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On the Γ-limit of weighted fractional energies

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Given $p \in [1,\infty)$ and a bounded open set $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary, we study the Γ-convergence of the weighted fractional seminorm

$$[u]^p_{s,p,f} = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|\tilde{u}(x)-\tilde{u}(y)|^p}{\|x-y\|^{d+sp}}\, f(x)\, f(y)\,\mathrm{d}x\,\mathrm{d}y,$$

as $s \to 1^-$ for $u \in L^p(\Omega)$, where $\tilde{u}=u$ on Ω and $\tilde{u}=0$ on $\mathbb{R}^d \setminus \Omega$. Assuming that $(f_s)_{s\in(0,1)} \subset L^\infty(\mathbb{R}^d;[0,\infty))$ and $f \in \mathrm{Lip}_b(\mathbb{R}^d;(0,\infty))$ are such that $f_s \to f$ in $L^\infty(\mathbb{R}^d)$ as $s \to 1^-$, we show that $(1-s)[u]^p_{s,p,f_s}$ Γ-converges to the Dirichlet p-energy weighted by f^2. In the case $p=2$, we also prove the convergence of the corresponding gradient flows.

Keywords: Γ-convergence; fractional gradient flows; Gagliardo seminorms; parabolic flows; weighted spaces

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1. Introduction

1.1. Framework

We let $d \in \mathbb{N}$, $s \in (0,1)$, and $p \in [1,\infty)$. Given a nonnegative *weight* $f \in L^\infty(\mathbb{R}^d;[0,\infty))$, our aim is to study the Γ-convergence as $s \to 1^-$ of the non-homogeneous (or *weighted* by f) s-fractional p-seminorm

$$[u]^p_{s,p,f} = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|u(x)-u(y)|^p}{\|x-y\|^{d+sp}}\, f(x)\, f(y)\,\mathrm{d}x\,\mathrm{d}y, \tag{1.1}$$

for $u \in L^p(\mathbb{R}^d)$.

The convergence as $s \to 1^-$ of (1.1) in the case $f \equiv 1$—for which we use the shorthand $[\,\cdot\,]^p_{s,p} = [\,\cdot\,]^p_{s,p,1}$—has been deeply studied in recent years, both in the pointwise and in the Γ-sense. Since the literature is very vast, here we limit ourselves to a non-comprehensive list of results which are closer to the spirit of the present work.

The pointwise limit of the seminorm $[\,\cdot\,]^p_{s,p}$ as $s \to 1^-$ is a notable instance of the celebrated *Bourgain–Brezis–Mironescu* (BBM, for short) *formula* [4, 10], yielding that $(1-s)[\,\cdot\,]_{s,p}$ converges to the Dirichlet p-energy up to a multiplicative constant. After the seminal contributions [4, 10], the BBM formula has been extensively studied in several directions, see [11, 26, 27] for more general results and [20, 21] for extensions to arbitrary domains. We also refer to [15, 23, 24] for anisotropic fractional energies and to [12, 16] for *sharp* conditions for the validity of the BBM formula.

The Γ-convergence of $(1-s)[\,\cdot\,]^p_{s,p}$ to the Dirichlet p-energy as $s \to 1^-$ has been established in [5] for every $p \in (1,\infty)$, in [9] only for $p = 2$, and in [16] for every $p \in [1,\infty)$. We also refer to [4, 26] for similar results on bounded open sets.

The geometric case $p = 1$ deserves special mention, due to the link with the (relative) fractional perimeter, see [1, 2, 16, 18, 22, 25] for closely related results in this direction. We also refer to [8, 13, 19] for higher-order convergence results.

Beyond the case $f \equiv 1$, the asymptotic behavior of (1.1) and of similarly-defined energies has been studied for some particular weights, see [7, 14] for the *Gaussian* framework and [17] for weights depending on negative powers of the distance from the boundary.

The aim of the present paper is to investigate the asymptotic behavior of the weighted seminorm (1.1) as $s \to 1^-$ as the weight f is also allowed to vary with respect to the parameter s. This is motivated by the recent interest in the extension of BBM-type formulas beyond the isotropic setting in order to address possible applications to non-isotropic frameworks [12]. Besides, our Γ-convergence result can be interpreted as a suitable extension to the weighted setting of the ones obtained in [5, 9].

Our main result, Theorem 1.1 below, deals with the Γ-convergence of the energy (1.1) with respect to a uniformly converging family of weights $(f_n)_{n\in\mathbb{N}}$ in $L^\infty(\mathbb{R}^d;[0,\infty))$, whose limit f is in $\mathrm{Lip}_b(\mathbb{R}^d;(0,\infty))$. Precisely, we prove that the Γ-limit is given by

$$u \mapsto \begin{cases} K_{d,p}\|\nabla u\|^p_{p,f^2} = K_{d,p}\displaystyle\int_\Omega f^2\,\|\nabla u\|^p\,\mathrm{d}x, & \text{for } p \in (1,\infty), \\ K_{d,1}\|Du\|_{1,f^2} = K_{d,1}\displaystyle\int_\Omega f^2\,\mathrm{d}|Du|, & \text{for } p = 1, \end{cases}$$

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where for every $p \in [1, \infty)$ (and here Γ being Euler's *Gamma function*),

$$K_{d,p} = \frac{1}{p} \int_{\partial B_1} |x \cdot \mathrm{e}_d|^p \, \mathrm{d}\mathcal{H}^{d-1}(x) = \frac{2\pi^{\frac{d-1}{2}}}{p} \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{N+p}{2}\right)}, \tag{1.2}$$

see [3, Lem. 2.1]. Here and below, given a measurable function $u \colon \Omega \to \mathbb{R}$ on an open set $\Omega \subset \mathbb{R}^d$, we define $\tilde{u} \colon \mathbb{R}^d \to \mathbb{R}$ be such that $\tilde{u} = u$ on Ω and $\tilde{u} = 0$ on $\mathbb{R}^d \setminus \Omega$.

THEOREM 1.1 (Γ-convergence with weights) *Let $p \in [1, \infty)$, $(f_n)_{n \in \mathbb{N}} \subset L^\infty(\mathbb{R}^d; [0, \infty))$ and $f \in \mathrm{Lip}_b(\mathbb{R}^d; (0, \infty))$ be such that $f_n \to f$ in $L^\infty(\mathbb{R}^d)$, $\Omega \subset \mathbb{R}^d$ be a bounded open set with Lipschitz boundary and $(s_n)_{n \in \mathbb{N}} \subset (0, 1)$ be such that $s_n \to 1^-$.*

(i) (Compactness) *If $(u^n)_{n \in \mathbb{N}} \subset L^p(\Omega)$ is such that*

$$\sup_{n \in \mathbb{N}} \left((1 - s_n) [\tilde{u}^n]^p_{s_n, p, f_n} + \|u^n\|^p_{L^p(\Omega)} \right) < \infty, \tag{1.3}$$

then, up to a subsequence, $u^n \to u$ in $L^p(\Omega)$ for some $u \in W^{1,p}_0(\Omega)$ if $p > 1$ or $u \in BV(\Omega)$ if $p = 1$.

(ii) (Γ-liminf inequality) *If $(u^n)_{n \in \mathbb{N}} \subset L^p(\Omega)$ is such that $u^n \to u$ in $L^p(\Omega)$ for some $u \in W^{1,p}_0(\Omega)$ if $p > 1$, or $u \in BV(\Omega)$ if $p = 1$, then*

$$\begin{aligned} K_{d,p} \|\nabla u\|^p_{p, f^2} &\le \liminf_{n \to \infty} (1 - s_n) [\tilde{u}^n]^p_{s_n, p, f_n} \quad \text{for } p > 1, \\ K_{d,1} \|Du\|_{1, f^2} &\le \liminf_{n \to \infty} (1 - s_n) [\tilde{u}^n]_{s_n, 1, f_n} \quad \text{for } p = 1. \end{aligned} \tag{1.4}$$

(iii) (Γ-limsup inequality) *If $u \in W^{1,p}_0(\Omega)$ if $p > 1$, or $u \in BV(\Omega)$ if $p = 1$, then there exists $(u^n)_{n \in \mathbb{N}} \subset L^p(\Omega)$ such that $u^n \to u$ in $L^p(\Omega)$ and*

$$\begin{aligned} K_{d,p} \|\nabla u\|^p_{p, f^2} &= \lim_{n \to \infty} (1 - s_n) [\tilde{u}^n]^p_{s_n, p, f_n} \quad \text{for } p > 1, \\ K_{d,1} \|Du\|_{1, f^2} &= \lim_{n \to \infty} (1 - s_n) [\tilde{u}^n]_{s_n, 1, f_n} \quad \text{for } p = 1. \end{aligned} \tag{1.5}$$

1.2. Convergence of flows

In the case $p = 2$, the Γ-convergence result obtained in Theorem 1.1 can be complemented with a stability result for the corresponding parabolic flows associated to the energies, see Theorem 1.2 below. Here and below, given a weight $f \in L^\infty(\mathbb{R}^d; [0, \infty))$, we define the weighted Laplacian of $u \in H^1_0(\Omega)$ as

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$$(-\mathfrak{D})^f u = -2\,\mathrm{div}(f^2 \nabla u), \tag{1.6}$$

in the distributional sense in duality with $C_c^\infty(\Omega)$ functions. Moreover, given $u \in L^2(\Omega)$ such that $[u]_{s,2,f} < \infty$, we define the weighted fractional s-Laplacian of u as

$$(-\mathfrak{D})^{s,f} u(x) = 4f(x)\,\mathrm{p.v.}\int_{\mathbb{R}^d} \frac{\tilde{u}(x) - \tilde{u}(y)}{\|x-y\|^{d+2s}}\, f(y)\,\mathrm{d}y, \tag{1.7}$$

again in the distributional sense in duality with $C_c^\infty(\Omega)$ functions. Note that, in the unweighted case $f \equiv 1$, up to a multiplicative constant, the operators (1.6) and (1.7) become the usual Laplacian and fractional s-Laplacian operators, respectively.

