it factors through two maps derived from the homotopy class η. The actual factorization, though, possibly alters $\Omega^3_0\overline{H}$ by some self-equivalence of $\Omega^4_0 S^3$ (this follows from the proof of Lemma 3.3).

We require several lemmas. To begin, let X be a space. The cofibration $S^2 \xrightarrow{2} S^2 \to P^3(2)$ gives a homotopy fibration

$$\mathrm{Map}_*\big(P^3(2), X\big) \longrightarrow \mathrm{Map}_*(S^2, X) \longrightarrow \mathrm{Map}_*(S^2, X)$$

which is equivalent to the homotopy fibration

$$\Omega^2 X\{2\} \longrightarrow \Omega^2 X \xrightarrow{2} \Omega^2 X.$$

The degree 2 map on the Moore space $P^3(2)$ factors as a composite

$$P^3(2) \xrightarrow{q} S^3 \xrightarrow{\eta} S^2 \xrightarrow{i} P^3(2),$$

where q is the pinch map onto the top cell and i is the inclusion of the bottom cell. This gives a composite of mapping spaces

$$\mathrm{Map}_*\big(P^3(2), X\big) \longrightarrow \mathrm{Map}_*(S^3, X) \xrightarrow{\eta^*} \mathrm{Map}_*(S^2, X) \longrightarrow \mathrm{Map}_*\big(P^3(2), X\big)$$

https://doi.org/10.4153/CMB-2004-013-x Published online by Cambridge University Press

which is equivalent to a factorization

$$\Omega X\{2\} \longrightarrow \Omega^2 X \xrightarrow{\eta^*} \Omega^3 X \longrightarrow \Omega^2 X\{2\}$$

of the H-space squaring map on $\Omega^2 X\{2\}$. In particular, with $X = S^5$ we have the following lemma.

Lemma 3.1 *The H-space squaring map on $\Omega^2 S^5\{2\}$ is homotopic to the composite*

$$\Omega^2 S^5\{2\} \longrightarrow \Omega^2 S^5 \xrightarrow{\eta^*} \Omega^3 S^5 \longrightarrow \Omega^2 S^5\{2\}. \qquad \blacksquare$$

Next, a special fibration can be obtained from the *EHP* fibration $S^2 \xrightarrow{E} \Omega S^3 \xrightarrow{H} \Omega S^5$. Taking two-connected covers gives a homotopy fibration sequence

$$\Omega^2 S^5 \xrightarrow{\phi} S^3 \xrightarrow{i} \Omega S^3\langle 3\rangle \xrightarrow{\overline{H}} \Omega S^5,$$

where i represents $\eta \in \pi_4(S^3)$. In particular, since ΩS^3 is a retract of ΩS^2, there is a homotopy commutative diagram

$$\begin{tikzcd}
\Omega^3 S^5 \arrow[r, "\Omega\phi"] \arrow[d, equal] & \Omega S^3 \arrow[d] \arrow[dr, equal] & \\
\Omega^3 S^5 \arrow[r, "\Omega P"] & \Omega S^2 \arrow[r, "H"] & \Omega S^3,
\end{tikzcd}$$

where P is the connecting map in the *EHP* sequence, and H is the second James-Hopf invariant. Thus $\Omega\phi \simeq H \circ \Omega P$. By [R] the composite $\Omega^3 S^5 \xrightarrow{\Omega P} \Omega S^2 \xrightarrow{H} \Omega S^3 \xrightarrow{\Omega E^2} \Omega^3 S^5$ is homotopic to the second power map. This proves the following lemma.

Lemma 3.2 *There is a homotopy commutative diagram*

$$\begin{tikzcd}
\Omega^3 S^5 \arrow[r, "\Omega\phi"] \arrow[d, "2"] & \Omega S^3 \arrow[d, "\Omega E^2"] \\
\Omega^3 S^5 \arrow[r, equal] & \Omega^3 S^5.
\end{tikzcd} \qquad \blacksquare$$

