ON MEROMORPHIC OPERATORS, II

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1. This paper forms a continuation of **(1)**, extending the concept of a meromorphic operator to not necessarily bounded, closed linear operators in complex Banach space. Let T denote such an operator with range and domain in Banach space X. We shall study the class of such operators T where $\lambda = 0$ and $\lambda = \infty$ are the only allowable points of accumulation of $\sigma(T)$ and every isolated point of $\sigma(T)$ is a pole of $R_\lambda(T)$. We shall write $\mathfrak{M}(0, \infty)$ to represent the class of such operators. If $\lambda = 0$ ($\lambda = \infty$) is the only allowable point of accumulation of $\sigma(T)$, we shall write $\mathfrak{M}(0)$ ($\mathfrak{M}(\infty)$) to denote the corresponding class of operators.

If the non-zero points of $\sigma(T)$ are eigenvalues of finite multiplicities, then we shall use the subscript "f" to denote the corresponding classes, e.g. $\mathfrak{M}_f(0, \infty)$, etc. We clearly have the inclusions

$$\mathfrak{M}(0, \infty) \supseteq \mathfrak{M}(0) \supseteq \mathfrak{M}, \qquad \mathfrak{M}(0, \infty) \supseteq \mathfrak{M}(\infty) \supseteq \mathfrak{M}_f(\infty),$$
$$\mathfrak{M}(0) \supseteq \mathfrak{M}_f(0) \supseteq \mathfrak{N},$$

where $\mathfrak{M}$ was defined in **(1)** and $\mathfrak{N}$ in **(2)**.

For any operator T, we define $n(T)$ as the dimension of $N(T)$ and $d(T)$ as the codimension of $R(T)$. We note that, since we are discussing poles of the resolvent, there is no ambiguity in speaking of "finite multiplicity." For if λ_0 is such a pole, it is customary to define $n(\lambda_0 - T)$ as the *algebraic multiplicity* and, if E_0 is the spectral projection associated with the single point λ_0, then the dimension of $R(E_0)$ is called the *spectral multiplicity* of λ_0. By **(3**, Theorem 5.8-A**)**, $R(E_0) = N[(\lambda_0 - T)^m]$ where $m = \alpha(\lambda_0)$, where $\alpha(\lambda_0) = \alpha(\lambda_0 - T)$, the ascent of $\lambda_0 - T$. Clearly

$$n(\lambda_0 - T) \leqslant \dim R(E_0)$$

and by **(3**, Lemma 1**)**

$$\dim R(E_0) \leqslant \alpha(\lambda_0) n(\lambda_0 - T).$$

Hence if one of the multiplicities is finite, so is the other.

2. Example. The study of certain differential operators gives rise to elements of $\mathfrak{M}(\infty)$. The following result is typical; for the proof, see **(7)**.

Let $X = L_p[a, b]$, $1 < p < \infty$, let α, β be fixed complex numbers, let $q(t) \in C[a, b]$. Define

$$D(T) = \{x \in X : x' \text{ is absolutely continuous and } x'' \in X; x(a)\cos\alpha + x'\sin\alpha = x(b)\cos\beta + x'(b)\sin\beta = 0\},$$
$$Tx = -x'' + q(t)x.$$

Then $T \in \mathfrak{M}(\infty)$.

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3. Decomposition of $\mathfrak{M}(0, \infty)$.

THEOREM 1. *Every operator T in $\mathfrak{M}(0, \infty)$ can be written as $T = T_1 + T_2$, $T_1 T_2 = T_2 T_1 = 0$, where $T_1 \in \mathfrak{M}$, $T_2 \in \mathfrak{M}(\infty)$ in such a way that*

$$R_\lambda(T) = R_\lambda(T_1) + R_\lambda(T_2) - 2I/\lambda. \tag{1}$$

The above assertion is also true if we replace $\mathfrak{M}(0, \infty)$, $\mathfrak{M}$, and $\mathfrak{M}(\infty)$ respectively by $\mathfrak{M}_f(0, \infty)$, $\mathfrak{N}$, and $\mathfrak{M}_f(\infty)$.

Proof. Choose $r > 0$ such that if $C = \{\lambda: |\lambda| = r\}$, then $C \cap \sigma(T) = \emptyset$. Define

$$\sigma_1 = \{\lambda: |\lambda| < r\} \cap \sigma(T),$$
$$\sigma_2 = [\{\lambda: |\lambda| > r\} \cap \sigma(T)] \cup \{\infty\}.$$

Then σ_1 and σ_2 are spectral sets of T. If E_1 and E_2 are, respectively, the associated spectral projections, then it is clear that $E_1 + E_2 = I$, $E_1 E_2 = E_2 E_1 = 0$.

Define $T_i = TE_i$, $i = 1, 2$. Then certainly $T = T_1 + T_2$ and

$$T_1 T_2 = T_2 T_1 = 0.$$

Now since σ_1 does not contain ∞, it is known from (**6**, Theorem 5.7-B), that $R(E_1) \subseteq D(T)$, so that T_1 is defined on all of X. It is simple to verify that T_1 is a closed operator so that, by the closed-graph theorem, it must be a member of $B(X)$.