THEOREM 1.2 (Stability of parabolic flows) *Let $(f_n)_{n\in\mathbb{N}}$, f, Ω, and $(s_n)_{n\in\mathbb{N}}$ be as in Theorem 1.1. If $(u_0^n)_{n\in\mathbb{N}} \subset L^2(\Omega)$ is such that $u_0^n \to u_0^\infty$ in $L^2(\Omega)$ for some function $u_0^\infty \in L^2(\Omega)$, $[\tilde{u}_0^n]_{s_n,2,f_n} < \infty$ for every $n \in \mathbb{N}$, and*

$$\sup_{n\in\mathbb{N}} (1-s_n)[\tilde{u}_0^n]^2_{s_n,2,f_n} < \infty,$$

then the following hold:

(i) *$u_0^\infty \in H_0^1(\Omega)$;*
(ii) *for every $T > 0$ and for every $n \in \mathbb{N}$, the problem*

$$\begin{cases} \dot{u}(t) = (1-s_n)(-\mathfrak{D})^{s_n,f_n} u(t), & \text{for a.e. } t \in (0,T), \\ u(0) = u_0^n, \end{cases}$$

admits a unique solution $u_n \in H^1([0,T]; L^2(\Omega))$ such that

$$(-\mathfrak{D})^{s_n,f_n} u_n(t) \in L^2(\Omega) \quad \textit{for a.e. } t \in (0,T);$$

(iii) *the problem*

$$\begin{cases} \dot{u}(t) = K_{d,2}(-\mathfrak{D})^f u(t), & \textit{for a.e. } t \in [0,\infty), \\ u(0) = u_0^\infty, \end{cases}$$

admits a unique solution $u_\infty \in H^1([0,T]; H_0^1(\Omega))$;
(iv) *$(u_n)_{n\in\mathbb{N}}$ weakly converges to u_∞ in $H^1([0,T]; L^2(\Omega))$.*

Moreover, if

$$\lim_{n\to\infty} (1-s_n)[\tilde{u}_0^n]^2_{s_n,2,f_n} = K_{d,2}\|\nabla u_0^\infty\|^2_{2,f^2},$$

then $(u_n)_{n\in\mathbb{N}}$ strongly converges to u_∞ in $H^1([0,T]; L^2(\Omega))$ and also

$$u_n(t) \xrightarrow{L^2} u_\infty(t) \quad \textit{and} \quad (1-s_n)[\tilde{u}_n(t)]_{s_n,2,f_n} \to K_{d,2}\|\nabla u_\infty(t)\|_{2,f^2}$$

for every $t \in [0,T]$.

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1.3. Organization of the paper

The paper is organized as follows. The notation and some useful preliminary results are detailed in Section 2. The proof of Theorem 1.1 is given in Section 3, while that of Theorem 1.2 can be found in Section 4.

2. Preliminaries

2.1. Notation

We briefly detail the main notation used throughout the paper.

The symbol $C(*, \cdots, *)$ indicates a generic positive constant that depends on $*, \cdots, *$ only and may change from line to line.

We let $d \in \mathbb{N}$ and work in the d-dimensional Euclidean space $\mathbb{R}^d$. We let $x \cdot y$ be the Euclidean inner product between $x, y \in \mathbb{R}^d$ and $\|x\|$ be the Euclidean norm of x.

We let $B_r(x)$ be the open ball in $\mathbb{R}^d$ of center $x \in \mathbb{R}^d$ and radius $r > 0$, and we use the shorthand $B_r = B_r(0)$. Given an open set $A \subset \mathbb{R}^d$, we let $A^c = \mathbb{R}^d \setminus A$ be the complement of A, ∂A be the topological boundary of A and, for every $t > 0$,

$$A_t = \{x \in \mathbb{R}^d : \operatorname{dist}(x; A) < t\}. \tag{2.1}$$

Throughout the paper, we let $\Omega \subset \mathbb{R}^d$ be a bounded open set with Lipschitz boundary.

We let $\mathcal{L}^d$ be the d-dimensional Lebesgue measure and $\mathcal{H}^\alpha$ be the α-dimensional Hausdorff measure for every $\alpha \in [0, d]$. We set $\omega_d = \mathcal{L}^d(B_1)$, so that $\mathcal{H}^{d-1}(\partial B_1) = d\,\omega_d$. Throughout the paper, all functions and sets are tacitly assumed to be $\mathcal{L}^d$-measurable.

Let $p \in [1, \infty)$ and $f \in L^\infty(\mathbb{R}^d; [0, \infty))$. Given $m \in \mathbb{N}$ and $v \colon \Omega \to \mathbb{R}^m$, we let

$$\|v\|_{p,f} = \left(\int_\Omega \|v(x)\|^p \, f(x) \, \mathrm{d}x \right)^{\frac{1}{p}} \in [0, \infty], \tag{2.2}$$

and we use the shorthand $\|v\|_p = \|v\|_{p,1}$. We thus let

$$[L^p_f(\Omega)]^m = \{v \colon \Omega \to \mathbb{R}^m : \|v\|_{p,f} < \infty\}.$$

When $m = 1$, we simply write $L^p_f(\Omega)$. We point out that if additionally f takes values in $(0, \infty)$, under our standing assumptions on f and Ω, the spaces $L^p(\Omega)$ and $L^p_f(\Omega)$ are equivalent, with

$$(\operatorname{ess\,inf}_\Omega f) \, \|v\|_p^p \le \|v\|_{p,f}^p \le \|f\|_\infty \|v\|_p^p. \tag{2.3}$$

Given $u \colon \Omega \to \mathbb{R}$, we define $\tilde{u} \colon \mathbb{R}^d \to \mathbb{R}$ by letting $\tilde{u} = u$ in Ω and $\tilde{u} = 0$ in $\mathbb{R}^d \setminus \Omega$. Thus, given $s \in (0, 1)$, we define

$$[u]_{s,p,f} = \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\tilde{u}(x) - \tilde{u}(y)|^p}{\|x - y\|^{d+sp}} \, f(x) \, f(y) \, \mathrm{d}x \, \mathrm{d}y \right)^{\frac{1}{p}}, \tag{2.4}$$

for every $u \colon \Omega \to \mathbb{R}$ and we use the shorthand $[u]_{s,p} = [u]_{s,p,1}$.

Finally, we let $W_0^{1,p}(\Omega)$ for $p>1$, be the closure of $C_c^\infty(\Omega)$ functions with respect to the Sobolev p-norm $u\mapsto \|u\|_p^p+\int_{\mathbb{R}^d}\|\nabla u\|_p^p\,\mathrm{d}x$, while $BV(\Omega)$ the weak* closure of $C_c^\infty(\Omega)$ with respect to the Sobolev 1-norm. We also set

$$\|Du\|_{1,f}=\int_{\mathbb{R}^d} f\,\mathrm{d}|Du|, \tag{2.5}$$

whenever $u\in BV(\Omega)$ and $f\in L^\infty(\mathbb{R}^d;(0,\infty))$.

2.2. Compactness and characterization

We recall the following well-known compactness result, see [6, Thm. 4.26] for example. Here and below, we let $\tau_h w(\,\cdot\,)=w(\,\cdot+h)$ for every $h\in\mathbb{R}^d$ and $w\in L^p(\mathbb{R}^d)$.

THEOREM 2.1 *Let $p\in[1,\infty)$. If $(v^n)_{n\in\mathbb{N}}\subset L^p(\Omega)$ is such that*

$$\sup_{n\in\mathbb{N}}\|v^n\|_p<\infty \quad \textit{and} \quad \lim_{h\to 0}\sup_{n\in\mathbb{N}}\|\tau_h\tilde{v}^n-\tilde{v}^n\|_{L^p(\mathbb{R}^d)}=0,$$

then, up to a subsequence, $v^n\to v$ in $L^p(\Omega)$ for some $v\in L^p(\Omega)$.

We also recall the following well-known characterization of Sobolev and BV functions, see [6, Prop. 9.3 and Rem. 6] for example.

THEOREM 2.2 *Let $p\in[1,\infty)$ and $v\in L^p(\mathbb{R}^d)$. The following are equivalent:*

(i) *$v\in W^{1,p}(\mathbb{R}^d)$ for $p>1$ or $v\in BV(\mathbb{R}^d)$ for $p=1$;*
(ii) *$\sup_{\|h\|\le 1}\|\tau_h v-v\|_p<\infty$.*

3. Proof of Theorem 1.1

Throughout this section, we let $p\in[1,\infty)$, $(s_n)_{n\in\mathbb{N}}\subset(0,1)$ be such that $s_n\to 1^-$, and $(f_n)_{n\in\mathbb{N}}\subset L^\infty(\mathbb{R}^d;[0,\infty))$ and $f\in \mathrm{Lip}_b(\mathbb{R}^d;(0,\infty))$ be such that $f_n\to f$ in $L^\infty(\mathbb{R}^d)$.

We preliminarily prove Theorem 1.1 in the case $f_n=f$ for every $n\in\mathbb{N}$. We restate our result in this particular case for better clarity.

