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A JORDAN-LIKE DECOMPOSITION IN THE NONCOMMUTATIVE SCHWARTZ SPACE

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Abstract

We show that every continuous self-adjoint functional on the noncommutative Schwartz space can be decomposed into a difference of two positive functionals. Moreover, this decomposition is minimal in the natural sense.

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1. Introduction

The aim of this paper is to investigate one concrete object, the so-called *noncommutative Schwartz space*—denoted by $\mathcal{S}$. We describe this Fréchet *-algebra in the next section. This object has been studied in several contexts and received reasonable attention, so far. It appears, for instance, in K-theory (see [3, 9]) and in cyclic cohomology for crossed products [6, 12]. Investigation of this object continues. Recently, Ciaś and the present author have obtained several further results. Ciaś, using purely Fréchet space tools, showed in [2] that the noncommutative Schwartz space admits a functional calculus and characterised closed, commutative *-subalgebras of $\mathcal{S}$. In [11], the present author showed that every positive linear functional on $\mathcal{S}$ as well as every derivation from $\mathcal{S}$ into any $\mathcal{S}$-bimodule is automatically continuous. This paper deals also with amenability properties of the noncommutative Schwartz space. Although $\mathcal{S}$ is not amenable (see [10, Theorem 9.7] and [11, Proposition 2]), it turns out that it is approximately amenable [11, Theorem 21]. The present paper is a continuation of this research. The aim is to provide a way of decomposing a continuous and self-adjoint functional on the noncommutative Schwartz space into a difference of two positive functionals.

The paper is divided into four parts. In Section 2, we recall the definition of the noncommutative Schwartz space and give basic notation. Section 3 deals with the

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dual of the noncommutative Schwartz space. Section 4 provides a construction of the above-mentioned decomposition.

For unexplained details we refer the reader to [8] for the structure theory of Fréchet spaces and to [4] for the 'algebraic-in-flavour' aspects of the paper.

2. Preliminaries

Throughout the paper we denote $\mathbb{N} := \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Next we recall that

$$s = \Big\{\xi = (\xi_j)_{j\in\mathbb{N}} \subset \mathbb{C}^{\mathbb{N}} : |\xi|_k^2 := \sum_{j=1}^{+\infty} |\xi_j|^2 j^{2k} < +\infty \text{ for all } k \in \mathbb{N}_0\Big\}$$

and its topological dual

$$s' = \Big\{\eta = (\eta_j)_{j\in\mathbb{N}} \subset \mathbb{C}^{\mathbb{N}} : |\eta|_k'^2 := \sum_{j=1}^{+\infty} |\eta_j|^2 j^{-2k} < +\infty \text{ for some } k \in \mathbb{N}_0\Big\}$$

are the so-called spaces of rapidly decreasing and slowly increasing sequences, respectively. We consider the space $\mathcal{S} := L(s', s)$ of linear and continuous operators from the dual of s into s with the topology of uniform convergence on bounded sets. It is possible to turn this space into a locally multiplicatively convex (LMC for short) Fréchet $*$-algebra by the use of the isomorphism

$$\mathcal{S} \simeq \mathcal{K}^\infty := \Big\{x = (x_{i,j})_{i,j\in\mathbb{N}} : \|x\|_n^2 := \sum_{i,j=1}^{\infty} |x_{i,j}|^2 (ij)^{2n} < +\infty \text{ for all } n \in \mathbb{N}_0\Big\}.$$

The algebra $\mathcal{S}$ will be called the *noncommutative Schwartz space* and we refer the reader to [11] for more information on the properties of this algebra.