We now link this information with the Lie group $SU(3)$. Taking three-connected covers there is a homotopy fibration sequence

$$\Omega S^5 \xrightarrow{\bar{\eta}} S^3\langle 3\rangle \longrightarrow SU(3)\langle 3\rangle \longrightarrow S^5,$$

https://doi.org/10.4153/CMB-2004-013-x Published online by Cambridge University Press

where $\bar{\eta}$ restricted to the bottom cell is η. Thus the composite $S^3 \xrightarrow{E^2} \Omega^2 S^5 \xrightarrow{\Omega\bar{\eta}} \Omega S^3\langle 3\rangle$ is the map i defined above. This results in a homotopy pullback diagram

$$\begin{tikzcd}
\Omega^2 S^5 \arrow[r, "\phi"] \arrow[d, "\epsilon"] & S^3 \arrow[r, "i"] \arrow[d, "E^2"] & \Omega S^3\langle 3\rangle \arrow[d, equal] \\
\Omega^2 SU(3)\langle 3\rangle \arrow[r] & \Omega^2 S^5 \arrow[r, "\Omega^2\bar{\eta}"] & \Omega S^3\langle 3\rangle,
\end{tikzcd}$$

for some map ϵ. Combining this pullback with Lemma 3.2 proves that the second power map on $\Omega^3 S^5$ factors through $\Omega^3 SU(3)$. The next lemma takes this one step further.

Lemma 3.3 *There is a homotopy pullback diagram*

$$\begin{tikzcd}
\Omega^4 S^5 \arrow[r] \arrow[d, equal] & \Omega^3 S^5\{2\} \arrow[r] \arrow[d, "\gamma"] & \Omega^3 S^5 \arrow[r, "2"] \arrow[d, "\Omega\epsilon"] & \Omega^3 S^5 \arrow[d, equal] \\
\Omega^4 S^5 \arrow[r, "\Omega^3\bar{\eta}"] & \Omega^3 S^3\langle 3\rangle \arrow[r] & \Omega^3 SU(3)\langle 3\rangle \arrow[r] & \Omega^3 S^5,
\end{tikzcd}$$

for some map γ which has a right homotopy inverse.

Proof It remains to show that γ has a right homotopy inverse. By [C1, 4.1], the Hopf invariant $\Omega S^3 \xrightarrow{H} \Omega S^5$ has order 2 when looped. Thus $2\Omega\overline{H}$ is null homotopic as well. This gives a lift of $\Omega\overline{H}$ to a map $f\colon \Omega^2 S^3\langle 3\rangle \to \Omega^2 S^5\{2\}$. Both γ and Ωf are the identity on the bottom cell and $\Omega^3 S^3\langle 3\rangle$ is atomic, so $\gamma \circ \Omega f$ is a homotopy equivalence. Thus γ must have a right homotopy inverse. ∎

Let $\delta\colon \Omega^3 S^3\langle 3\rangle \to \Omega^3 S^5\{2\}$ be a right homotopy inverse of the map γ appearing in Lemma 3.3.

Proposition 3.4 *The second power map on $\Omega_0^4 S^3$ is homotopic to the composite*

$$\Omega_0^4 S^3 \xrightarrow{\Omega\delta} \Omega_0^4 S^5\{2\} \longrightarrow \Omega^4 S^5 \xrightarrow{\Omega^2\eta^*} \Omega_0^5 S^5 \xrightarrow{\Omega^4\bar{\eta}} \Omega_0^4 S^3.$$

Proof Consider the diagram

$$\begin{tikzcd}
\Omega^3 S^5\{2\} \arrow[r] & \Omega^3 S^5 \arrow[r, "\Omega\eta^*"] & \Omega^4 S^5 \arrow[r] \arrow[d, "\Omega^2\bar{\eta}"] & \Omega^3 S^5\{2\} \arrow[d, "\gamma"] \\
\Omega^3 S^3\langle 3\rangle \arrow[u, "\delta"] & & \Omega^3 S^3\langle 3\rangle \arrow[r, equal] & \Omega^3 S^3\langle 3\rangle.
\end{tikzcd}$$

The right square homotopy commutes by Lemma 3.3. The top row is the second power map on $\Omega^3 S^5\{2\}$ by Lemma 3.1. Looping so that γ is multiplicative, the lemma follows. ■

Remark 3.5 The factorization in Proposition 3.4 can be delooped if the map γ in Lemma 3.3 can be chosen to be an H-map.