THEOREM 3.1 *The following hold.*

(i) (Compactness) *If $(u^n)_{n\in\mathbb{N}}\subset L^p(\Omega)$ is such that*

$$\sup_{n\in\mathbb{N}}\Big((1-s_n)[\tilde{u}^n]^p_{s_n,p,f}+\|u^n\|^p_{L^p(\Omega)}\Big)<\infty, \tag{3.1}$$

then, up to a subsequence, $u^n\to u$ in $L^p(\Omega)$ for some $u\in W_0^{1,p}(\Omega)$ if $p\in(1,\infty)$ or $u\in BV(\Omega)$ if $p=1$.

(ii) (Γ-liminf inequality) *If* $(u_n)_{n\in\mathbb{N}} \subset L^p(\Omega)$ *is such that* $u_n \to u$ *in* $L^p(\Omega)$ *for some* $u \in L^p(\Omega)$, *then*

$$\begin{aligned} K_{d,p}\|\nabla u\|_{p,f^2}^p &\le \liminf_{n\to\infty}(1-s_n)[\tilde{u}^n]_{s_n,p,f}^p \quad \textit{for } p \in (1,\infty), \\ K_{d,1}\|Du\|_{1,f^2} &\le \liminf_{n\to\infty}(1-s_n)[\tilde{u}^n]_{s_n,1,f} \quad \textit{for } p = 1. \end{aligned} \tag{3.2}$$

(iii) (Γ-limsup inequality) *If* $u \in W_0^{1,p}(\Omega)$ *if* $p>1$, *or* $u \in BV(\Omega)$ *if* $p=1$, *then there exists* $(u^n)_{n\in\mathbb{N}} \subset L^p(\Omega)$ *such that* $u_n \to u$ *in* $L^p(\Omega)$ *and*

$$\begin{aligned} K_{d,p}\|\nabla u\|_{p,f^2}^p &= \lim_{n\to\infty}(1-s_n)[\tilde{u}^n]_{s_n,p,f}^p \quad \textit{for } p \in (1,\infty), \\ K_{d,1}\|Du\|_{1,f^2} &= \lim_{n\to\infty}(1-s_n)[\tilde{u}^n]_{s_n,1,f} \quad \textit{for } p = 1. \end{aligned} \tag{3.3}$$

The proof of the three statements (i), (ii), and (iii) of Theorem 3.1 is split across Sections 3.1, 3.2, and 3.3. The proof of Theorem 1.1 is given in Section 3.4.

3.1. Proof of Theorem 3.1(i)

We adapt the strategy of [1] to our setting. To this aim, we need two preliminary results. The first one is the following, which generalizes [1, Prop. 5] to any $p \in [1,\infty)$ and weighted L^p norms. We also refer to [9, Prop. 2.4] for the case $p=2$ without weights.

Proposition 3.2. *Let* $f \in \mathrm{Lip}_b(\mathbb{R}^d;(0,\infty))$. *There exists* $C = C(d,p) > 0$ *such that*

$$\|\tau_h v - v\|_{L_f^p(E)}^p \le C\frac{\|h\|^p}{\varrho^{d+p}}\int_{B_\varrho}\|\tau_y v - v\|_{L_f^p(E_{\|h\|})}^p \,\mathrm{d}y \tag{3.4}$$

for every $v \in L^p(\mathbb{R}^d)$, $h \in \mathbb{R}^d$, $\varrho \in (0,\|h\|]$, *and every bounded open set* $E \subset \mathbb{R}^d$, *where* $E_{\|h\|}$ *is defined according to the notation in* (2.1).

Proof. The proof closely follows the one of [1, Prop. 5]. Let $\varphi \in C_c^1(B_1)$ be such that

$$\varphi \ge 0 \quad \text{and} \quad \int_{B_1}\varphi(x)\,\mathrm{d}x = 1. \tag{3.5}$$

For every $\varrho > 0$, we let U_ϱ and V_ϱ be defined as

$$\begin{aligned} U_\varrho(x) &= \frac{1}{\varrho^d}\int_{B_\varrho} v(x+y)\varphi\left(\frac{y}{\varrho}\right)\mathrm{d}y, \\ V_\varrho(x) &= \frac{1}{\varrho^d}\int_{B_\varrho} (v(x)-v(x+y))\varphi\left(\frac{y}{\varrho}\right)\mathrm{d}y, \end{aligned}$$

for every $x \in \mathbb{R}^d$. Owing to (3.5), we have that $v(x) = U_\varrho(x) + V_\varrho(x)$ for every $\varrho > 0$ and $x \in \mathbb{R}^d$, so that

$$|\tau_h v(x) - v(x)|^p \le 3^p\big(|U_\varrho(x+h) - U_\varrho(x)|^p + |V_\varrho(x)|^p + |V_\varrho(x+h)|^p\big). \tag{3.6}$$

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We now estimate each term in the right-hand side of (3.6) separately. Concerning the second and third term, by Jensen's inequality, we can estimate

$$|V_\varrho(\xi)|^p \le \frac{\omega_d}{\varrho^d}\,\|\varphi\|_\infty^p \int_{B_\varrho} |v(\xi) - \tau_y v(\xi)|^p \,\mathrm{d}y, \tag{3.7}$$

for every $\xi \in \mathbb{R}^d$. Instead, concerning the first term, by the change of variables $z = x + y$, we can rewrite

$$U_\varrho(x) = \frac{1}{\varrho^d} \int_{B_\varrho(x)} v(z)\,\varphi\left(\frac{z-x}{\varrho}\right) \mathrm{d}z.$$

Thus, owing to the fact that $\varphi((z-\cdot)\varrho^{-1}) \in C_c^1(B_\varrho(x))$, we can integrate by parts and get

$$\begin{aligned}\nabla U_\varrho(x) &= -\frac{1}{\varrho^{d+1}} \int_{B_\varrho(x)} v(z)\,\nabla\varphi\left(\frac{z-x}{\varrho}\right) \mathrm{d}z \\ &= -\frac{1}{\varrho^{d+1}} \int_{B_\varrho(x)} (v(z) - v(x))\,\nabla\varphi\left(\frac{z-x}{\varrho}\right) \mathrm{d}z \\ &= -\frac{1}{\varrho^{d+1}} \int_{B_\varrho} (v(x+y) - v(x))\,\nabla\varphi\left(\frac{y}{\varrho}\right) \mathrm{d}y.\end{aligned}$$

Therefore, by the Fundamental Theorem of Calculus and by Jensen's inequality, we obtain

$$\begin{aligned}|U_\varrho(x+h) - U_\varrho(x)|^p &\le \|h\|^p \int_0^1 |\nabla U_\varrho(x+th)|^p \,\mathrm{d}t \\ &\le \omega_d^{p-1} \frac{\|h\|^p}{\varrho^{d+p}} \|\nabla\varphi\|_\infty^p \int_0^1 \int_{B_\varrho} |\tau_y v(x+th) - v(x+th)|^p \,\mathrm{d}y\,\mathrm{d}t.\end{aligned} \tag{3.8}$$

Now, using that $\varrho < \|h\|$ and combining (3.6), (3.7), and (3.8), we get that

$$\begin{aligned}|\tau_h v(x) - v(x)|^p \le{}& C\frac{\|h\|^p}{\varrho^{d+p}} \int_0^1 \int_{B_\varrho} |\tau_y v(x+th) - v(x+th)|^p \,\mathrm{d}y\,\mathrm{d}t \\ &+ C\frac{\|h\|^p}{\varrho^{d+p}} \int_{B_\varrho} |\tau_y v(x) - v(x)|^p \,\mathrm{d}y \\ &+ C\frac{\|h\|^p}{\varrho^{d+p}} \int_{B_\varrho} |\tau_y v(x+h) - v(x+h)|^p \,\mathrm{d}y,\end{aligned} \tag{3.9}$$

where we have set

$$C = C(d,p) = (3\max\{\|\varphi\|_\infty; \|\nabla\varphi\|_\infty\})^p\,\omega_d.$$

Multiplying inequality (3.9) by $f(x)$ and integrating with respect to $x \in E$, the claim immediately follows by Fubini's Theorem. We omit the simple details. □

We can now pass to the following result, which extends [1, Prop. 4] to any $p \in [1,\infty)$ also in the case of weighted L^p norms.

PROPOSITION 3.3. *Let $f \in \mathrm{Lip}_b(\mathbb{R}^d;(0,\infty))$. There exists $C = C(d,p) > 0$ such that*

$$\|\tau_h v - v\|^p_{L^p_f(E)} \leq C(1-s)\|h\|^{sp} \int_{B_{\|h\|}} \frac{\|\tau_y v - v\|^p_{L^p_f(E_{\|h\|})}}{\|y\|^{d+sp}} \,\mathrm{d}y$$

for every $v \in L^p(\mathbb{R}^d)$, $h \in \mathbb{R}^d$, and every bounded open set $E \subset \mathbb{R}^d$, where $E_{\|h\|}$ is defined according to the notation in (2.1).

The proof of Proposition 3.3 requires the following Hardy-type inequality, which is taken from [1, Prop. 6].

LEMMA 3.4. *If $\varphi\colon \mathbb{R} \to [0,\infty)$ is a Borel function, then*

$$\int_0^r \frac{1}{\varrho^{d+l+1}} \int_0^\varrho \varphi(t)\,\mathrm{d}t\,\mathrm{d}\varrho \leq \frac{1}{d+l} \int_0^r \frac{\varphi(t)}{t^{d+l}}\,\mathrm{d}t, \tag{3.10}$$

for every $l, r \geq 0$.