We now apply the operational calculus for unbounded operators, as discussed in (**6**, pp. 287–296), to deduce the remaining assertions of the theorem. Let D, D_1, and D_2 be Cauchy domains such that $D \supseteq \sigma(T)$, $D_i \supseteq \sigma_i$, $i = 1, 2$, $\bar{D}_1 \cap \bar{D}_2 = \emptyset$, and $D_1 \cup D_2 = D$. Let $f_i(\lambda)$ be defined to equal 1 when $\lambda \in \bar{D}_i$ and to equal zero elsewhere. We shall write $B(D)$ to denote the boundary of any Cauchy domain D and $+B(D)$ for the positively oriented boundary.

Then, for any $\mu \notin D_1$, we can write, using the above-mentioned operational calculus,

$$\begin{aligned} R_\mu(T_1) = R_\mu(TE_1) &= \frac{I}{\mu} + \frac{1}{2\pi i}\oint_{+B(D)} \frac{1}{\mu - \lambda f_1(\lambda)} R_\lambda(T)d\lambda \\ &= \frac{I}{\mu} + \frac{1}{2\pi i}\oint_{+B(D_1)} \frac{1}{\mu - \lambda} R_\lambda(T)d\lambda + \frac{1}{2\pi i}\oint_{+B(D_2)} \frac{1}{\mu} R_\lambda(T)d\lambda \\ &= \frac{I + E_2}{\mu} + \frac{1}{2\pi i}\oint_{+B(D)} \frac{f_1(\lambda)}{\mu - \lambda} R_\lambda(T)d\lambda \\ &= \frac{I + E_2}{\mu} + \left[\frac{1}{2\pi i}\oint_{+B(D)} f_1(\lambda)R_\lambda(T)d\lambda\right] \\ &\qquad \times \left[\frac{1}{2\pi i}\oint_{+B(D)} \frac{1}{\mu - \lambda} R_\lambda(T)d\lambda\right] \\ &= \frac{I + E_2}{\mu} + E_1 R_\mu(T). \end{aligned} \tag{2}$$

Similarly

$$(3)\qquad \begin{aligned} R_\mu(T_2) &= R_\mu(TE_2)\\ &= \frac{1}{2\pi i}\oint_{+B(D)} \frac{1}{\mu - \lambda f_2(\lambda)} R_\lambda(T)d\lambda\\ &= \frac{1}{2\pi i}\oint_{+B(D_1)} \frac{1}{\mu} R_\lambda(T)d\lambda + \frac{1}{2\pi i}\oint_{+B(D_2)} \frac{1}{\mu - \lambda} R_\lambda(T)d\lambda\\ &= \frac{E_1}{\mu} + \frac{1}{2\pi i}\oint_{+B(D)} \frac{1}{\mu - \lambda} f_2(\lambda)R_\lambda(T)d\lambda\\ &= \frac{E_1}{\mu} + \left[\frac{1}{2\pi i}\oint_{+B(D)} \frac{1}{\mu - \lambda} R_\lambda(T)d\lambda\right]\cdot f_2(T)\\ &= \frac{E_1}{\mu} + E_2\, R_\mu(T). \end{aligned}$$

Adding (2) and (3) and rearranging, we get (1).

Finally, suppose that every non-zero point of $\sigma(T)$ is an eigenvalue of finite multiplicity for T. Now, it is not difficult to show that

$$(4)\qquad \sigma(T_i) = \sigma_i \cup \{0\} \qquad (i = 1, 2).$$

For consider $\lambda_0 \in \sigma(T)$, $\lambda_0 \neq 0$. Then, if we write E_0 for the corresponding spectral projection,

$$(5)\qquad E_i\, E_0 = \begin{cases} 0 & \text{if } \lambda_0 \notin \sigma_i, \\ E_0 & \text{if } \lambda_0 \in \sigma_i, \end{cases} \quad i = 1, 2.$$

For $E_i\, E_0 = f_i(T)f_0(T)$, where we define $f_0(\lambda)$ to be 1 near λ_0 and zero on the remaining points of $\sigma(T)$. Now $f_i(T)f_0(T) = (f_i f_0)(T)$ and the result follows from the definition of f_i.

It is clear from (2) and (3) that the only possible points in $\sigma(T_i)$ are $\lambda = 0$ or points of $\sigma(T)$. Also, if we consider $R_\lambda(T)$ near λ_0, then $R_\lambda(T)$ has principal part

$$\sum_{n=1}^{\alpha(\lambda_0)} \frac{B_n}{(\lambda - \lambda_0)^n} \qquad \text{with } B_n = (T - \lambda_0)^{n-1}E_0.$$

See **(6**, p. 306**)**. From this, in conjunction with (2), (3), and (5)**,** it follows that (4) is valid and that the principal part of $R_\lambda(T_i)$ equals that of $R_\lambda(T)$ at any $\lambda \in \sigma(T_i)$, $\lambda \neq 0$. In particular, if every non-zero point of $\sigma(T)$ is an eigenvalue of finite multiplicity for $R_\lambda(T)$, the same must be true for T_i. This concludes the proof.