3. Dual of the noncommutative Schwartz space

The topological dual of $\mathcal{S}$ has several natural representations. Observe first that by [8, Proposition 28.16], s is nuclear and so by [7, 21.2.2] it has the approximation property. Consequently, finite-rank operators are dense in $L(s', s)$. Therefore, by [7, 15.3.4 and 16.1.4] the map

$$x \otimes y \mapsto (x' \mapsto \langle x, x'\rangle y)$$

extends to a topological isomorphism

$$\chi : s \otimes s \to \mathcal{S}.$$

Now, applying [7, 16.1.7], we can observe that

$$\mathcal{S}' = L(s', s)' = (s \otimes s)' = s' \otimes s' = L(s, s').$$

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We can also view $\mathcal{S}'$ as the space of matrices. Recall that $\mathcal{S} \simeq \mathcal{K}^\infty$ consists of the so-called rapidly decreasing matrices, that is an infinite matrix $x = (x_{ij})_{i,j\in\mathbb{N}}$ belongs to $\mathcal{S}$ if and only if $\sup\{|x_{ij}|(ij)^k : i, j \in \mathbb{N}\}$ is finite for every $k \in \mathbb{N}_0$. Since, by [8, definition on page 326], $\mathcal{S}$ is a Köthe sequence space, we can use [8, Lemma 27.11] to observe that $\mathcal{S}'$ is again a space of matrices, the so-called slowly increasing ones. More precisely,

$$\mathcal{S}' = \{\phi = (\phi_{ij})_{i,j\in\mathbb{N}} \mid \sup\{|\phi_{ij}|(ij)^{-k} : i, j \in \mathbb{N}\} < +\infty \text{ for some } k \in \mathbb{N}_0\}.$$

The duality in the matrix language is given by the trace, that is, if $x \in \mathcal{S}, \phi \in \mathcal{S}'$, then

$$\phi(x) := \langle\langle x, \phi\rangle\rangle = \sum_{i,j=1}^{+\infty} x_{ij}\overline{\phi}_{ij}.$$

Analogously to the continuous inclusion $s \hookrightarrow s'$, also for matrices we easily observe that $\mathcal{S} \hookrightarrow \mathcal{S}'$ continuously. This shows in particular that every rapidly decreasing matrix is a functional on $\mathcal{S}$.

Observe now that the order in $\mathcal{S}$ is inherited from $B(\ell_2)$. This is a consequence of a continuous inclusion $\mathcal{S} \hookrightarrow B(\ell_2)$ and [1, Proposition A.2.8] (see [5, Corollary 2.5]). Therefore, we can use this order to define positive functionals on the noncommutative Schwartz space. To this end, let $\phi \in \mathcal{S}'$. We say that it is *positive* if it maps positive elements into nonnegative numbers, that is, $\phi(x) \geq 0$ for every $x \geq 0$ in $\mathcal{S}$. By $\mathcal{S}'_+$, we denote the cone of positive functionals on the noncommutative Schwartz space. We can also define self-adjoint functionals in the usual manner. First we define

$$\phi^*(x) := \overline{\phi(x^*)}, \quad x \in \mathcal{S}$$

and we say that ϕ is *self adjoint* if $\phi = \phi^*$. As in the C*-algebra case, we can easily show that ϕ is self adjoint if and only if $\phi(x)$ is real for any self-adjoint $x \in \mathcal{S}$. If we represent $\phi \in \mathcal{S}'$ as a matrix, then ϕ^* is represented by the transposed complex-conjugate matrix. Self adjointness of $\phi \in \mathcal{S}'$ means that the representing matrix is self adjoint.

Let us now give several 'easy-to-obtain' consequences of the above definition. In what follows, u_n stands for the infinite matrix $\left(\begin{smallmatrix} I_n & 0 \\ 0 & 0 \end{smallmatrix}\right)$ with I_n being the $n \times n$ identity map. Consequently, $u_n\phi u_n$ (being the matrix multiplication) is the nth truncation of ϕ.