Actually, in the proof of Proposition 3.3 we use the weaker estimate

$$\int_0^r \frac{1}{\varrho^{d+l+1}} \int_0^\varrho \varphi(t)\,\mathrm{d}t\,\mathrm{d}\varrho \leq \frac{1}{d} \int_0^r \frac{\varphi(t)}{t^{d+l}}\,\mathrm{d}t, \tag{3.11}$$

that is, we can ignore the dependence on l in the prefactor in the right-hand side of (3.10).

Proof of Proposition 3.3. We let $\varphi_v\colon [0,\|h\|] \to \mathbb{R}$ be defined as

$$\varphi_v(t) = \int_{\partial B_t} \|\tau_y v - v\|^p_{L^p_f(E_{\|h\|})}\,\mathrm{d}\mathcal{H}^{d-1}(y), \tag{3.12}$$

for all $t > 0$. Owing to (3.4) and to the definition in (3.12), we can estimate

$$\|\tau_h v - v\|^p_{L^p_f(E)} \leq C\frac{\|h\|^p}{\varrho^{d+p}} \int_0^\varrho \varphi_v(t)\,\mathrm{d}t, \tag{3.13}$$

for some $C = C(d,p) > 0$. We now multiply both sides of (3.13) by $\varrho^{-1+p-sp}$ and integrate in the interval $[0,\|h\|]$ with respect to ϱ, getting

$$\|\tau_h v - v\|^p_{L^p_f(E)} \leq Cp(1-s)\frac{\|h\|^p}{\|h\|^{p-sp}} \int_0^{\|h\|} \frac{1}{\varrho^{d+sp+1}} \int_0^\varrho \varphi_v(t)\,\mathrm{d}t\,\mathrm{d}\varrho.$$

By exploiting (3.11) with $l = sp$ and $\varphi = \varphi_v$, we thus obtain that

$$\|\tau_h v - v\|^p_{L^p_f(E)} \leq C(1-s)\|h\|^{sp} \int_0^{\|h\|} \frac{\varphi_v(t)}{t^{d+sp}}\,\mathrm{d}t,$$

and the conclusion follows from the very definition of φ_v. □

We are now ready to detail the proof of the compactness statement (i) in Theorem 3.1.

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Proof of Theorem 3.1(i). Given $h \in \mathbb{R}^d$ such that $\|h\| < 1$, we have $\Omega_{\|h\|} \Subset \Omega_1 \Subset (\Omega_1)_{\|h\|}$ (recall the notation in (2.1)). We can hence set

$$c = c(\Omega, f) = \inf_{(\Omega_1)_{\|h\|}} f > 0,$$

and observe that

$$c^2 \|\tau_h \tilde{u}^n - \tilde{u}^n\|^p_{L^p(\mathbb{R}^d)} = c^2 \|\tau_h \tilde{u}^n - \tilde{u}^n\|^p_{L^p(\Omega_{\|h\|})} \le c \|\tau_h \tilde{u}^n - \tilde{u}^n\|^p_{L^p_f(\Omega_1)}.$$

By Proposition 3.3 applied on Ω_1 and by the previous inequality, we have

$$c^2 \|\tau_h \tilde{u}^n - \tilde{u}^n\|^p_{L^p(\mathbb{R}^d)} \le C(1 - s_n) \|h\|^{s_n p} \int_{B_{\|h\|}} \frac{c \|\tau_y \tilde{u}^n - \tilde{u}^n\|^p_{L^p_f((\Omega_1)_{\|h\|})}}{\|y\|^{d + s_n p}} \, \mathrm{d}y,$$

where $C = C(d, p) > 0$. Now, explicitly writing down the L^p_f norm on the right-hand side, swapping order of integration, performing the change of variables $y = \xi - x$, and bounding c with $f(\xi)$, we obtain that

$$\begin{aligned} c^2 \|\tau_h \tilde{u}^n - \tilde{u}^n\|^p_{L^p(\mathbb{R}^d)} & \\ \le C(1 - s_n) \|h\|^{s_n p} \int_{B_{\|h\|}} \int_{(\Omega_1)_{\|h\|}} \frac{|\tilde{u}^n(x + y) - \tilde{u}^n(x)|^p}{\|y\|^{d + s_n p}} c f(x) \, \mathrm{d}x \, \mathrm{d}y & \\ \le C(1 - s_n) \|h\|^{s_n p} \int_{(\Omega_1)_{\|h\|}} \int_{B_{\|h\|}(x)} \frac{|\tilde{u}^n(\xi) - \tilde{u}^n(x)|^p}{\|\xi - x\|^{d + s_n p}} f(\xi) f(x) \, \mathrm{d}\xi \, \mathrm{d}x & \\ \le C(1 - s_n) \|h\|^{s_n p} [\tilde{u}^n]^p_{s_n, p, f}. & \end{aligned}$$

Dividing by c^2 and owing to our equiboundedness assumption (3.1), we get that

$$\|\tau_h \tilde{u}^n - \tilde{u}^n\|_{L^p(\mathbb{R}^d)} \le C(d, p, M, \Omega, f) \|h\|^{s_n}, \tag{3.14}$$

for all $h \in \mathbb{R}^d$ such that $\|h\| \le 1$. Thus, owing to (3.1) and to (3.14), we can apply Theorem 2.1 and find $u \in L^p(\mathbb{R}^d)$ such that, up to a subsequence, $\tilde{u}^n \to u$ in $L^p(\mathbb{R}^d)$. Furthermore, since $\tilde{u}^n = 0$ for all $n \in \mathbb{N}$ on $\mathbb{R}^d \setminus \Omega$, we also have that $u = 0$ on $\mathbb{R}^d \setminus \Omega$. Finally, letting $n \to \infty$ in (3.14), we have

$$\|\tau_h u - u\|_{L^p(\mathbb{R}^d)} \le C(d, p, M, \Omega, f) \|h\|,$$

for all h with $\|h\| \le 1$, so that $u \in W^{1,p}(\mathbb{R}^d)$ for $p > 1$ or $u \in BV(\mathbb{R}^d)$ for $p = 1$ by Theorem 2.2. Since $u = 0$ on $\mathbb{R}^d \setminus \Omega$ and Ω has Lipschitz boundary, we get that $u|_\Omega \in W^{1,p}_0(\Omega)$ for $p > 1$, or $u|_\Omega \in BV(\Omega)$ for $p = 1$, concluding the proof. □

3.2. Proof of Theorem 3.1(ii)

We adapt the strategy of the proof of [9, Thm. 2.1] to our setting. To this aim, we need some preliminaries.

Let us begin with some notation. We let $Q = (-1,1)^d$. Consequently, given $\gamma > 0$, for every $i \in \gamma\mathbb{Z}^d$ and $a \geq 0$, we let $Q_i^a = i + aQ$. Note that, if $a = \gamma$, then the family of cubes $(Q_i^\gamma)_i$ is a tiling of $\mathbb{R}^d$. Moreover, since Ω is bounded, the set

$$I_\gamma = \{i \in \gamma\mathbb{Z}^d : \mathcal{L}^d(Q_i^\gamma \cap \Omega) > 0\}$$

is finite. In addition, given $f\colon \mathbb{R}^d \to \mathbb{R}$, we let $f_i^a = \inf_{Q_i^a} f$. Notice that, whenever $f \in \operatorname{Lip}(\mathbb{R}^d;(0,\infty))$, then $f_i^a > 0$. Finally, we let $\eta \in C_c^\infty(B_1)$ be such that $\eta \geq 0$ and $\int_{B_1} \eta \,\mathrm{d}x = 1$ and, for every $\varepsilon > 0$, we set $\eta_\varepsilon(\,\cdot\,) = \varepsilon^{-d}\eta(\,\cdot\,/\varepsilon)$. Accordingly, we let $u_\varepsilon = u * \eta_\varepsilon$ for every $\varepsilon > 0$ and $u \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$.

We can now prove the following preliminary estimate.

LEMMA 3.5. *Let $p \in [1,\infty)$, $f \in \operatorname{Lip}_b(\mathbb{R}^d;(0,\infty))$. There exist $\varepsilon, \beta, \gamma > 0$ with $\varepsilon \ll \beta \ll \gamma$ such that*

$$\begin{aligned}\int_{Q_i^{(1-\beta)\gamma}} \int_{Q_i^{(1-\beta)\gamma}} &\frac{|\tilde{u}_\varepsilon(x) - \tilde{u}_\varepsilon(y)|^p}{\|x-y\|^{d+sp}} (f_i^\gamma)^2 \,\mathrm{d}x\,\mathrm{d}y \\ &\leq \int_{Q_i^\gamma}\int_{Q_i^\gamma} \frac{|\tilde{u}(x) - \tilde{u}(y)|^p}{\|x-y\|^{d+sp}} f(x)f(y)\,\mathrm{d}x\,\mathrm{d}y\end{aligned} \tag{3.15}$$

holds for all $i \in \gamma\mathbb{Z}^d$ and $u \in L^p(\Omega)$.