COROLLARY. *Every operator in $\mathfrak{M}(0)$ ($\mathfrak{M}_f(0)$) can be written as the sum of an operator in $\mathfrak{M}$ ($\mathfrak{N}$) and an operator whose spectrum has no non-zero points.*

Proof. Let $T \in \mathfrak{M}(0)$. Since $\sigma(T)$ is bounded, we can choose r so that $\sigma_2 = \{\infty\}$. Our assertion then follows. Similarly for $\mathfrak{M}_f(0)$.

THEOREM 2. *If $T \in \mathfrak{M}(0, \infty)$, then there exist Banach spaces X_1, X_2 which completely reduce T in the sense that*

(i) $T(D(T) \cap X_i) \subseteq X_i$,

(ii) $X = X_1 \oplus X_2$,

(iii) *if E_i is the projection of X onto X_i, then E_i is continuous and*

$$E_i D(T) \subseteq D(T)$$

for $i = 1, 2$.

Moreover, it is possible to choose X_i so that if we write the restriction of T to X_i as $T^{(i)}$, then $T^{(1)} \in \mathfrak{M}$ and $T^{(2)} \in \mathfrak{M}(\infty)$ and if $x = x_1 + x_2$ is the decomposition of x with $x_i \in X_i$, then

$$R_\lambda(T)x = R_\lambda(T^{(1)})x_1 + R_\lambda(T^{(2)})x_2.$$

Proof. We define $X_i = R(E_i)$ where E_i is defined in the proof of Theorem 1, so that certainly X_1 and X_2 completely reduce T as **(6**, p. 299**)** shows. By the restriction of T to X_i, we mean, of course, that $D(T^{(i)}) = X_i \cap D(T)$ and $T^{(i)}x = Tx$ for $x \in D(T^{(i)})$, $i = 1, 2$.

Again, from **(6**, p. 299**)**, we see that $X_1 \subseteq D(T)$ since σ_1 does not contain $\lambda = \infty$. Hence $D(T^{(1)}) = X_1$, and since this subspace X_1 is closed, we can deduce from the closed-graph theorem that $T^{(1)} \in B(X_1)$. Also $\sigma(T^{(i)}) = \sigma_i$ so that we must now show that each point of $\sigma(T^{(i)})$ is a pole of $R_\lambda(T^{(i)})$. Now $R_\lambda(T^{(i)}) \in B(X_i)$, and it is not difficult to show that $R_\lambda(T^{(i)})$ is the restriction of $R_\lambda(T)$ to X_i. For if $x_i \in X_i$ and $\lambda \in \rho(T) \subseteq \rho(T^{(i)})$

$$\begin{aligned}
&[(\lambda - T)^{-1} - (\lambda - T^{(i)})^{-1}]x_i \\
&\quad = (\lambda - T)^{-1}[(\lambda - T^{(i)}) - (\lambda - T)](\lambda - T^{(i)})^{-1}x_i \\
&\quad = (\lambda - T)^{-1}[T - T^{(i)}](\lambda - T^{(i)})^{-1}x_i = 0
\end{aligned}$$

since $(\lambda - T^{(i)})^{-1}x_i \in D(T^{(i)})$.

If we now take $\lambda_0 \in \sigma_i$ with $\lambda_0 \neq 0$, and consider the principal part of $R_\lambda(T)$ at $\lambda = \lambda_0$, then it is clear that $R_\lambda(T^{(i)})$ has principal part at λ_0 consisting of a finite number of terms. Hence $T^{(1)} \in \mathfrak{M}$ and $T^{(2)} \in \mathfrak{M}(\infty)$. Finally

$$R_\lambda(T)x = R_\lambda(T)(x_1 + x_2) = R_\lambda(T^{(1)})x_1 + R_\lambda(T^{(2)})x_2$$

for $\lambda \in \rho(T)$ and $x \in X$. This concludes the proof.

4. LEMMA 1. *Let $\alpha(T)$, $\delta(T)$, and $n(T)$ be finite and suppose that $p = \delta(T)$. Then, if $D(T^p)$ has finite codimension in X, $d(T)$ is finite.*

Proof. By **(6**, p. 273**)**, we can write

$$D(T^p) = [R(T^p) \cap D(T^p)] \oplus N(T^p).$$

Now $n(T^p) \leqslant pn(T)$ according to **(3**, Lemma 1**)**. Hence $R(T^p) \cap D(T^p)$ has finite codimension in $D(T^p)$ so that $R(T^p)$ has finite codimension in X. This implies that $d(T)$ is finite, for $d(T) \leqslant d(T^p)$.