Proposition 3.1. *Let ϕ be a functional on the noncommutative Schwartz space.*

(i) *If ϕ is a rapidly decreasing matrix, then $\phi \geq 0$ in $\mathcal{S}'$ if and only if $\phi \geq 0$ in $\mathcal{S}$.*
(ii) *$\phi \geq 0$ if and only if $u_n\phi u_n \geq 0$ for every $n \in \mathbb{N}$.*

Proof. (i) Suppose that $\phi \geq 0$ in $\mathcal{S}$ and take a positive $y \in \mathcal{S}$. By [11, Proposition 3(ii)], $y = xx^*$ for some $x \in \mathcal{S}$. Since $xx^* = (\sum_k x_{ik}\overline{x}_{jk})_{i,j}$,

$$\phi(xx^*) = \sum_{i,j=1}^{+\infty} \overline{\phi}_{ij} \sum_{k=1}^{+\infty} x_{ik}\overline{x}_{jk} = \sum_{k=1}^{+\infty} \langle \xi^k, \phi\xi^k\rangle,$$

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where $\xi^k = (x_{jk})_{j\in\mathbb{N}} \in s'$. By [11, Proposition 3(viii)], $\langle \xi, \phi\xi \rangle \geq 0$ for every $\xi \in s'$ and, finally, $\phi \geq 0$ in $\mathcal{S}'$.

Let now $\phi \geq 0$ in $\mathcal{S}'$ and take an arbitrary $\xi \in s'$. Then

$$\langle \phi\xi, \xi \rangle = \lim_{n\to+\infty} \langle \phi u_n \xi, u_n \xi \rangle.$$

Now, for every $n \in \mathbb{N}$, define the infinite matrix $x_n \in \mathcal{S}$ by

$$x_n := \begin{pmatrix} \xi_1 & \dots & \xi_n & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

By assumption, $\phi \geq 0$ in $\mathcal{S}'$ and therefore $\phi(x_n^* x_n) \geq 0$ for every $n \in \mathbb{N}$. Since $\phi(x_n^* x_n) = \langle \phi u_n \xi, u_n \xi \rangle$, we obtain $\langle \phi\xi, \xi \rangle \geq 0$. Consequently, by [11, Proposition 3(viii)], $\phi \geq 0$ in $\mathcal{S}$.

(ii) Suppose that $\phi \geq 0$. Then, for any $n \in \mathbb{N}$, $u_n \phi u_n$ and arbitrary $x \geq 0$ in $\mathcal{S}$,

$$\langle\!\langle u_n \phi u_n, x \rangle\!\rangle = \langle\!\langle \phi, u_n x u_n \rangle\!\rangle.$$

By [11, Proposition 3(viii)], $x \geq 0$ if and only if $u_n x u_n \geq 0$ for all $n \in \mathbb{N}$ and, consequently, $u_n \phi u_n \geq 0$ in $\mathcal{S}'$. Suppose now that the converse holds. Then for any $x \in \mathcal{S}$ we have

$$\langle\!\langle \phi, x \rangle\!\rangle = \lim_{n\to+\infty} \langle\!\langle u_n \phi u_n, x \rangle\!\rangle.$$

Applying this to $x \geq 0$, we get the conclusion. □

Recall that an infinite matrix $\phi \in \mathcal{S}'$ if and only if

$$\|\phi\|_k^* := \sup\{|\phi_{ij}|(ij)^{-k} : i, j \in \mathbb{N}\} < +\infty$$

for some $k \in \mathbb{N}_0$. Repeating the proof of [11, Lemma 5], we get the following result.

Proposition 3.2. *Suppose that $\phi = (\phi_{ij})_{i,j\in\mathbb{N}}$ is a positive functional on the noncommutative Schwartz space. Then*

$$\|\phi\|_k^* = \sup\{\phi_{jj} j^{-2k} : j \in \mathbb{N}\}.$$

4. Construction

In this final section we provide a way of decomposing a self-adjoint functional on the noncommutative Schwartz space into a difference of two positive functionals. We will also show that it is minimal in the following, natural sense. Suppose that $\phi = \phi_+ - \phi_-$ is such a decomposition. We will say that it is *minimal* if any other decomposition $\phi = \phi_1 - \phi_2$ of a self-adjoint $\phi \in \mathcal{S}'$ into a difference of two positive functionals $\phi_1, \phi_2 \in \mathcal{S}'$ with the additional property $\phi_1 \leq \phi_+$, $\phi_2 \leq \phi_-$ implies that $\phi_1 =$