4 Bounding the Order of Characteristic Maps

Recall that the characteristic maps $S^5 \to SU(3)$, $S^7 \to SU(4)$, and $S^7 \to Sp(2)$ are of degrees 2, 2, and 4 respectively. The purpose of this section is to calculate bounds for the order of these maps (see Proposition 4.5 for the precise statement). The characteristic map for G_2 is a bit different, so we begin by describing it.

Start with the standard fibration $SU(3) \to G_2 \xrightarrow{g} S^6$. Use a second James-Hopf invariant to obtain a homotopy pullback diagram

$$\begin{tikzcd}
X \arrow[r, "a"] \arrow[d] & \Omega G_2 \arrow[r, "b"] \arrow[d, "\Omega g"] & \Omega S^{11} \arrow[d, equal] \\
S^5 \arrow[r, "E"] \arrow[d] & \Omega S^6 \arrow[r, "H"] \arrow[d] & \Omega S^{11} \\
SU(3)\langle 3\rangle \arrow[r, equal] & SU(3)\langle 3\rangle, &
\end{tikzcd}$$

where b is defined as the composite $H \circ \Omega g$. Note that the space X is the same as the one in Corollary 4.2.

Lemma 4.1 *There is a 'characteristic' map $j\colon S^{11} \to G_2$ such that $b \circ \Omega j \simeq \Omega \underline{2}$.*

Proof By [MT] the Whitehead product $S^{11} \xrightarrow{[\iota,\iota]} S^6$ factors as a composite $S^{11} \xrightarrow{j} G_2 \xrightarrow{g} S^6$ for some map j. By [C2], the composite $\Omega S^{11} \xrightarrow{\Omega[\iota,\iota]} \Omega S^6 \xrightarrow{H} \Omega S^{11}$ is homotopic to $\Omega\underline{2}$. Thus $b \circ \Omega j = H \circ \Omega g \circ \Omega j \simeq H \circ \Omega[\iota,\iota] \simeq \Omega\underline{2}$. ■

We will need H-exponent information about X. Observe that X fits in another homotopy pullback diagram,

https://doi.org/10.4153/CMB-2004-013-x Published online by Cambridge University Press

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d] & X \arrow[d] & \\
S^5\{\underline{2}\} \arrow[r] \arrow[d] & S^5 \arrow[r, "\underline{2}"] \arrow[d] & S^5 \arrow[d, equal] \\
S^3\langle 3\rangle \arrow[r] & SU(3)\langle 3\rangle \arrow[r] & S^5.
\end{tikzcd}$$

Lemma 4.2 *$H\exp(\Omega^3 X) \le 2^5$.*

Proof Consider the homotopy fibration $X \longrightarrow S^5\{\underline{2}\} \longrightarrow S^3\langle 3\rangle$. Recall from the introduction that $H\exp(\Omega^2 S^3\langle 3\rangle) = 4$, and it is known that $H\exp(\Omega^3 S^{2n+1}\{\underline{2}\}) \le 8$, so the lemma follows. ■

We need a couple of additional lemmas before proceeding to Proposition 4.5. Both give information about the constituent maps in the factorization of the second power map on $\Omega_0^4 S^3$ in Proposition 3.4.

Lemma 4.3 *There is a homotopy commutative diagram*

$$\begin{tikzcd}
\Omega S^5 \arrow[r, "\bar{\eta}"] \arrow[d, "\Omega\eta"] & S^3 \arrow[d, "E^3"] \\
\Omega S^4 \arrow[r, "\Omega E^2"] & \Omega^3 S^6.
\end{tikzcd}$$

In particular, $E^3 \circ \bar{\eta}$ is an H-map.