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THEOREM 3. *Let T be a closed linear operator with $D(T^k)$ of finite codimension in X for each $k = 1, 2, \ldots$. Let X be a space of infinite dimension and $\emptyset \neq \sigma(T) \neq \mathfrak{C}$ where $\mathfrak{C}$ denotes the complex plane. Write Φ_T to denote the Fredholm region of T; that is, Φ_T is the set of complex numbers λ such that $n(\lambda - T)$ and $a(\lambda - T)$ are finite. Then $T \in \mathfrak{M}_f(0)$ if and only if $\Phi_T = \mathfrak{C} - \{0\}$.*

Proof. Let $T \in \mathfrak{M}_f(0)$; by definition, $n(\lambda - T)$ is finite for all $\lambda \neq 0$. Moreover, by (**6**, Theorem 5.8-A), $\alpha(\lambda - T)$ and $\delta(\lambda - T)$ are finite for all $\lambda \neq 0$ since such λ are either in $\rho(T)$ or are poles of $R_\lambda(T)$. By Lemma 1, $d(\lambda - T)$ is finite for all $\lambda \neq 0$. Hence $\Phi_T \supseteq \mathfrak{C} - \{0\}$. But Φ_T cannot be the entire complex plane; for it was shown in (**4**) that this would entail that X were finite dimensional.

Conversely, if $\Phi_T = \mathfrak{C} - \{0\}$, then by (**5**, Theorem 3.3) $n(\lambda)$ has a constant value K on Φ_T except at certain isolated points at which $n(\lambda) > K$. Since by assumption $\rho(T) \neq \emptyset$, it is clear that $\Phi_T \cap \rho(T)$ is an open set so that $K = 0$. Moreover, by (**5**, Theorem 3.1), $d(\lambda) - n(\lambda)$ is constant on Φ_T. Hence we can deduce that $n(\lambda) = d(\lambda) = 0$ for all non-zero λ except some isolated points. Hence the non-zero points of $\sigma(T)$ are isolated. Let λ_0 be such a point and E_0 be the corresponding spectral projection. Then it is known (**5**, p. 313) that λ_0 is a pole of $R_\lambda(T)$ if $R(E_0)$ is finite dimensional.

We shall denote $R(E_0)$ by X_0 and since E_0 is continuous, X_0 is closed and can therefore be considered as a Banach space. By (**6**, Theorem 5.7-B), $X_0 \subseteq D(T)$ since λ_0 is a finite point. Moreover, if T_0 is the restriction of T to X_0, then $R(T_0) \subseteq X_0$ and so, by the closed-graph theorem, we can consider T_0 as a member of $B(X_0)$ and $\sigma(T_0) = \{\lambda_0\}$. We shall show that $\Phi_{T_0} = \mathfrak{C}$. Then by (**5**, Theorem 3.2), we can deduce that X_0 is finite dimensional and so conclude the proof.

Now we have $X = X_0 \oplus N(E_0)$ from which we can easily deduce that

$$D(T) = X_0 \oplus [N(E_0) \cap D(T)]$$

and

$$R(T - \lambda_0) = (T - \lambda_0)X_0 \oplus (T - \lambda_0)[N(E_0) \cap D(T)]. \tag{6}$$

Now the restriction of T to $N(E_0)$ has spectrum $\sigma(T) - \{\lambda_0\}$, so that $T - \lambda_0$ maps $N(E_0) \cap D(T)$ onto $N(E_0)$. Thus (6) becomes

$$R(T - \lambda_0) = R(T_0 - \lambda_0) \oplus N(E_0). \tag{7}$$

Suppose now that $X_0 = R(T_0 - \lambda_0) \oplus Y$. Then

$$X = R(T_0 - \lambda_0) \oplus Y \oplus N(E_0),$$

which, by (7), becomes $X = R(T - \lambda_0) \oplus Y$. Hence, since $d(T - \lambda_0)$ is finite, Y is finite dimensional. Hence $d(T_0 - \lambda_0)$ is finite. Also $n(T_0 - \lambda_0) \leqslant n(T - \lambda_0)$ so that $\lambda_0 \in \Phi_{T_0}$. Since all other λ are in $\rho(T_0)$, $\Phi_{T_0} = \mathfrak{C}$. This completes the proof.

COROLLARY. *Let T have the properties assumed in the statement of the theorem. Then if $T \in \mathfrak{M}_f(\infty)$, X is finite dimensional.*

For if $T \in M_f(\infty)$, $\Phi(T) = \mathfrak{C}$. By **(4)**, this implies that X is finite dimensional.

5. An extension of the operational calculus. The operational calculus for closed linear operators with non-empty resolvent set which we have used so far is defined as follows:

We define $\tilde{\mathfrak{A}}_\infty(T)$ to be the class of functions which are analytic on some neighbourhood of $\sigma(T)$ and on some neighbourhood of $\lambda = \infty$. If two such functions take equal values on some open set containing $\sigma(T)$ and $\lambda = \infty$, we consider them to be in the same equivalence class; the family of such equivalence classes is denoted by $\mathfrak{A}_\infty(T)$. The operational calculus as described in **(6**, pp. 287–296**)** defines an algebraic homomorphism $f \to f(T)$ of the algebra $\mathfrak{A}_\infty(T)$ into $B(X)$ by the formula

$$f(T) = f(\infty)I + \frac{1}{2\pi i}\oint_{+B(D)} f(\lambda)R_\lambda(T)d\lambda.$$

However, this mapping has some unfortunate features:

(i) We begin with possibly unbounded T and obtain $f(T) \in B(X)$.