Proof. Clearly, we can choose ε, β, and γ such that

$$Q_i^{(1-\beta)\gamma} + z \subset Q_i^\gamma \tag{3.16}$$

for all $z \in B_\varepsilon$ and all $i \in \gamma\mathbb{Z}^d$. Indeed,

$$Q_i^{(1-\beta)\gamma} + z \subset \{y \in \mathbb{R}^d : \operatorname{dist}(y, Q_i^{(1-\beta)\gamma}) < \varepsilon\} \subset Q_i^{(1-\beta)\gamma+\varepsilon}$$

and therefore, to get (3.16), it is enough to choose $\varepsilon < \beta\gamma$ to ensure $(1-\beta)\gamma + \varepsilon < \gamma$. Using the definition of convolution, Jensen's inequality, changing order of integration, performing the change of variables $x - z = \xi$ and $y - z = \zeta$, owing to (3.16), changing once again order of integration, using the fact that η_ε has unit L^1 norm, and finally owing to the definition of $f_i^\gamma = \inf_{Q_i^\gamma} f$, we obtain

$$\begin{aligned}\int_{Q_i^{(1-\beta)\gamma}} \int_{Q_i^{(1-\beta)\gamma}} &\frac{|\tilde{u}_\varepsilon(x) - \tilde{u}_\varepsilon(y)|^p}{\|x-y\|^{d+sp}} (f_i^\gamma)^2 \,\mathrm{d}x\,\mathrm{d}y \\ &\leq \int_{Q_i^{(1-\beta)\gamma}} \int_{Q_i^{(1-\beta)\gamma}} \int_{B_\varepsilon} \frac{|\tilde{u}(x-z) - \tilde{u}(y-z)|^p}{\|x-z-(y-z)\|^{d+sp}} \eta_\varepsilon(z)(f_i^\gamma)^2 \,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y \\ &\leq \int_{B_\varepsilon} \int_{Q_i^{(1-\beta)\gamma}+z} \int_{Q_i^{(1-\beta)\gamma}+z} \frac{|\tilde{u}(\xi) - \tilde{u}(\zeta)|^p}{\|\xi-\zeta\|^{d+sp}} \eta_\varepsilon(z)(f_i^\gamma)^2 \,\mathrm{d}\xi\,\mathrm{d}\zeta\,\mathrm{d}z \\ &\leq \int_{Q_i^\gamma}\int_{Q_i^\gamma} \frac{|\tilde{u}(\xi) - \tilde{u}(\zeta)|^p}{\|\xi-\zeta\|^{d+sp}} \, f(\xi)\, f(\zeta)\,\mathrm{d}\xi\,\mathrm{d}\zeta,\end{aligned}$$

concluding the proof. □

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We are now ready to prove the lim inf statement (ii) in Theorem 3.1.

Proof of Theorem 3.1(ii). Let $(u^n)_{n\in\mathbb{N}} \subset L^p(\Omega)$ be such that $u^n \to u$ in $L^p(\Omega)$ for some $u \in L^p(\Omega)$. As a consequence, $\sup_{n\in\mathbb{N}} \|u^n\|_p < \infty$. Furthermore, we can also assume that

$$\liminf_{n\to\infty}(1-s_n)[\tilde{u}^n]^p_{s_n,p,f} < \infty,$$

otherwise inequality (3.2) is trivially true. Therefore, we can assume the validity of (3.1), which, in turn, implies that $u \in W^{1,p}_0(\Omega)$ for $p > 1$, or $u \in BV(\Omega)$ for $p = 1$. By Lemma 3.5, there exist $\varepsilon, \beta, \gamma > 0$ with $\varepsilon \ll \beta \ll \gamma$ such that

$$\begin{aligned}\int_{Q_i^\gamma}\int_{Q_i^\gamma} \frac{|\tilde{u}^n(x)-\tilde{u}^n(y)|^p}{\|x-y\|^{d+s_np}} f(x)f(y)\,\mathrm{d}y\,\mathrm{d}x \\ \geq \int_{Q_i^{(1-\beta)\gamma}}\int_{Q_i^{(1-\beta)\gamma}} \frac{|(\tilde{u}^n)_\varepsilon(x)-(\tilde{u}^n)_\varepsilon(y)|^p}{\|x-y\|^{d+s_np}} (f_i^\gamma)^2\,\mathrm{d}y\,\mathrm{d}x,\end{aligned} \tag{3.17}$$

for every $i \in \gamma\mathbb{Z}^d$ and $n \in \mathbb{N}$. We now perform a first-order Taylor expansion of $(\tilde{u}^n)_\varepsilon$. Precisely, owing to the uniform bound on the p-norms granted by (3.1) and the boundedness of Ω, we can estimate

$$\begin{aligned}|\nabla(\tilde{u}^n)_\varepsilon(x)\cdot(x-y)| &\leq |(\tilde{u}^n)_\varepsilon(x)-(\tilde{u}^n)_\varepsilon(y)| + \frac{1}{2}|\langle D^2(\tilde{u}^n)_\varepsilon(\xi)(x-y),(x-y)\rangle| \\ &\leq |(\tilde{u}^n)_\varepsilon(x)-(\tilde{u}^n)_\varepsilon(y)| + C(\Omega)\|u^n\|_1\,\|\eta_\varepsilon\|_{C^2(\mathbb{R}^d)}\,\|x-y\|^2 \\ &\leq |(\tilde{u}^n)_\varepsilon(x)-(\tilde{u}^n)_\varepsilon(y)| + C(\varepsilon,M,\Omega)\|x-y\|^2,\end{aligned} \tag{3.18}$$

where ξ belongs to the segment from x to y. Now, assuming $\|x-y\|$ small enough (which is always possible by taking γ small enough), since $\tilde{u}^n_\varepsilon$ is locally Lipschitz, we have

$$\begin{aligned}&|(\tilde{u}^n)_\varepsilon(x)-(\tilde{u}^n)_\varepsilon(y)| + C(\varepsilon,M,\Omega)\|x-y\|^2 \\ &\qquad\leq \|\nabla(\tilde{u}^n)_\varepsilon\|_\infty\|x-y\| + C(\varepsilon,M,\Omega)\|x-y\|^2 \\ &\qquad= C(\Omega)\|u^n\|_1\|\eta_\varepsilon\|_{C^1(\mathbb{R}^d)}\|x-y\| + C(\varepsilon,M,\Omega)\|x-y\|^2 \\ &\qquad= C(\varepsilon,M,\Omega)\|x-y\|.\end{aligned} \tag{3.19}$$

Taking p-th powers in (3.18), using that $(t_0+t)^p = t_0^p + p(t_0+\tau)^{p-1}t$ with $\tau \in (0,t)$, and finally owing to (3.19), we get

$$|\nabla(\tilde{u}^n)_\varepsilon(x)\cdot(x-y)|^p \leq |(\tilde{u}^n)_\varepsilon(x)-(\tilde{u}^n)_\varepsilon(y)|^p + C(p,\varepsilon,M,\Omega)\|x-y\|^{p+1}.$$

Therefore, plugging the above inequality in the inner integral in the right-hand side of (3.17), we get that

$$\int_{Q_i^{(1-\beta)\gamma}} \frac{|(\tilde{u}^n)_\varepsilon(x)-(\tilde{u}^n)_\varepsilon(y)|^p}{\|x-y\|^{d+s_np}}(f_i^\gamma)^2\,\mathrm{d}y \geq I' + I'', \tag{3.20}$$

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having set

$$\begin{aligned} I' &= \int_{Q_i^{(1-\beta)\gamma}} \frac{|\nabla(\tilde{u}^n)_\varepsilon(x)\cdot(x-y)|^p}{\|x-y\|^{d+s_np}}(f_i^\gamma)^2\,\mathrm{d}y, \\ I'' &= -C\int_{Q_i^{(1-\beta)\gamma}} \frac{\|x-y\|^{p+1}}{\|x-y\|^{d+s_np}}(f_i^\gamma)^2\,\mathrm{d}y. \end{aligned} \tag{3.21}$$

We now estimate the two terms in (3.21) separately. On the one hand, we have

$$\begin{aligned} I'' &\geq -C(p,\varepsilon,M,\Omega)\|f\|_\infty^2 \int_{B_{\delta_1}} \frac{\mathrm{d}\xi}{\|\xi\|^{d+s_np-p-1}} \\ &= -C(p,\varepsilon,M,\Omega,f)d\omega_d\int_0^{\delta_1}\frac{\mathrm{d}\varrho}{\varrho^{s_np-p}} = -\frac{C(d,p,\varepsilon,M,\Omega,f)}{1+p-s_np}\delta_1^{1+p-s_np} \end{aligned}$$

for $x \in Q_i^{(1-\beta)\gamma}$, where we have set $\delta_1 = 2\sqrt{d}(1-\beta)\gamma$. Integrating the above inequality over the cube $Q_i^{(1-\beta)\gamma}$ with respect to x, we obtain

$$\int_{Q_i^{(1-\beta)\gamma}} I''\,\mathrm{d}x \geq -\frac{C(d,p,\varepsilon,M,\Omega,f)}{1+p-s_np}|Q_i^{(1-\beta)\gamma}|\delta_1^{1+p-s_np},$$

so that

$$\liminf_{n\to\infty}(1-s_n)\int_{Q_i^{(1-\beta)\gamma}} I''\,\mathrm{d}x \geq 0. \tag{3.22}$$