Proof We first set up a homotopy equivalence $\Omega S^4 \simeq S^3 \times \Omega S^7$ which has the properties we want. There is a homotopy pullback diagram

$$\begin{tikzcd}
SU(3) \arrow[r] \arrow[d] & S^5 \arrow[r] \arrow[d, "\eta"] & BS^3 \arrow[d, equal] \\
S^7 \arrow[r, "\nu"] & S^4 \arrow[r, "i"] & BS^3,
\end{tikzcd}$$

where i is the inclusion of the bottom cell. In particular $r\colon \Omega S^4 \xrightarrow{\Omega i} S^3$ is a choice of retration such that $r \circ \Omega\eta \simeq \bar{\eta}$. Also, the Hopf invariant of ν is the identity. We now have a composite (H is the James-Hopf invariant)

$$S^3 \times \Omega S^7 \xrightarrow{E\cdot\Omega\nu} \Omega S^4 \xrightarrow{r\times H} S^3 \times \Omega S^7$$

https://doi.org/10.4153/CMB-2004-013-x Published online by Cambridge University Press

which is homotopic to the identity. Therefore

$$\Omega S^4 \xrightarrow{r\times H} S^3 \times \Omega S^7 \xrightarrow{E\cdot\Omega\nu} \Omega S^4$$

is homotopic to the identity.

Now consider the composite $\Omega S^5 \xrightarrow{\Omega\eta} \Omega S^4 \xrightarrow{\Omega E^2} \Omega^3 S^6$. The homotopy for the identity map on ΩS^4 ending the previous paragraph implies that $\Omega E^2 \circ (E\cdot\Omega\nu) \circ (r\times H) \circ \Omega\eta \simeq (\Omega E^2 \circ E \circ r \circ \Omega\eta)\cdot(\Omega E^2 \circ \Omega\nu \circ H \circ \Omega\eta)$. On the one hand, $r \circ \Omega\eta \simeq \bar\eta$ so $\Omega E^2 \circ E \circ r \circ \Omega\eta \simeq E^3 \circ \Omega\bar\eta$. We want to show that on the other hand, $\Omega E^2 \circ \Omega\nu \circ H \circ \Omega\eta \simeq *$. This would prove the lemma.

Since $S^5 \xrightarrow{\eta} S^4$ is a suspension, the naturality of the James–Hopf invariant implies the composite $\Omega S^5 \xrightarrow{\Omega\eta} \Omega S^4 \xrightarrow{H} \Omega S^7$ is homotopic to the composite $\Omega S^5 \xrightarrow{H} \Omega S^9 \xrightarrow{\Omega\eta^2} \Omega S^7$. This leads to considering the composite $\Omega f\colon \Omega S^9 \xrightarrow{\Omega\eta^2} \Omega S^7 \xrightarrow{\Omega\nu} \Omega S^4 \xrightarrow{\Omega E^2} \Omega^3 S^6$. But f has finite order while $\pi_{11}(S^6) = \mathbf{Z}$, so $f \simeq *$. Thus $\Omega E^2 \circ \Omega\nu \circ H \circ \Omega\eta \simeq *$ and we are done. ∎

Next, for the $SU(3)$ case, we need to be able to work with the map η^* which, as described in Section 3, is the map $S^3 \xrightarrow{\eta} S^2$ induces on mapping spaces.

Let A, B, C, and X be spaces. Suppose there is a map $f\colon A \to B$. Then there is an induced map $f^*\colon \operatorname{Map}_*(B,X) \to \operatorname{Map}_*(A,X)$ which is natural in the X variable. As well, the exponential law gives equivalences

$$\operatorname{Map}_*\big(A, \operatorname{Map}_*(C,X)\big) \xrightarrow{\cong} \operatorname{Map}_*(A \wedge C, X) \xrightarrow{\cong} \operatorname{Map}_*\big(C, \operatorname{Map}_*(A,X)\big)$$

which are natural in all three variables. In particular, suppose C is the circle. Then

$$\operatorname{Map}_*\big(A, \operatorname{Map}_*(C,X)\big) = \operatorname{Map}_*(A, \Omega X) \quad \text{and}$$
$$\operatorname{Map}_*\big(C, \operatorname{Map}_*(A,X)\big) = \Omega \operatorname{Map}_*(A,X).$$

Naturality in the A variable then gives the following lemma.