(ii) $\mathfrak{A}_\infty(T)$ is so restrictive that no non-constant entire function is included.

(iii) For our purposes, we wish to take T from $\mathfrak{M}(0, \infty)$ and obtain $f(T) \in \mathfrak{M}(0, \infty)$. As in **(1)**, this will necessitate $f(0) = 0$. But if $f \in \mathfrak{A}_\infty(T)$ and $\sigma(T)$ has point of accumulation at $\lambda = \infty$, then $\sigma[f(T)]$ is found to have point of accumulation at $f(\infty)$.

A simple step removes these disadvantages. We shall write $\mathfrak{A}_\infty^{(p)}(T)$ to denote the class obtained by applying the above equivalence relation to the family of functions, analytic on some neighbourhood of $\sigma(T)$ but allowing a pole at $\lambda = \infty$. For $f \in \mathfrak{A}_\infty^{(p)}(T)$, we shall write $n(f)$ to denote the order of the pole at infinity. Since, by assumption, there exists λ_0 in $\rho(T)$, we observe that $f_0(\lambda) \in \mathfrak{A}_\infty(T)$ where

$$f_0(\lambda) = \frac{f(\lambda)}{(\lambda - \lambda_0)^{n(f)+1}}.$$

We define

$$f(T) = f_0(T)(T - \lambda_0)^{n(f)+1}, \tag{8}$$

where $f_0(T)$ is defined by the operational calculus already described. We shall show that (8) defines an algebraic homomorphism from the algebra $\mathfrak{A}_\infty^{(p)}(T)$ into the class of closed linear operators. To begin with, we show that $f(T)$ is a closed linear operator with domain $D(T^{n(f)+1})$. It is easy to prove from the fact that T is closed that $T - \lambda_0$ is also closed. Moreover, so is $(T - \lambda_0)^s$ for each positive integer s. If we assume that we have shown that $(T - \lambda_0)^{s-1}$ is closed, then we consider a sequence $\{x_k\}$ in $D(T^s)$ such that $x_k \to x$ and $(T - \lambda_0)^s x_k \to y$. Now by assumption, $T - \lambda_0$ has a bounded inverse defined on X. Hence $(T - \lambda_0)^{s-1}x_k \to (T - \lambda_0)^{-1}y$ and by the inductive hypothesis, we conclude that $x \in D(T^{s-1})$ and $(T - \lambda_0)^{s-1}x = (T - \lambda_0)^{-1}y$, i.e. $x \in D(T^s)$ and $(T - \lambda_0)^s x = y$. Hence, in particular, $(T - \lambda_0)^{n(f)+1}$ is closed. Finally, suppose $\{x_k\}$ is a sequence in $D(T^{n(f)+1})$ such that $x_k \to x$ and $f(T)x_k \to y$. Then $f_0(T)x_k \to f_0(T)x$ and $(T - \lambda_0)^{n(f)+1}f_0(T)x_k \to y$ as $k \to \infty$. Thus $f_0(T)x \in D(T^{n(f)+1})$ and $(T - \lambda_0)^{n(f)+1}f_0(T)x = y$. But since $f_0(T)$ commutes with T, this means that $f(T)x = y$. Hence $f(T)$ is a closed linear operator.

We observe next that $f(T)$, defined by (8), is independent of λ_0. For suppose $\lambda_1 \in \rho(T)$ and that we write $k = n(f) + 1$; then

$$(9) \quad \left[\frac{1}{2\pi i}\oint_{+B(D)} \frac{f(\lambda)}{(\lambda - \lambda_1)^k} R_\lambda(T)d\lambda\right](T - \lambda_1)^k$$
$$= \left[\frac{1}{2\pi i}\oint_{+B(D)} \frac{f(\lambda)}{(\lambda - \lambda_0)^k}\left(\frac{\lambda - \lambda_0}{\lambda - \lambda_1}\right)^k R_\lambda(T)d\lambda\right](T - \lambda_1)^k$$
$$= \left[\frac{1}{2\pi i}\oint_{+B(D)} \frac{f(\lambda)}{(\lambda - \lambda_0)^k} R_\lambda(T)d\lambda\right]$$
$$\times\left[I + \frac{1}{2\pi i}\oint_{+B(D)} \left(\frac{\lambda - \lambda_0}{\lambda - \lambda_1}\right)^k R_\lambda(T)d\lambda\right](T - \lambda_1)^k.$$