On the other hand, calling $\delta_2 = \delta_2(x) = \mathrm{dist}\left(x;\partial Q_i^{(1-\beta)\gamma}\right)$, taking the normalization $\nu(x) = \nabla(\tilde{u}^n)_\varepsilon(x)/\|\nabla(\tilde{u}^n)_\varepsilon(x)\|$, applying the change of variables $z = x-y$, exploiting the invariance of the integral on B_{δ_2} with respect to rotations, applying the change of variable $z \mapsto z\delta_2^{-1}$, and observing that, for every $\nu \in \mathbb{S}^{d-1}$

$$(1-s_n)\int_{B_1}\frac{|z\cdot\nu|^p}{\|z\|^{d+s_np}}\,\mathrm{d}z = (1-s_n)\int_0^1 \varrho^{p-s_np-1}\int_{\partial B_1}|\eta\cdot\nu|^p\,\mathrm{d}\eta\,\mathrm{d}\varrho = K_{d,p}, \tag{3.23}$$

we get that

$$\begin{aligned} I' &= |\nabla(\tilde{u}^n)_\varepsilon(x)|^p(f_i^\gamma)^2\int_{Q_i^{(1-\beta)\gamma}}\frac{|\nu(x)\cdot(x-y)|^p}{\|x-y\|^{d+s_np}}\,\mathrm{d}y \\ &\geq |\nabla(\tilde{u}^n)_\varepsilon(x)|^p(f_i^\gamma)^2\int_{B_{\delta_2}}\frac{|\nu(x)\cdot z|^p}{\|z\|^{d+s_np}}\,\mathrm{d}z = \frac{K_{d,p}|\nabla(\tilde{u}^n)_\varepsilon(x)|^p\delta_2^{p(1-s_n)}(f_i^\gamma)^2}{1-s_n}. \end{aligned}$$

Multiplying the above inequality by $(1-s_n)$, integrating over the cube $Q_i^{(1-\beta)\gamma}$ with respect to x, taking the $\liminf$ as $n \to \infty$, owing to Fatou's lemma, we get that

$$\liminf_{n\to\infty}(1-s_n)\int_{Q_i^{(1-\beta)\gamma}} I'\,\mathrm{d}x \geq K_{d,p}\int_{Q_i^{(1-\beta)\gamma}} \liminf_{n\to\infty}\left(|\nabla(\tilde{u}^n)_\varepsilon(x)|^p\delta_2^{p(1-s_n)}\right)(f_i^\gamma)^2\,\mathrm{d}x.$$

Since $(u^n)_\varepsilon \to u_\varepsilon$ in any Sobolev norm as $n\to\infty$, up to passing to a suitable subsequence, $\nabla(\tilde{u}^n)_\varepsilon(x)\to\nabla\tilde{u}_\varepsilon(x)$ for a.e. $x\in\mathbb{R}^d$. Therefore, we get that

$$\liminf_{n\to\infty}(1-s_n)\int_{Q_i^{(1-\beta)\gamma}} I'\,\mathrm{d}x \geq K_{d,p}\int_{Q_i^{(1-\beta)\gamma}} |\nabla\tilde{u}_\varepsilon(x)|^p(f_i^\gamma)^2\,\mathrm{d}x. \tag{3.24}$$

Thence, by taking the lim inf as $n\to\infty$ of $(1-s_n)$ times (3.17) and owing to (3.20)–(3.24), we have

$$\liminf_{n\to\infty}(1-s_n)\int_{Q_i^\gamma}\int_{Q_i^\gamma}\frac{|\tilde{u}^n(x)-\tilde{u}^n(y)|^p}{\|x-y\|^{d+sp}}f(x)f(y)\,\mathrm{d}y\,\mathrm{d}x \geq K_{d,p}\int_{Q_i^{(1-\beta)\gamma}}|\nabla\tilde{u}_\varepsilon(x)|^p(f_i^\gamma)^2\,\mathrm{d}x$$

whenever $\varepsilon\ll\beta\ll\gamma\ll 1$. Now first letting $\varepsilon\to 0^+$, and then $\beta\to 0^+$, we get

$$\begin{aligned}
&\liminf_{n\to\infty}(1-s_n)\int_{Q_i^\gamma}\int_{Q_i^\gamma}\frac{|\tilde{u}^n(x)-\tilde{u}^n(y)|^p}{\|x-y\|^{d+sp}}f(x)f(y)\,\mathrm{d}y\,\mathrm{d}x\\
&\qquad\geq K_{d,p}\int_{Q_i^\gamma}|\nabla\tilde{u}(x)|^p(f_i^\gamma)^2\,\mathrm{d}x, \quad \text{if } p\in(1,\infty),\\
&\liminf_{n\to\infty}(1-s_n)\int_{Q_i^\gamma}\int_{Q_i^\gamma}\frac{|\tilde{u}^n(x)-\tilde{u}^n(y)|}{\|x-y\|^{d+s}}f(x)f(y)\,\mathrm{d}y\,\mathrm{d}x\\
&\qquad\geq K_{d,1}\int_{Q_i^\gamma}(f_i^\gamma)^2\,\mathrm{d}|Du|(x), \quad \text{if } p=1.
\end{aligned} \tag{3.25}$$

We notice now that, by the Lebesgue's Dominated Convergence Theorem

$$\lim_{\gamma\to 0^+}\sum_{i\in I_\gamma}K_{d,p}\int_{Q_i^\gamma}|\nabla\tilde{u}(x)|^p(f_i^\gamma)^2\,\mathrm{d}x = K_{d,p}\|\nabla u\|_{p,f^2}^p$$
$$\text{and}\ \lim_{\gamma\to 0^+}\sum_{i\in I_\gamma}K_{d,1}\int_{Q_i^\gamma}(f_i^\gamma)^2\,\mathrm{d}|Du| = K_{d,1}\|Du\|_{1,f^2}.$$

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Pairing this with (3.25) allows to conclude. Indeed, if $p \in (1, \infty)$ (the case $p = 1$ being analogous and thus omitted), then we have

$$
\begin{aligned}
& K_{d,p} \int_{\Omega} |\nabla u(x)|^p f^2(x) \, \mathrm{d}x \\
& \quad \leq \lim_{\gamma \to 0^+} \sum_{i \in I_\gamma} \liminf_{n \to \infty} (1 - s_n) \int_{Q_i^\gamma} \int_{Q_i^\gamma} \frac{|\tilde{u}^n(x) - \tilde{u}^n(y)|^p}{\|x - y\|^{d + s_n p}} f(x) f(y) \, \mathrm{d}y \, \mathrm{d}x \\
& \quad = \lim_{\gamma \to 0^+} \liminf_{n \to \infty} \sum_{i \in I_\gamma} (1 - s_n) \int_{Q_i^\gamma} \int_{Q_i^\gamma} \frac{|\tilde{u}^n(x) - \tilde{u}^n(y)|^p}{\|x - y\|^{d + s_n p}} f(x) f(y) \, \mathrm{d}y \, \mathrm{d}x \\
& \quad \leq \lim_{\gamma \to 0^+} \liminf_{n \to \infty} (1 - s_n) \int_{\Omega} \int_{\Omega} \frac{|\tilde{u}^n(x) - \tilde{u}^n(y)|^p}{\|x - y\|^{d + s_n p}} f(x) f(y) \, \mathrm{d}y \, \mathrm{d}x \\
& \quad = \liminf_{n \to \infty} (1 - s_n) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\tilde{u}^n(x) - \tilde{u}^n(y)|^p}{\|x - y\|^{d + s_n p}} f(x) f(y) \, \mathrm{d}y \, \mathrm{d}x,
\end{aligned}
$$

which is exactly the claimed inequality. □

3.3. Proof of Theorem 3.1(iii)

We begin with the following preliminary result, establishing the lim sup inequality (iii) in Theorem 3.1 for smooth functions supported in Ω.

Theorem 3.6 *If $v \in C_c^\infty(\mathbb{R}^d)$, then*

$$
\begin{aligned}
\lim_{s \to 1^-} (1 - s) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} & \frac{|v(x) - v(y)|^p}{\|x - y\|^{d + sp}} f(x) f(y) \, \mathrm{d}x \, \mathrm{d}y \\
& = K_{d,p} \int_{\mathbb{R}^d} \|\nabla v(z)\|^p f^2(z) \, \mathrm{d}z.
\end{aligned} \tag{3.26}
$$

Proof. Let us write

$$
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|v(x) - v(y)|^p}{\|x - y\|^{d + sp}} f(x) f(y) \, \mathrm{d}x \, \mathrm{d}y = I_1 + I_2 + I_3, \tag{3.27}
$$

where

$$
\begin{aligned}
I_1 &= \int_{\mathbb{R}^d} \int_{\{\|x - y\| < 1\}} \frac{|v(x) - v(y)|^p}{\|x - y\|^{d + sp}} f^2(y) \, \mathrm{d}x \, \mathrm{d}y, \\
I_2 &= \int_{\mathbb{R}^d} \int_{\{\|x - y\| < 1\}} \frac{|v(x) - v(y)|^p}{\|x - y\|^{d + sp}} f(y)(f(x) - f(y)) \, \mathrm{d}x \, \mathrm{d}y, \\
I_3 &= \int_{\mathbb{R}^d} \int_{\{\|x - y\| \geq 1\}} \frac{|v(x) - v(y)|^p}{\|x - y\|^{d + sp}} f(x) f(y) \, \mathrm{d}x \, \mathrm{d}y.
\end{aligned} \tag{3.28}
$$

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We estimate the three terms separately. Concerning I_3, we have that

$$\begin{aligned} I_3 &\le 2^p \|f\|_\infty^2 \int_{\mathbb{R}^d} \int_{\{\|x-y\|\ge 1\}} \frac{|v(x)|^p + |v(y)|^p}{\|x-y\|^{d+sp}} \,\mathrm{d}x\,\mathrm{d}y \\ &\le 2^{p+1}\|f\|_\infty^2 \int_{\mathbb{R}^d}\int_{B_1(y)^c} \frac{|v(y)|^p}{\|x-y\|^{d+sp}}\,\mathrm{d}x\,\mathrm{d}y \\ &= 2^{p+1}\|f\|_\infty^2 \|v\|_p^p \int_1^\infty \frac{d\omega_d}{\varrho^{1+sp}}\,\mathrm{d}\varrho = \frac{2^{p+1}}{sp}\|f\|_\infty^2\|v\|_p^p. \end{aligned} \tag{3.29}$$