Lemma 4.4 *There is a commutative diagram*

$$\begin{array}{ccc}
\operatorname{Map}_*(B,\Omega X) & \xrightarrow{\cong} & \Omega \operatorname{Map}_*(B,X) \\
\downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle \Omega f^*} \\
\operatorname{Map}_*(A,\Omega X) & \xrightarrow{\cong} & \Omega \operatorname{Map}_*(A,X).
\end{array}$$

∎

We need one more piece of information about factoring power maps on the loops of odd dimensional spheres. Recall from [R] that the H-space squaring map on $\Omega^3 S^{4n+1}$ factors through the double suspension, and from [C1, 5.9] that the 4-th-power map on $\Omega^3 S^{4n-1}$ factors through the double suspension. In particular, the 2-nd-power map on $\Omega^3 S^5$ factors through $\Omega S^3 \xrightarrow{\Omega E^2} \Omega^3 S^5$, the 8-th-power map on $\Omega^5 S^7$ factors through $\Omega S^3 \xrightarrow{\Omega E^4} \Omega^5 S^7$, and the 2^6-power map on $\Omega^9 S^{11}$ factors through $\Omega S^3 \xrightarrow{\Omega E^8} \Omega^9 S^{11}$.

Proposition 4.5 *The following hold for the characteristic maps:*

(a) $\Omega_0^6 S^5 \to \Omega_0^6 SU(3)$ *has order* 4*,*
(b) $\Omega_0^8 S^7 \to \Omega_0^8 SU(4)$ *has order* ≤ 16*,*
(c) $\Omega_0^8 S^7 \to \Omega_0^8 Sp(2)$ *has order* ≤ 16*,*
(d) $\Omega_0^{12} S^{11} \to \Omega_0^{12} G_2$ *has order* $\leq 2^7$*.*

Proof The homotopy group calculations in what follows come from [MT] for $SU(3)$, $SU(4)$, and $Sp(2)$, and from [M] for G_2. The characteristic map for each Lie group will commonly be denoted by j.

We begin with (b), (c) and (d) as they are most immediate. For (b), the sixteenth power map on $\Omega^5 S^7$ factors through the second power map on ΩS^3. By Proposition 3.4, the second power map on $\Omega_0^4 S^3$ factors through the (basepoint component of the) fourth loop of the map $\Omega S^5 \xrightarrow{\bar{\eta}} S^3$. Thus it suffices to show that the composite

$$f\colon \Omega S^5 \xrightarrow{\bar{\eta}} S^3 \xrightarrow{E^4} \Omega^4 S^7 \xrightarrow{\Omega^4 j} \Omega^4 SU(4)$$

is null homotopic. By Lemma 4.3, $E^4 \circ \bar{\eta}$ is an H-map so to show that f is trivial it suffices to check it is on the bottom cell. The generator of $\pi_8\big(SU(4)\big) = \mathbb{Z}/2\mathbb{Z}$ is a class which composes to $\nu \oplus \eta$ under the map $g\colon SU(4) \to S^5 \times S^7$ whose fiber is S^3. The characteristic map $S^7 \xrightarrow{j} SU(4)$ can be chosen to satisfy $g \circ j \simeq * \oplus \underline{2}$. Thus $g \circ j \circ \eta \simeq *$, and so f is trivial on the bottom cell.

Parts (c) and (d) proceed similarly, and are even easier, because $\pi_8\big(Sp(2)\big) = 0$ and $\pi_{12}(G_2) = 0$.

Part (a) is a little trickier. Now $\pi_6\big(SU(3)\big) = \mathbb{Z}/2\mathbb{Z}$ is generated by the composite $S^4 \xrightarrow{\eta} S^3 \xrightarrow{E^2} \Omega^2 S^5 \xrightarrow{\Omega^2 j} \Omega^2 SU(3)$, so we cannot simply repeat the argument in part (b). Instead we have to dig a little deeper into the factorization of the second power map on $\Omega_0^4 S^3$ and bring into play the map η^*.

First, because $SU(3)$ is a loop space, the characteristic map is homotopic to the composite $S^5 \xrightarrow{E} \Omega S^6 \xrightarrow{\Omega j'} \Omega BSU(3)$, where j' is a representative of the generator of $\pi_6\big(BSU(3)\big) = \mathbf{Z}$. Next, the naturality of η^* implies that there is a homotopy commutative diagram

$$\begin{array}{ccccccc}
\Omega^2 S^5 & \xrightarrow{\Omega^2 \eta} & \Omega^2 S^4 & \xrightarrow{\Omega^2 E^2} & \Omega^4 S^6 & \xrightarrow{\Omega^4 j'} & \Omega^4 BSU(3) \\
\downarrow{\scriptstyle \eta^*} & & \downarrow{\scriptstyle \eta^*} & & \downarrow{\scriptstyle \eta^*} & & \downarrow{\scriptstyle \eta^*} \\
\Omega^3 S^5 & \xrightarrow{\Omega^3 \eta} & \Omega^3 S^4 & \xrightarrow{\Omega^3 E^2} & \Omega^5 S^6 & \xrightarrow{\Omega^5 j'} & \Omega^5 BSU(3).
\end{array}$$