Now

$$(10) \quad I + \frac{1}{2\pi i}\oint_{+B(D)} \left(\frac{\lambda - \lambda_0}{\lambda - \lambda_1}\right)^k R_\lambda(T)d\lambda$$
$$= I + \frac{1}{2\pi i}\oint_{+B(D)} \sum_{s=0}^{k} \binom{k}{s}\left(\frac{\lambda_1 - \lambda_0}{\lambda - \lambda_1}\right)^s R_\lambda(T)d\lambda$$
$$= I + \sum_{s=0}^{k} \binom{k}{s}(\lambda_1 - \lambda_0)^s \cdot \frac{1}{2\pi i}\oint_{+B(D)} \frac{R_\lambda(T)}{(\lambda - \lambda_1)^s} d\lambda$$
$$= I + \frac{1}{2\pi i}\oint_{+B(D)} R_\lambda(T)d\lambda$$
$$+ \sum_{s=1}^{k} \binom{k}{s}(\lambda_1 - \lambda_0)^s \left[\frac{1}{2\pi i}\oint_{+B(D)} \frac{R_\lambda(T)}{\lambda - \lambda_1} d\lambda\right]^s$$
$$= \sum_{s=0}^{k} \binom{k}{s}(\lambda_1 - \lambda_0)^s [R_{\lambda_1}(T)]^s \qquad \text{using } (\mathbf{6}, \text{Theorem 5.6-G})$$
$$= (T - \lambda_0)^k [R_{\lambda_1}(T)]^k.$$

Substituting (10) into (9), we get

$$\left[\frac{1}{2\pi i}\oint_{+B(D)} \frac{f(\lambda)}{(\lambda - \lambda_1)^k} R_\lambda(T)d\lambda\right](T - \lambda_1)^k$$
$$= \left[\frac{1}{2\pi i}\oint \frac{f(\lambda)}{(\lambda - \lambda_0)^k} R_\lambda(T)d\lambda\right](T - \lambda_0)^k,$$

and thereby show that our definition of $f(T)$ is independent of the choice of λ_0.

We next observe that if $f(\lambda)$ is analytic at $\lambda = \infty$, then $f(T)$ given by our method agrees with that given by the conventional operational calculus for both unbounded and bounded operators. Moreover, by **(6**, Theorem 5.6-G**)**, if f is a polynomial,

$$\text{say } f(\lambda) = \sum_0^k a_s \lambda^s, \qquad \text{then } f(T) = \sum_0^k a_s T^s.$$

We next show in routine fashion that the map $f \rightarrow f(T)$ is an algebraic homomorphism of the algebra of equivalence classes of $\mathfrak{A}_\infty^{(p)}$ into the class of closed linear operators with domain and range in X.

Consider $f, g \in \mathfrak{A}_\infty^{(p)}$ and suppose that $n(f) \geqslant n(g)$. Then it is apparent that both $f(T) + g(T)$ and $(f+g)(T)$ have the same domain, namely $D(T^{n(f)+1})$. Moreover,

$$(11) \qquad f(T) + g(T) = \frac{1}{2\pi i}\oint_{+B(D)} \left[f(\lambda)\left(\frac{T-\lambda_0}{\lambda-\lambda_0}\right)^{n(f)+1} + g(\lambda)\left(\frac{T-\lambda_0}{\lambda-\lambda_0}\right)^{n(g)+1}\right] R_\lambda(T) d\lambda$$

$$= (f+g)(T) + \frac{1}{2\pi i}\oint_{+B(D)} g(\lambda)\left[\left(\frac{T-\lambda_0}{\lambda-\lambda_0}\right)^{n(g)+1} - \left(\frac{T-\lambda_0}{\lambda-\lambda_0}\right)^{n(f)+1}\right] R_\lambda(T) d\lambda$$

$$= (f+g)(T) + \frac{1}{2\pi i}\oint_{+B(D)} g(\lambda)\left(\frac{T-\lambda_0}{\lambda-\lambda_0}\right)^{n(g)+1} R_\lambda(T) d\lambda - \left[\frac{1}{2\pi i}\oint_{+B(D)} g(\lambda)\left(\frac{T-\lambda_0}{\lambda-\lambda_0}\right)^{n(g)+1} R_\lambda(T) d\lambda\right] \times \left[\frac{1}{2\pi i}\oint \frac{R_\lambda(T)}{(\lambda-\lambda_0)^{n(f)-n(g)}} d\lambda\right](T-\lambda_0)^{n(f)-n(g)}.$$

Now

$$\frac{1}{2\pi i}\oint_{+B(D)} \frac{R_\lambda(T)}{(\lambda-\lambda_0)^k} d\lambda = [R_{\lambda_0}(T)]^k,$$

as we saw in deriving (10) from (9).

This fact, in conjunction with (11), gives the result.

Next, it is quite obvious that $(\alpha f)(T) = \alpha f(T)$. Finally, we consider the operators $f(T)g(T)$ and $(fg)T$; clearly the latter has domain $D(T^{n(f)+n(g)+1})$. It is not difficult to show that $f(T)g(T)$ is also defined on this domain; for if $x \in D(T^{n(f)+n(g)+1})$, then $(T-\lambda_0)^{n(g)+1}x \in D(T^{n(f)})$. We now make use of **(6**, Lemma 5.6-E**)**, observing that $g(\lambda)/\{(\lambda-\lambda_0)^{n(g)}\} \in \mathfrak{A}_\infty(T)$. Hence $g(T)x \in D(T^{n(f)+1})$ and $f(T)g(T)x$ is well defined. Thus

$$D(f(T)g(T)) \supseteq D(T^{n(f)+n(g)+1}).$$

It is not exactly clear that these two domains coincide, but this does not cause any problems. We show, in fact, that, for $x \in D(T^{n(f)+n(g)+1})$,