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ϕ_+, $\phi_2 = \phi_-$. For the purpose of this construction, we denote by $\mathbf{0}_{m,n}$, $1 \leq m, n \leq \infty$, the $m \times n$ matrix of zeros. An element $\phi \in \mathcal{S}'$ of the form

$$\left(\begin{array}{c|cccc} \mathbf{0}_{k-1,k-1} & & \mathbf{0}_{k-1,\infty} & & \\ \hline & \xi_{kk} & \xi_{k,k+1} & \xi_{k,k+2} & \cdots \\ \mathbf{0}_{\infty,k-1} & \xi_{k+1,k} & & & \\ & \xi_{k+2,k} & & \mathbf{0}_{\infty,\infty} & \\ & \vdots & & & \end{array}\right),$$

where nonzero elements run east and south of the (k, k)th entry, will be called a *corner matrix*.

Step 1. Particular case. We start our construction with a self-adjoint corner matrix

$$\phi = \begin{pmatrix} \xi_1 & \xi_2 & \xi_3 & \cdots \\ \overline{\xi}_2 & & & \\ \overline{\xi}_3 & & \mathbf{0}_{\infty,\infty} & \\ \vdots & & & \end{pmatrix}$$

for some $\xi \in s'$. Now we define

$$\psi := (\psi_{ij})_{i,j\in\mathbb{N}} = \begin{pmatrix} \max\{1, \xi_1^2\} & \xi_2 & \xi_3 & \cdots \\ \overline{\xi}_2 & & & \\ \overline{\xi}_3 & & (\overline{\xi}_i \xi_j)_{i,j>1} & \\ \vdots & & & \end{pmatrix}.$$

Obviously, $\psi \in \mathcal{S}'$, since

$$\|\psi\|_k^* \leq \max\{|\xi|_k', |\xi|_k'^2\}$$

and $\xi \in s'$. Let now $\psi^n \in \mathcal{S}$ be the nth truncation of ψ and take an arbitrary $\eta \in s'$. Then

$$\langle \psi^n \eta, \eta \rangle \geq \Big| \sum_{j=1}^{n} \psi_{1j} \eta_j \Big|^2 \geq 0.$$

Therefore, $\psi^n = u_n \psi u_n \geq 0$ for every $n \in \mathbb{N}$ (as an element of $\mathcal{S}$). Consequently, by Proposition 3.1, $\psi \geq 0$. Similarly, we can show that $\psi - \phi \geq 0$. Finally,

$$\phi = \psi - (\psi - \phi)$$

is a decomposition of a corner matrix into a difference of two positive functionals.

Step 2. General case. Let now $\phi = (\phi_{ij})_{i,j\in\mathbb{N}}$ be an arbitrary self-adjoint functional on the noncommutative Schwartz space and denote by $(e_{ij})_{i,j\in\mathbb{N}}$ the sequence of matrix units. That is, e_{ij} is an infinite matrix with one in the (i, j)th entry and zeros elsewhere. We now represent ϕ as an infinite sum of self-adjoint corner matrices. More precisely,

$$\phi = \sum_{k=1}^{+\infty} \Big(\phi_{kk} e_{kk} + \sum_{j=k+1}^{+\infty} (\phi_{kj} e_{kj} + \phi_{jk} e_{jk}) \Big) =: \sum_{k=1}^{+\infty} \phi^k,$$