The exponential law in Lemma 4.4 implies that $\Omega^4 BSU(3) \xrightarrow{\eta^*} \Omega^5 BSU(3)$ is a double loop map. Thus the upper direction around the diagram is homotopic to a double loop map. Since $\pi_7\big(SU(3)\big) = 0$, the upper direction around the diagram is null homotopic. Let $\psi\colon \Omega^3 S^5 \to \Omega^5 BSU(3)$ be the composite defined by the lower row in the diagram. Then $\psi \circ \eta^* \simeq *$. But the homotopy in Lemma 4.3 and the definition of j' implies that ψ is homotopic to the composite

$$\theta\colon \Omega^3 S^5 \xrightarrow{\Omega^2 \bar{\eta}} \Omega^2 S^3 \xrightarrow{\Omega^2 E^2} \Omega^2 S^5 \xrightarrow{\Omega^2 j} \Omega^2 SU(3),$$

and so $\theta \circ \eta^* \simeq *$. The first two maps in $\theta \circ \eta^*$, namely $\Omega^2\bar{\eta} \circ \eta^*$, appear (looped twice) in the factorization of the second power map on $\Omega_0^4 S^3$, which in turn factors through the fourth power map on $\Omega_0^6 S^5$. Thus $\Omega_0^6 S^5 \xrightarrow{4} \Omega_0^6 S^5 \xrightarrow{\Omega_0^6 j} \Omega_0^6 SU(3)$ is null homotopic. (Note that this is best possible as the characteristic map in this case has degree 2 while the order of the identity map on $\Omega^4 S^5\langle 5\rangle$ is 8.) ■

5 H-Exponent Bounds

We now combine the fibrations in Lemmas 2.2 and 2.4 with the orders of the characteristic maps in Proposition 4.5 to prove Theorem 1.1.

Proof of Theorem 1.1 For $SU(3)$, there is a fibration $S^3 \xrightarrow{i} SU(3) \to S^5$ and the characteristic map $S^5 \xrightarrow{j} SU(3)$ is degree 2. Using Lemma 2.4 we obtain a homotopy fibration

$$\Omega_0^3 S^3 \times \Omega^3 S^5 \xrightarrow{\phi} \Omega_0^3 SU(3) \longrightarrow \Omega^2 S^5\{\underline{4}\}$$

where $\phi = \Omega_0^3 i\langle 3\rangle \cdot (-2\Omega^3 j)$. Looping often enough to align with Proposition 4.5 (a), consider the diagram:

$$\begin{tikzcd}
 & \Omega_0^6 SU(3) \arrow[dl, "\lambda"'] \arrow[d, "4"] & \\
\Omega_0^6 S^3 \times \Omega_0^6 S^5 \arrow[r, "\Omega_0^3\phi"] \arrow[d, "2"] & \Omega_0^6 SU(3) \arrow[r] \arrow[d, "2"] & \Omega_0^5 S^5\{\underline{4}\} \arrow[d, "2"] \\
\Omega_0^6 S^3 \times \Omega_0^6 S^5 \arrow[r, "\Omega_0^3\phi"] & \Omega_0^6 SU(3) \arrow[r] & \Omega_0^5 S^5\{\underline{4}\}.
\end{tikzcd}$$