$$[(fg)T](x) = f(T)g(T)x. \tag{12}$$

Now

$$\begin{aligned} f(T)g(T)x &= \left[\frac{1}{2\pi i}\oint_{+B(D)} \frac{f(\lambda)}{(\lambda-\lambda_0)^{n(f)+1}} R_\lambda(T)d\lambda\right](T-\lambda_0)^{n(f)+1} \\ &\quad \times \left[\frac{1}{2\pi i}\oint_{+B(D)} \frac{g(\lambda)}{(\lambda-\lambda_0)^{n(g)+1}} R_\lambda(T)d\lambda\right](T-\lambda_0)^{n(g)+1} x \\ &= \left[\frac{1}{2\pi i}\oint_{+B(D)} \frac{R_\lambda(T)}{\lambda-\lambda_0} d\lambda\right] \\ &\quad \times \left[\frac{1}{2\pi i}\oint_{+B(D)} \frac{f(\lambda)g(\lambda)}{(\lambda-\lambda_0)^{n(f)+n(g)+1}} R_\lambda(T)d\lambda\right](T-\lambda_0)^{n(f)+n(g)+2} x \\ &= R_{\lambda_0}(T)(fg)(T)(T-\lambda_0)x = (fg)(T)x. \end{aligned}$$

It should be observed that, for any $h \in \mathfrak{A}_\infty$, $h(T)$ commutes with T and hence the above rearrangements are valid.

6; We now apply the operational calculus defined in §5 to operators in classes considered in §1. By virtue of Theorem 1, we may confine our attention to the classes $\mathfrak{M}(\infty)$ and $\mathfrak{M}_f(\infty)$. For, if $T \in \mathfrak{M}(0, \infty)$, then using the notation of Theorem 1, we have, if we write $k = n(f) + 1$,

$$\begin{aligned} f(T) = \Bigg[\frac{1}{2\pi i}\oint_{+B(D)} \frac{f(\lambda)}{(\lambda-\lambda_0)^k} R_\lambda(T_1)d\lambda + \frac{1}{2\pi i}\oint_{+B(D)} \frac{f(\lambda)}{(\lambda-\lambda_0)^k} R_\lambda(T_2)d\lambda \\ - \frac{1}{2\pi i}\oint_{+B(D)} \frac{2f(\lambda)}{\lambda(\lambda-\lambda_0)^k} d\lambda\Bigg](T-\lambda_0)^k. \end{aligned}$$

Now $T_1 \in \mathfrak{M}$ so that the first integral is $f(T_1)$. The properties of $f(T_1)$ were studied in **(1)**. The third integral is evidently a scalar. Hence we can write

$$f(T) = f(T_1) + f(T_2) - \alpha(f)(T-\lambda_0)^k$$

where

$$\alpha(f) = \frac{1}{2\pi i}\oint \frac{2f(\lambda)}{\lambda(\lambda-\lambda_0)^k} d\lambda.$$

Therefore only the nature of $f(T_2)$ requires elucidation.

THEOREM 4. *Let $T \in \mathfrak{M}(\infty)$ and $f \in \mathfrak{A}_\infty^{(p)} - \mathfrak{A}_\infty$. Then $f(T) \in \mathfrak{M}(\infty)$. If μ_0 is a non-zero point in the spectrum of $f(T)$ and E_0 is the corresponding spectral projection, then*

$$E_0 = \sum_{n \in S} E_n$$

where E_n is the spectral projection associated with $\lambda_n \in \sigma(T)$ and T and $S = \{n : f(\lambda_n) = \mu_0\}$.

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Proof. If q_n is the order of the pole of $R_\lambda(T)$ at $\lambda = \lambda_n$, then by **(7)**, it is possible to develop $R_\lambda(T)$ in a Mittag-Leffler expansion similar to that obtained when T was in $\mathfrak{M}$. Without loss of generality, let us assume that $\lambda_0 = 0$ so that $T^{-1} \in B(X)$. Then we can find integers $\{p_n\}$, operator-valued polynomials $P_n^{(p)}(\lambda)$, and operators $Q_n \in B(X)$ such that

$$(13) \qquad R_\lambda(T) = \sum_{n=1}^{\infty} [S_n(\lambda) - P_n^{(p_n)}(\lambda)] + \sum_{n=0}^{\infty} Q_n \lambda^n$$

where

$$S_n(\lambda) = \sum_{k=1}^{q_n} \frac{(T - \lambda_n)^{k-1}}{(\lambda - \lambda_n)^k} E_n$$

and

$$P_n^{(p)}(\lambda) = -\sum_{i=1}^{p} T^{-i} \lambda^{i-1} E_n.$$

It is shown in **(7)** that by suitable choice of $\{p_n\}$ we can obtain uniform convergence of (13) for $\lambda \in B(D)$. Proceeding as in **(1**, Theorem 6**)**, we write

$$(14) \qquad R_\mu(f(T)) = \frac{1}{2\pi i} \oint_{+B(D)} [\mu - f(\lambda)]^{-1} R_\lambda(T) d\lambda$$

$$= \sum_{n=1}^{\infty} \left[\sum_{k=1}^{q_n} I_{n,k}(T - \lambda_n)^{k-1} E_n + \sum_{k=1}^{p_n} I_{k-1} T^{-k} E_n \right] + \sum_{n=0}^{\infty} Q_n I_n,$$

defining

$$I_{n,k} = \frac{1}{2\pi i} \oint_{+B(D)} [\mu - f(\lambda)]^{-k} (\lambda - \lambda_n)^{-k} d\lambda,$$

$$I_k = \frac{1}{2\pi i} \oint_{+B(D)} [\mu - f(\lambda)]^{-1} \lambda^k d\lambda.$$