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where each ϕ^k is a corner matrix. We now apply to those corner matrices the procedure from Step 1. This leads to

$$\phi^k_+ = \left(\begin{array}{c|cccc} \mathbf{0}_{k-1,k-1} & & \mathbf{0}_{k-1,\infty} & & \\ \hline & \max\{1,\phi^2_{kk}\} & \phi_{k,k+1} & \phi_{k,k+2} & \dots \\ \mathbf{0}_{\infty,k-1} & \phi_{k+1,k} & & & \\ & \phi_{k+2,k} & (\phi_{ik}\phi_{kj})_{i,j>k} & & \\ & \vdots & & & \end{array}\right)$$

and

$$\phi^k_- = \left(\begin{array}{c|cccc} \mathbf{0}_{k-1,k-1} & & \mathbf{0}_{k-1,\infty} & & \\ \hline & \max\{1,\phi^2_{kk}\} - \phi_{kk} & 0 & 0 & \dots \\ \mathbf{0}_{\infty,k-1} & 0 & & & \\ & 0 & (\phi_{ik}\phi_{kj})_{i,j>k} & & \\ & \vdots & & & \end{array}\right),$$

where, for all $k \in \mathbb{N}$, we have $\phi^k_+, \phi^k_- \geq 0$ and $\phi^k = \phi^k_+ - \phi^k_-$. Finally, we define

$$\phi_+ := \sum_{k=1}^{+\infty} \phi^k_+ \quad \text{and} \quad \phi_- := \sum_{k=1}^{+\infty} \phi^k_-.$$

Obviously, ϕ_+ and ϕ_- are positive, as sums of positive functionals. If we show that these two matrices are slowly increasing, then we will obtain a decomposition $\phi = \phi_+ - \phi_-$ into a difference of two positive functionals. To this end, we rewrite $\phi_+ = (\psi_{ij})_{i,j\in\mathbb{N}}$ in the following form:

$$\psi_{ij} = \begin{cases} \phi_{1j} & \text{if } i = 1, \\ \phi_{i1} & \text{if } j = 1, \\ \max\{1, \phi^2_{jj}\} + \sum_{k=1}^{j-1} |\phi_{jk}|^2 & \text{if } i = j > 1, \\ \phi_{ij} + \sum_{k=1}^{\min\{i,j\}-1} \phi_{ik}\phi_{kj} & \text{if } i, j > 1,\ i \neq j. \end{cases}$$

Since $\phi \in \mathcal{S}'$, there is an $m \in \mathbb{N}$ such that

$$\|\phi\|^*_m = \sup_{i,j\in\mathbb{N}} \{|\phi_{ij}|(ij)^{-m}\} < +\infty.$$

Equivalently, there exist $m \in \mathbb{N}$ and a constant $C \geq 1$ such that, for all $i, j \in \mathbb{N}$,

$$|\phi_{ij}| \leq C(ij)^m.$$

We divide the calculations into three cases:

(i) $i = 1$ or $j = 1$:

$$|\psi_{1j}| = |\phi_{1j}| \leq Cj^m, \quad |\psi_{i1}| = |\phi_{i1}| \leq Ci^m;$$

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(ii) $i = j > 1$:

$$|\psi_{jj}| \le \max\{1, \phi_{jj}^2\} + C^2 \sum_{k=1}^{j-1} (jk)^m \le C^2(j^{4m} + j^{4m+1}) \le 2C^2 j^{4m+1};$$

(iii) $i \ne j,\ i, j > 1$:

$$|\psi_{ij}| \le |\phi_{ij}| + C^2 \sum_{k=1}^{\min\{i,j\}} (ik)^m (kj)^m \le C^2((ij)^m + ii^{2m} j^{2m}) \le 2C^2 (ij)^{2m+1}.$$

In all cases we get $|\psi_{ij}| \le 2C^2(ij)^{2m+1}$. Therefore, $\|\phi_+\|_{2m+1}^* < +\infty$ and, consequently, $\phi_+ \in \mathcal{S}'$. Finally, $\phi_- = \phi_+ - \phi \in \mathcal{S}'$ and we get the desired decomposition.