Multiplication by 2 commutes with loop maps so both lower squares homotopy commute. Since the 4-th-power map on $\Omega_0^5 S^5\{\underline{4}\}$ is null homotopic, the 4-th-power map on $\Omega_0^6 SU(3)$ composes trivially into $\Omega_0^5 S^5\{\underline{4}\}$ and so lifts through $\Omega_0^3\phi$. Choose a lift and call it λ; then by definition the upper triangle in the diagram above homotopy commutes and so the entire diagram homotopy commutes. Thus the 8-th-power map on $\Omega_0^6 SU(3)$ is homotopic to $\Omega_0^3\phi \circ 2 \circ \lambda$. But $\Omega_0^3\phi = \Omega_0^6 i \cdot (-2\Omega_0^6 j)$ has order 2

https://doi.org/10.4153/CMB-2004-013-x Published online by Cambridge University Press

because on the one hand $\Omega_0^6 i$ has order 2 by [C1, 5.4] while on the other hand $\Omega_0^6 j$ has order 4 by Proposition 4.5 (a). Thus $\Omega_0^3\phi\circ 2$ is null homotopic and so the 8-th-power map on $\Omega_0^6 SU(3)$ is null homotopic.

For $SU(4)$, use the fibration $S^3 \to SU(4) \to S^5 \times S^7$. The characteristic maps $S^5 \xrightarrow{j_1} SU(4)$ and $S^7 \xrightarrow{j_2} SU(4)$ are both degree 2. By Lemma 2.4 we obtain a homotopy fibration

$$\Omega_0^3 S^3 \times \Omega^3 S^5 \times \Omega^3 S^7 \xrightarrow{\psi} \Omega_0^3 SU(4) \longrightarrow \Omega^2 S^5\{\underline{4}\} \times \Omega^2 S^7\{\underline{4}\},$$

where $\psi = \Omega_0^3 i \cdot (-2\Omega^3 j_1) \cdot (-2\Omega^3 j_2)$. Now proceed as in the $SU(3)$ case.

For $Sp(2)$, use the fibration $S^3 \xrightarrow{i} Sp(2) \to S^7$. The characteristic map $S^7 \xrightarrow{j}$ $Sp(2)$ has degree 4. By Lemma 2.2 there is a homotopy fibration

$$\Omega S^3 \times \Omega S^7 \xrightarrow{\Omega i\cdot(-\Omega j)} \Omega Sp(2) \longrightarrow S^7\{\underline{4}\}.$$

Now proceed as in the first case, using the fact that $H\exp(\Omega^2 S^3\langle 3\rangle) = 4$.

For G_2, by Lemmas 4.1 and 2.4 we obtain a homotopy fibration

$$\Omega^2 X \times \Omega^3 S^{11} \xrightarrow{\Omega a\cdot(-2\Omega^3 j)} \Omega^3 G_2 \longrightarrow \Omega^2 S^{11}\{\underline{4}\},$$

Now proceed as in the first case, noting that $H\exp(\Omega^3 X) \leq 2^5$ by Lemma 4.2. ∎

References

[C1] F. R. Cohen, *A short course in some aspects of classical homotopy theory.* Lecture Notes in Math. **1286**, Springer-Verlag, 1987, 1–92.

[C2] ———, *Applications of loop spaces to some problems in topology.* London Math. Soc. Lecture Notes Series **139**(1989), 11–20.

[DM1] D. M. Davis and M. Mahowald, *Three contributions to the homotopy theory of the exceptional Lie groups G_2 and F_4.* J. Math. Soc. Japan **43**(1991), 661–671.

[DM2] ———, *Some remarks on v_1-periodic homotopy groups.* London Math. Soc. Lecture Notes Series **176**(1992), 55–72.

[M] M. Mimura, *The homotopy groups of Lie groups of low rank.* J. Math. Kyoto Univ. **6**(1967), 131–176.

[MT] M. Mimura and H. Toda, *Homotopy groups of $SU(3)$, $SU(4)$, and $Sp(2)$.* J. Math. Kyoto Univ. **3**(1964), 217–250.

[N] J. A. Neisendorfer, *Properties of certain H-spaces.* Quart. J. Math. Oxford **34**(1983), 201–209.

[R] W. Richter, *The H-space squaring map on $\Omega^3 S^{4n+1}$ factors through the double suspension.* Proc. Amer. Math. Soc. **123**(1995), 3889–3900.

Department of Mathematics
University of Virginia
Charlottesville, VA 22904
USA

Current address:
Department of Mathematical Sciences
University of Aberdeen
Aberdeen AB24 3UE
United Kingdom
email: s.theriault@maths.abdn.ac.uk

https://doi.org/10.4153/CMB-2004-013-x Published online by Cambridge University Press