As shown in **(1)**, $I_{n,k}$ is analytic except for a pole of order not greater than k at $\mu = f(\lambda_n)$. On the other hand, $\lambda^k/(\mu - f(\lambda))$ is analytic in D except possibly at $\lambda = \infty$.

If $\lambda = \infty$ is not a singularity of $\lambda^k/(\mu - f(\lambda))$, then $I_k = 0$. If, however, $f(\lambda)$ has a pole of order p at $\lambda = \infty$, we can write, for large $|\lambda|$,

$$f(\lambda) = s\left(\frac{1}{\lambda}\right) + \sum_{t=0}^{p} a_t \lambda^t,$$

where $s(\lambda)$ is an entire function. Hence

$$\mu - f(\lambda) = -S\left(\frac{1}{\lambda}\right) + \mu - \sum_{t=0}^{p} a_t \lambda^t.$$

Now

$$\frac{1}{2\pi i} \oint_{+B(D)} [\mu - f(\lambda)]^{-1} \lambda^k d\lambda = -\frac{1}{2\pi i} \oint_{F} [\mu - f(1/\zeta)]^{-1} \zeta^{-k-2} d\zeta,$$

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when we make the substitution $\lambda = 1/\zeta$ and write F for the image of $+B(D)$.
Now

$$\zeta^{k+2}[\mu - f(1/\zeta)] = -\zeta^{k+2}s(\zeta) + (\mu - a_0)\zeta^{k+2} - \sum_{t=1}^{p} a_t \zeta^{k-t+2}$$

will have a zero at $\zeta = 0$ if $k + 2 > p$. Hence $[\mu - f(1/\zeta)]^{-1}\zeta^{-k-2}$ will have a pole of order $k + 2 - p$ at $\zeta = 0$. Therefore

$$I_k = \frac{1}{k + 2 - p!} D^{k+1-p} [[\mu - f(1/\zeta)]^{-1} \zeta^{-p}]_{\zeta=0},$$

writing $D = d/d\zeta$.

If we write $\Phi = \zeta^p[\mu - f(1/\zeta)]$ and $\Theta = \Phi^{-1}$, then we can easily calculate I_k by using the determinantal expression obtained in the proof of **(1**, Theorem 6**)** and, in so doing, we find that I_k is a polynomial in μ. Hence when we now examine (14), we can conclude that $R_\mu[f(T)]$ has poles at $f(\lambda_n)$. If $\sigma(T)$ is finite, then $\sigma[f(T)]$ consists of a finite number of poles; if $\sigma(T)$ is infinite, $\lambda_n \to \infty$ and by choice of f, $f(\lambda_n) \to \infty$. Hence in either case $f(T) \in \mathfrak{M}(\infty)$.

The remaining assertions of the theorem can now be proved exactly as in **(1)** since only the $I_{n,k}$ enter into the argument and the definitions of $I_{n,k}$ are the same in both cases. This concludes the proof.

COROLLARY. *If* $T \in \mathfrak{M}_f(\infty)$ *and* $f \in \mathfrak{A}_\infty^{(p)} - \mathfrak{A}_\infty$, *then* $f(T) \in \mathfrak{M}_f(\infty)$.

Proof. Every E_n has finite-dimensional range, so since S is obviously a finite set for each $\mu_0 \in \sigma[f(T)]$, the spectral projection E_0 associated with μ_0 and $f(T)$ has finite-dimensional range. Since $N(\mu_0 - f(T)) \subseteq R(E_0)$, the conclusion follows.

COROLLARY. *Let* $T \in \mathfrak{M}(\infty)$ $(\mathfrak{M}_f(\infty))$. *Then, for each* $\lambda \in \rho(T)$, $R_\lambda(T) \in \mathfrak{M}$ $(\mathfrak{N})$. *In particular if* $0 \in \rho(T)$, $T^{-1} \in \mathfrak{M}$ $(\mathfrak{N})$.

Proof.

$$R_\lambda(T) = \frac{1}{2\pi i} \oint_{+B(D)} \frac{1}{\lambda - \mu} R_\mu(T) d\mu.$$

By the spectral mapping theorem, **(6**, p. 302**)**,

$$\sigma[R_\lambda(T)] = \{1/(\lambda - \mu): \mu \in \sigma_e(T)\}.$$

The result now follows.

Remark. The above corollary shows that the class $\mathfrak{M}(\infty)$ includes operators with compact resolvent.

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