Step 3. Minimality. Let $\phi \in \mathcal{S}'$ be a self-adjoint functional on the noncommutative Schwartz space and define

$$Z_\phi := \{(\phi_1, \phi_2) : \phi_1, \phi_2 \in \mathcal{S}'_+, \phi = \phi_1 - \phi_2\}.$$

By Step 2 of the above construction, Z_ϕ is nonempty for every $\phi \in \mathcal{S}'$. We define in Z_ϕ a partial order relation as follows:

$$(\phi_1, \phi_2) \le (\psi_1, \psi_2) \Leftrightarrow \phi_1 \le \psi_1 \wedge \phi_2 \le \psi_2.$$

Let now $(\psi_\alpha, \psi_\alpha)_\alpha$ be a chain in Z_ϕ. For every $x \in \mathcal{S}_+$, the net $(\phi_\alpha(x))_\alpha$ is nonincreasing and bounded from below (by zero). Consequently, $\lim_\alpha \phi_\alpha(x)$ exists for every positive element x in the noncommutative Schwartz space. By [11, Proposition 3(v)], positive elements span the whole of $\mathcal{S}$ and therefore we may define

$$\phi_+(y) := \lim_\alpha \phi_\alpha(y), \quad y \in \mathcal{S}.$$

Similarly,

$$\phi_-(y) := \lim_\alpha \psi_\alpha(y), \quad y \in \mathcal{S}.$$

Obviously, $\phi = \phi_+ - \phi_-$ and $\phi_+, \phi_- \ge 0$. It is also not difficult to see that $(\phi_+, \phi_-) \in Z_\phi$ is an upper bound for the chain $(\psi_\alpha, \psi_\alpha)_\alpha$. Now an application of the Kuratowski–Zorn lemma gives us a minimal element in Z_ϕ.

We may now state the main result of this section.

Theorem 4.1. *Every continuous, linear and self-adjoint functional on the noncommutative Schwartz space admits a minimal decomposition into a difference of two positive functionals.*

Remark. The above construction can by no means be thought of as unique. For, if we take an $n \times n$ matrix

$$\phi := \begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \\ & & \dots & & \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix},$$

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then Step 2 of our construction leads to

$$\phi = \begin{pmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 1 & \dots & 1 \\ & & \dots & & \\ 0 & 1 & 1 & \dots & 1 \end{pmatrix}.$$

This decomposition is not minimal, since we also have

$$\phi = \begin{pmatrix} \frac{1}{2} & \dots & \frac{1}{2} \\ \vdots & \ddots & \vdots \\ \frac{1}{2} & \dots & \frac{1}{2} \end{pmatrix} - \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \dots & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \dots & \frac{1}{2} \\ & & \dots & & \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \dots & \frac{1}{2} \end{pmatrix}.$$

Some easy (but tedious) calculations show that this last decomposition is minimal. However, the spectral decomposition (which is also minimal) gives us

$$\phi = \begin{pmatrix} \frac{\sqrt{n-1}}{2} & \frac{1}{2} & \frac{1}{2} & \dots & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2\sqrt{n-1}} & \frac{1}{2\sqrt{n-1}} & \dots & \frac{1}{2\sqrt{n-1}} \\ & & \dots & & \\ \frac{1}{2} & \frac{1}{2\sqrt{n-1}} & \frac{1}{2\sqrt{n-1}} & \dots & \frac{1}{2\sqrt{n-1}} \end{pmatrix}$$

$$- \begin{pmatrix} \frac{\sqrt{n-1}}{2} & -\frac{1}{2} & -\frac{1}{2} & \dots & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2\sqrt{n-1}} & \frac{1}{2\sqrt{n-1}} & \dots & \frac{1}{2\sqrt{n-1}} \\ & & \dots & & \\ -\frac{1}{2} & \frac{1}{2\sqrt{n-1}} & \frac{1}{2\sqrt{n-1}} & \dots & \frac{1}{2\sqrt{n-1}} \end{pmatrix}.$$

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