For I_2 instead, we can write, recalling the notation in (2.1),

$$\begin{aligned} I_2 &\le \|f\|_\infty \|\nabla f\|_\infty \int_{\mathbb{R}^d}\int_{B_1(y)} \frac{|v(x)-v(y)|^p}{\|x-y\|^{d+sp-1}}\,\mathrm{d}x\,\mathrm{d}y \\ &\le C(f)\|\nabla v\|_\infty^p \int_{(\operatorname{spt} v)_1}\int_{B_1(y)} \frac{1}{\|x-y\|^{d-1-p+sp}}\,\mathrm{d}x\,\mathrm{d}y \\ &\le C(f,v) d\omega_d |(\operatorname{spt} v)_1| \int_0^1 \frac{1}{\varrho^{-p+sp}}\,\mathrm{d}\varrho \\ &\le \frac{C(d,f,v)}{p-sp+1} \le C(d,p,f,v). \end{aligned} \tag{3.30}$$

We are thus left with estimating I_1. To this aim, let us observe that

$$|v(x)-v(y)|^p \le |\nabla v(y)\cdot(x-y)|^p + \|D^2 v\|_\infty^{2p}\,\|x-y\|^{2p}.$$

Thus $I_1 \le I_1' + I_1''$, where, owing to the fact that v has compact support and recalling the notation in (2.1),

$$\begin{aligned} I_1' &= \int_{\mathbb{R}^d}\int_{B_1(y)} \frac{|\nabla v(y)\cdot(x-y)|^p}{\|x-y\|^{d+sp}} f^2(y)\,\mathrm{d}x\,\mathrm{d}y, \\ I_1'' &= \int_{(\operatorname{spt} v)_1}\int_{B_1(y)} \frac{\|D^2 v\|_\infty^{2p}}{\|x-y\|^{d+sp-2p}} f^2(y)\,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

Now, on the one hand, we have that

$$\begin{aligned} I_1'' &\le \|f\|_\infty^2 \|D^2 v\|_\infty^{2p} \int_{(\operatorname{spt} v)_1}\int_{B_1(y)} \frac{1}{\|x-y\|^{d+sp-2p}}\,\mathrm{d}x\,\mathrm{d}y \\ &\le C(f,v) d\omega_d |(\operatorname{spt} v)_1| \int_0^1 \varrho^{p-1+p(1-s)}\,\mathrm{d}\varrho \\ &= \frac{C(d,f,v)}{2p-sp} \le C(d,p,f,v). \end{aligned} \tag{3.31}$$

By the non-negativity of I_1'', I_2, I_3, and by (3.29)–(3.31), we thus get that

$$\lim_{s\to 1^-} (1-s)(I_1'' + I_2 + I_3) = 0. \tag{3.32}$$

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Therefore, in order to conclude, we are left with showing that $\lim_{s\to 1^-}(1-s)I_1'$ equals the right hand side of (3.26). Indeed, by the change of variables $z = x - y$, by setting $\nu(y) = \nabla v(y)/\|\nabla v(y)\|$, and using (3.23), we have

$$\begin{aligned}
I'_1 &= \int_{\mathbb{R}^d}\int_{B_1} \frac{|\nabla v(y)\cdot z|^p}{\|z\|^{d+sp}} f^2(y)\,\mathrm{d}z\,\mathrm{d}y \\
&= \int_{\mathbb{R}^d} \|\nabla v(y)\|^p f^2(y) \int_{B_1} \frac{|\nu(y)\cdot z|^p}{\|z\|^{d+sp}}\,\mathrm{d}z\,\mathrm{d}y \\
&= \frac{K_{d,p}}{(1-s)} \int_{\mathbb{R}^d} \|\nabla v(x)\|^p f^2(y)\,\mathrm{d}y. \qquad (3.33)
\end{aligned}$$

The conclusion hence follows by combining (3.32) and (3.33). □

REMARK 3.7. We underline that, in the chain of inequalities leading to (3.30), the Lipschitz regularity of the weight f is crucial in order to ensure the finiteness of the integral in ϱ. If f were only α-Hölder, for some $\alpha \in (0,1)$, one would end up with ϱ to the power $-(\alpha - 1 - p + sp)$, which is not integrable in a neighborhood of the origin.

We are now ready to prove the lim sup statement (iii) in Theorem 3.1.

Proof of Theorem 3.1(iii). Let $u \in W_0^{1,p}(\Omega)$ for $p > 1$, or $u \in BV(\Omega)$ for $p = 1$. By the density of $C_c^\infty(\Omega)$ in $W_0^{1,p}(\Omega)$ for $p > 1$, or in $BV(\Omega)$ for $p = 1$, we can find $(v^k)_{k\in\mathbb{N}} \subset C_c^\infty(\Omega)$ such that $v^k \to u$ in $W_0^{1,p}(\Omega)$ for $p > 1$, or in energy in $BV(\Omega)$ for $p = 1$. In view of Theorem 3.6, we thus get that

$$\begin{aligned}
\lim_{n\to\infty}(1-s_n)[\tilde{v}^k]^p_{s_n,p,f} &= \lim_{n\to\infty}(1-s_n)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|\tilde{v}^k(x)-\tilde{v}^k(y)|^p}{\|x-y\|^{d+s_np}} f(x)f(y)\,\mathrm{d}x\,\mathrm{d}y \\
&= K_{d,p}\int_{\mathbb{R}^d} \|\nabla\tilde{v}^k(x)\|^2 f^2(x)\,\mathrm{d}x \\
&= K_{d,p}\int_{\Omega} \|\nabla v^k(x)\|^p f^2(x)\,\mathrm{d}x
\end{aligned}$$

for every $k \in \mathbb{N}$. The conclusion hence follows by a standard diagonal argument. □

3.4. Proof of Theorem 1.1

We can now prove Theorem 1.1 in full generality. Indeed, the result easily follows by combining Theorem 3.1 with Lemmas 3.8 and 3.9 below (under the same standing assumptions as stated at the very beginning of Section 3).

LEMMA 3.8. *If $(u^n)_{n\in\mathbb{N}} \subset L^p(\Omega)$ is such that*

$$\sup_{n\in\mathbb{N}}\Big((1-s_n)[\tilde{u}^n]^p_{s_n,p} + \|u^n\|^p_{L^p(\Omega)}\Big) < \infty, \qquad (3.34)$$

then

$$\lim_{n\to\infty}(1-s_n)\big|[\tilde{u}^n]^p_{s_n,p,f_n} - [\tilde{u}^n]^p_{s_n,p,f}\big| = 0. \qquad (3.35)$$

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Proof. Since we can estimate

$$\left|[\tilde{u}^n]^p_{s_n,p,f_n} - [\tilde{u}^n]^p_{s_n,p,f}\right| \leq [\tilde{u}^n]^p_{s_n,p} \sup_{(x,y)\in\mathbb{R}^{2d}} |f_n(x)f_n(y) - f(x)f(y)|$$
$$\leq 2[\tilde{u}^n]^p_{s_n,p} \sup\{\|f_n\|_\infty : n \in \mathbb{N}\}\|f_n - f\|_\infty,$$

the conclusion immediately follows from (3.34). □

Lemma 3.9. *If $(u^n)_{n\in\mathbb{N}} \subset L^p(\Omega)$ is such that*

$$\sup_{n\in\mathbb{N}} \Big((1-s_n)[\tilde{u}^n]^p_{s_n,p,f_n} + \|u^n\|^p_{L^p(\Omega)}\Big) < \infty, \tag{3.36}$$

then

$$\sup_{n\in\mathbb{N}} (1-s_n)[\tilde{u}^n]^p_{s_n,p} + \|u^n\|^p_{L^p(\Omega)} < \infty.$$

Proof. On the one hand, by the inequality $(a+b)^p \leq 2^p(a^p+b^p)$ for $a,b \geq 0$, exploiting symmetry, and passing to the d-dimensional spherical coordinates, we can estimate

$$\begin{aligned}\int_{\mathbb{R}^d}\int_{B_1(y)^c} \frac{|\tilde{u}^n(x)-\tilde{u}^n(y)|^p}{\|x-y\|^{d+s_np}}\,\mathrm{d}x\,\mathrm{d}y &\leq 2^p \int_{\mathbb{R}^d}\int_{B_1(y)^c} \frac{|\tilde{u}^n(x)|^p + |\tilde{u}^n(y)|^p}{\|x-y\|^{d+s_np}}\,\mathrm{d}x\,\mathrm{d}y \\ &\leq 2^{p+1}\int_{\mathbb{R}^d} |\tilde{u}^n(y)|^p \int_{B_1(y)^c} \frac{\mathrm{d}x}{\|x-y\|^{d+s_np}}\,\mathrm{d}y \\ &= 2^{p+1}\|u^n\|^p_p \int_1^\infty \frac{d\omega_d}{\varrho^{1+s_np}}\,\mathrm{d}\varrho = \frac{C(d,p)\|u^n\|^p_p}{s_n} \leq C(d,p)\|u^n\|^p_p\end{aligned} \tag{3.37}$$

for all $n \in \mathbb{N}$, since $s_n \to 1^-$, thus we can assume $s_n > \frac{1}{2}$. On the other hand, recalling the notation in (2.1), since $\tilde{u}^n(y)$ is supported on Ω, then $\tilde{u}^n(x)$ for $x \in B_1(y)$ is supported on Ω_1, and since $f_n \to f$ uniformly and $f > 0$, we have

$$\begin{aligned}\int_{\mathbb{R}^d}\int_{B_1(y)} \frac{|\tilde{u}^n(x)-\tilde{u}^n(y)|^p}{\|x-y\|^{d+s_np}}\,\mathrm{d}x\,\mathrm{d}y &= \int_{\Omega_1}\int_{\Omega_1} \frac{|\tilde{u}^n(x)-\tilde{u}^n(y)|^p}{\|x-y\|^{d+s_np}}\,\mathrm{d}x\,\mathrm{d}y \\ &\leq \int_{\Omega_1}\int_{\Omega_1} \frac{|\tilde{u}^n(x)-\tilde{u}^n(y)|^p}{\|x-y\|^{d+s_np}} \frac{f_n(x)f_n(y)}{\inf_{(x,n)\in\Omega_1\times\mathbb{N}} f_n^2(x)}\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{[\tilde{u}^n]^p_{s_n,p,f_n}}{\inf_{(x,n)\in\Omega_1\times\mathbb{N}} f_n^2(x)} = C(\Omega_1, f_n)[\tilde{u}^n]^p_{s_n,p,f_n}.\end{aligned} \tag{3.38}$$

