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ON CHARACTERS OF HEIGHT ZERO

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Abstract

Every irreducible ordinary character in a p-block of a finite metabelian group is of height 0 if and only if the defect group of the p-block is abelian.

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Brauer conjectured that all ordinary characters in a p-block B of a finite group G have height 0 if and only if the defect group of B is abelian. Fong in [2], [3], [4], and [5] has given proofs of various cases of this conjecture. In this note we prove this for the metabelian groups.

THEOREM. *Let G be a finite metabelian group and B be a p-block of G. Then every ordinary character of B has height* 0 *if and only if the defect group of B is abelian.*

PROOF. We use the results in [1]. Let Q be the p-Sylow subgroup of the commutator group G', then $G' = Q \times A$, where $p \nmid |A|$. Let H be a subgroup of G', $H \supseteq Q$, such that G'/H is cyclic. Then $H = Q \times \Lambda$, $p \nmid |\Lambda|$. For any subgroup L of G' let $K(L) \supseteq G'$, and $K(L)/L$ be a maximal abelian subgroup of $N(L)/L$. If $\Lambda \subseteq L \subseteq H$, we may pick $K(\Lambda) \subseteq K(L) \subseteq K(H)$. Let σ be a linear modular representation of $K(\Lambda)$ with $\ker \sigma \cap G' = H$ and $B(\sigma, H)$ be the collection of all ordinary representations T'^G where T' is a linear representation of $K(L) \supseteq K(\Lambda)$, $L \subseteq H$, G'/L cyclic, H/L a p-group, with $\ker T' \cap G' = L$

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and $\bar{T}'_{K(\Lambda)}$ G-conjugate to σ. See [1, §2]. All these representations T'^G are irreducible. Include in $B(\sigma, H)$ the characters of T'^G and the irreducible composition factors of $\bar{T}'^G$. From [1, §4], $B(\sigma, H)$ is a p-block and the p-Sylow subgroup P of $K(H)$ is its defect group. [Any p-block of G is given by $B(\sigma, H)$, σ and H as described above.] Note that $P \cap G' = Q$.

First assume P is abelian and let $\pi \in P$. Although this follows from the results in [3] and [4], we give below an easy proof for the special case. Since $K(H)/H$ is abelian, $\pi^{-1}k\pi \equiv k (\text{mod } H)$ for all $k \in A$. But $\Lambda = H \cap A$, and hence $\pi^{-1}k\pi \equiv k (\text{mod } \Lambda)$ for all $k \in A$. Since $\pi^{-1}k\pi = k$ for all $k \in Q$, it follows that $\pi^{-1}k\pi \equiv k (\text{mod } \Lambda)$ for all $k \in G'$. Thus $P \subseteq K(\Lambda)$ and $p \nmid |K(H)/K(\Lambda)|$. Since every (irreducible) representation T'^G in $B(\sigma, H)$ is induced by a linear representation T' of $K(L) \supseteq K(\Lambda)$, of some L, it follows that the degree of T'^G divides $|G/K(\Lambda)|$ but is divisible by $|G/K(H)|$. Thus every ordinary character in $B(\sigma, H)$ has height 0.

Now assume P is non-abelian. We shall construct an irreducible character in $B(\sigma, H)$ of height greater than 0. Let $R = P \cap K(\Lambda)$. If $k \in K(\Lambda)$, $\pi \in R$, then $k^{-1}\pi k \equiv \pi (\text{mod } \Lambda)$. But $\Lambda \cap R = 1$, and thus $k^{-1}\pi k = \pi$ for all $\pi \in R$ and all $k \in K(\Lambda)$. Thus P is not contained in $K(\Lambda)$, that is, $R \subset P$, R abelian, and $K(\Lambda) = R \times K_1$, $p \nmid |K_1|$, with $\Lambda \subseteq K_1$, K_1/Λ abelian. There is a linear ordinary representation V of $K(\Lambda)$, $V(\pi) = 1$ for all $\pi \in R$ and $\bar{V} = \sigma$, ker V = ker σ. Since $1 \neq P' \subseteq R$ there is a linear ordinary representation W_0 of P', ker $W_0 = L_0$ and $|P'/L_0| > 1$. Since R/L_0 is abelian, an extension W_1 of W_0 to R exists. Here ker $W_1 \cap P' = L_0$. Define the linear representation W of $K(\Lambda)$ by $W(\pi) = W_1(\pi)$ for all $\pi \in R$ and $W(k) = 1$ for all $k \in K_1$. Let $T(k) = V(k)W(k)$ for all $k \in K(\Lambda)$. Then T is a linear representation of $K(\Lambda)$ and $\bar{T} = \sigma$. Let $L = \ker T \cap G'$ and $K(L) \supseteq K(\Lambda)$. Then $L \cap P' = L_0$ and thus there are $\pi_1 \in P'$ and $\pi \in P$ such that $\pi^{-1}\pi_1\pi \not\equiv \pi_1 (\text{mod } L_0)$. This means that $\pi^{-1}\pi_1\pi \not\equiv \pi_1 (\text{mod } L)$ or P is not contained in $K(L)$. Let T' be an extension of T to $K(L)$, then T'^G is irreducible and since $\bar{T}'_{K(\Lambda)} = \sigma$, $T'^G \in B(\sigma, H)$. Now since T'^G is of degree $|G/K(L)|$ and $p \mid |K(H)/K(L)|$, it follows that its character χ is of positive height. This completes the proof of the theorem.

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