Since $f_n \to f$ uniformly and $f > 0$, the constant C can be made independent of n, and only dependent on the uniform limit f. The conclusion hence follows by combining (3.37) and (3.38), multiplying by $(1-s_n)$, and owing to the assumption (3.36). We omit the plain details. □

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4. Proof of Theorem 1.2

4.1. Stability of Hilbertian gradient flows

We briefly recall some abstract machinery from [9] concerning Hilbertian gradient flows.

Let $\mathscr{H}$ be a Hilbert space endowed with scalar product $\langle\cdot,\cdot\rangle_{\mathscr{H}}$ and norm $\|\cdot\|_{\mathscr{H}}$. Given $\mathscr{F}\colon \mathscr{H}\to(-\infty,+\infty]$, we let $\mathscr{D}(\mathscr{F})=\{\,x\in\mathscr{H}\colon \mathscr{F}(x)<+\infty\,\}$ and

$$\partial\mathscr{F}(x)=\left\{\,v\in\mathscr{H}\colon \liminf_{y\to x}\frac{\mathscr{F}(y)-\mathscr{F}(x)-\langle v,y-x\rangle_{\mathscr{H}}}{\|y-x\|_{\mathscr{H}}}\geq 0\,\right\}$$

be the *subdifferential* of $\mathscr{F}$ at $x\in\mathscr{D}(\mathscr{F})$.

We recall the following result, which is a particular case of [9, Prop. 3.7]. Here and below, given any vector space $\mathscr{V}$, we let $\mathscr{V}^*$ be the algebraic dual space of $\mathscr{V}$ and $\mathscr{V}'$ the topological dual space of $\mathscr{V}$.

PROPOSITION 4.1. *Let $\mathscr{K}$ be a dense subspace of $\mathscr{H}$. Let $\mathscr{F}\colon\mathscr{H}\to(-\infty,+\infty]$ be a proper, convex, and strongly lower semicontinuous functional and let $x\in\mathscr{D}(\mathscr{F})$. If there exists $T\in\mathscr{K}^*$ such that*

$$\lim_{t\to 0}\frac{\mathscr{F}(x+ty)-\mathscr{F}(x)}{t}=T(y),\quad \text{for every } y\in\mathscr{K},$$

then either $\partial\mathscr{F}(x)=\emptyset$ or $\partial\mathscr{F}(x)=\{v\}$, where v is the (unique) element in $\mathscr{H}$ satisfying $T(y)=\langle v,y\rangle_{\mathscr{H}}$ for every $y\in\mathscr{K}$. In particular, $T\in\mathscr{K}'$ and v is its unique continuous extension to $\mathscr{H}'$.

We also recall the following stability result, which is contained in [9, Thm. 3.8].

THEOREM 4.2 *Let $\mathscr{F}_n\colon\mathscr{H}\to(-\infty,+\infty]$ be a proper, strongly lower semicontinuous, convex, and positive functional for every $n\in\mathbb{N}$. Assume the following:*

(a) $(\mathscr{F}_n)_{n\in\mathbb{N}}$ Γ-converges to some proper functional $\mathscr{F}_\infty\colon\mathscr{H}\to(-\infty,+\infty]$ with respect to the strong $\mathscr{H}$-convergence;
(b) any bounded sequence $(x_n)_{n\in\mathbb{N}}\subset\mathscr{H}$ such that $\sup_{n\in\mathbb{N}}\mathscr{F}_n(x_n)<\infty$ admits a strongly $\mathscr{H}$-convergent subsequence.

If $(x_0^n)_{n\in\mathbb{N}}\subset\mathscr{H}$ is such that $x_0^n\in\mathscr{D}(\mathscr{F}_n)$ for every $n\in\mathbb{N}$, $\sup_{n\in\mathbb{N}}\mathscr{F}_n(x_0^n)<\infty$ and $x_0^n\to x_0^\infty$ strongly in $\mathscr{H}$ for some $x_0^\infty\in\mathscr{H}$, then the following hold:

(i) *$x_0^\infty\in\mathscr{D}(\mathscr{F}_\infty)$;*
(ii) *for every $T>0$, the problem*

$$\begin{cases}\dot{x}(t)\in-\partial\mathscr{F}_n(x(t)), & \textit{for a.e. } t\in(0,T),\\ x(0)=x_0^n,\end{cases}$$

admits a unique solution $x_n\in H^1([0,T];\mathscr{H})$ for every $n\in\mathbb{N}\cup\{\infty\}$;

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(iii) $(x_n)_{n\in\mathbb{N}}$ *weakly converges to* x_∞ *in* $H^1([0,T];\mathscr{H})$.

Moreover, if $\lim_{n\to\infty}\mathscr{F}^n(x_0^n)=\mathscr{F}_\infty(x_0^\infty)$, *then actually* $(x_n)_{n\in\mathbb{N}}$ *strongly converges to* x_∞ *in* $H^1([0,T];\mathscr{H})$ *and also*

$$x_n(t)\xrightarrow{\mathscr{H}}x_\infty(t)\quad\textit{and}\quad\mathscr{F}_n(x_n(t))\to\mathscr{F}_\infty(x_\infty(t))\quad\textit{for every } t\in[0,T].$$

4.2. Proof of Theorem 1.2

The validity of Theorem 1.2 follows by combining the abstract results above with the following proposition. Here and below, $\langle\cdot,\cdot\rangle$ denotes the standard scalar product in $L^2(\mathbb{R}^d)$.

PROPOSITION 4.3. *Let* $\varphi\in C_c^\infty(\Omega)$ *and* $u\in L^2(\Omega)$. *The following hold:*

(i) *if* $u\in H_0^1(\Omega)$, *then*

$$\lim_{t\to0}\frac{\|\nabla(u+t\varphi)\|_{2,f^2}^2-\|\nabla u\|_{2,f^2}^2}{t}=\langle(-\mathfrak{D})^f u,\varphi\rangle;\tag{4.1}$$

(ii) *if* $[\tilde{u}]_{s,2,f}<\infty$ *for some* $s\in(0,1)$, *then*

$$\lim_{t\to0}\frac{[u+t\varphi]_{s,2,f}^2-[u]_{s,2,f}^2}{t}=\langle(-\mathfrak{D})^{s,f}u,\varphi\rangle.\tag{4.2}$$

Proof. We only prove (4.2), the proof of (4.1) being straightforward. We note that

$$\begin{aligned}[u+t\varphi]_{s,2,f}^2&=[u]_{s,2,f}^2+t^2[\varphi]_{s,2,f}^2\\&\quad+2t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{\|x-y\|^{d+2s}}f(x)f(y)\,\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

for every $t\in\mathbb{R}$, and thus we easily get that

$$\begin{aligned}\lim_{t\to0}\frac{[u+t\varphi]_{s,2,f}^2-[u]_{s,2,f}^2}{t}&=2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{\|x-y\|^{d+2s}}f(x)f(y)\,\mathrm{d}x\,\mathrm{d}y\\&=4\int_{\mathbb{R}^d}u(x)f(x)\lim_{r\to0^+}\int_{\mathbb{R}^d\setminus B_r}\frac{\varphi(x)-\varphi(y)}{\|x-y\|^{d+2s}}f(y)\,\mathrm{d}y\,\mathrm{d}x\\&=\langle u,(-\mathfrak{D})^{s,f}\varphi\rangle=\langle(-\mathfrak{D})^{s,f}u,\varphi\rangle\end{aligned}$$

in virtue of the distributional definition in (1.7), concluding the proof. □

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Since $f_n\to f$ uniformly and $f>0$, the functionals given by $\mathscr{F}_n(u)=(1-s_n)[\tilde{u}]_{s_n,2,f_n}^2$ are positive for $n\gg1$. Further, they are easily shown to be convex. By Theorem 1.1(ii)–(iii), $(\mathscr{F}_n)_{n\in\mathbb{N}}$ Γ-converges to the functional $\mathscr{F}(u)=K_{d,2}\|\nabla u\|_{2,f^2}^2$ in the strong topology of $L^2(\Omega)$, whereas by Theorem 1.1(i) every

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sequence $(u_n)_{n\in\mathbb{N}} \subset L^2(\Omega)$ such that $\sup_{n\in\mathbb{N}} \mathscr{F}_n(u_n) < \infty$ admits a strong L^2-limit $u \in L^2(\Omega)$. The conclusion hence follows by Theorem 4.2 and Proposition 4.1 combined with Proposition 4.3. □

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