A NOTE ON SOME PERFECT SQUARED SQUARES

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1. Introduction. In a recent paper [5], general methods were described for the dissection of a square into a finite number n of unequal non-overlapping squares. In this note, examples of such perfect squares are given in which the sides and elements are relatively small integers; in particular, a dissection of a square into 24 different elements, which is believed to be the squaring of least order known at the present time. All the dissections which follow make use of auxiliary rectangles; that is to say, the squarings are compound. The following terminology, introduced by the authors of [2], will be used. A *dissection* of a rectangle R into a finite number n of non-overlapping squares is called a *squaring* of R of *order* n; and the n squares are the *elements* of the dissection. If the elements are all unequal and $n > 1$, the squaring is *perfect* and R is a *perfect rectangle.* A squared rectangle which contains a smaller squared rectangle is called *compound*, all others being *simple.* Two squared rectangles which have the same shape (i.e. proportional sides) but are not merely rigid displacements of each other are called *conformal;* two conformal rectangles are said to be *totally different* if C_2 times an element of the first is never equal to C_1 times an element of the second, where C_1 and C_2 are their respective corresponding sides.

Complete dissections will be expressed in the notation of C. J. Bouwkamp [3], which consists in enclosing within brackets the lengths of the sides of those squares whose upper sides lie in the same horizontal segment, the brackets being read in order from the top to the bottom of the rectangle.

2. Methods for constructing perfect squares. The following methods are based on the combination in various ways of certain related pairs of rectangles. A squared rectangle of sides x and y will be written (x, y) and if the squaring is perfect will be denoted by P (x, y). A rectangle which has but two elements equal, one (or both) of which is in a corner, will be called *quasi-perfect* and written Q (x, y). The imperfection of such rectangles may be either trivial or non-trivial, two equal squares constituting a *trivial* imperfection if they extend between the same two horizontal (or vertical) segments in the dissection [2, (5.22)]. In what follows, when reference is made to two perfect rectangles, it is to be understood that they are totally different, unless the contrary is stated.

2.1. *From two rectangles* P (x, y) *and two squares* x *and* y. An example of order 28 and reduced side 1015 is given in [1]. One of order 34 and reduced side 960 is derived from two P (479, 481). It is (218, 124, 137, 481), (49, 53, 22),

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(9, 128), (31), (45, 4), (88), (263), (216), (479, 240, 241), (239, 1), (76, 67, 99), (11, 24, 32), (59, 15, 2), (13), (44, 8), (139), (103).

2.11. *From two rectangles* P (x, y) *and two squares* $x - A$ *and* $y - A$ *where* A *is a corner element of one* P (x, y). In the resulting squaring the element A is overlapped. In favourable cases eight different perfect squares may be formed, but some arrangements of the border squares of the auxiliary rectangles yield a lesser number. Examples of order 33 and reduced sides 821, 823, 857, 861 and 884 are derived from two P (479, 481). These rectangles are (218, 124, 137), (49, 53, 22), (9, 128), (31), (45, 4), (88), (263), (216) and (76, 67, 99, 239), (11, 24, 32), (59, 15, 2), (13), (44, 8), (139), (103), (241, 1), (240) and the P-squares are obtained by making A successively, 139, 137, 103, 99, 76. (As the second of these rectangles is compound, reorientation of its auxiliary rectangle is necessary to make the squares 76, 99, 103, 139 successively occupy a corner.)

Note (a). This method fails if both rectangles are such that no two adjacent sides can be chosen each having more than two elements, e.g., the rectangles XIII, 1015 g and h in the classification of [3].

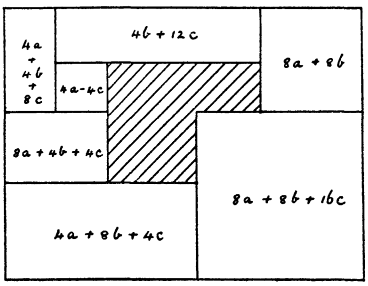

FIG. A

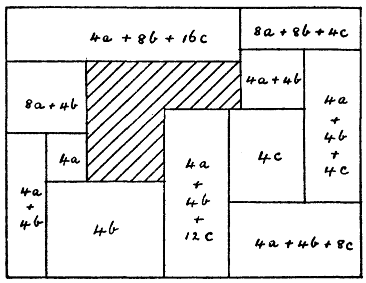

FIG. B

Note (b). This method can be used however if (i) the two rectangles P(x, y) have one corner element A in common. The structure of a square of 39th order of this type was given in [2], or if (ii) a P-rectangle and a Q-rectangle are used instead of two P-rectangles. An example of order 24 and reduced side 175 is (55, 39, 81), (16, 9, 14), (4, 5), (3, 1), (20), (56, 18), (38), (30, 51), (64, 31, 29), (8, 43), (2, 35), (33). One of order 26 and reduced side 492 is (57, 59, 56, 95, 225), (3, 14, 39), (55, 2,), (53, 11), (25), (17, 142), (125), (24, 60, 141), (255, 12), (36), (96), (15, 126), (111). The first of these was published in [4].

2.2. *From rectangles* P(x, y) *and* P$(x, x+y)$ *and a square* y. An example of order 26 and reduced side 608 was given in [2].

2.21. *From rectangles* P(x, y) *and* P$(x, x+y)$ *and squares* $x, x-A, x+y-A$ *where* A *is a corner element of a rectangle, and is overlapped as before.* Using the auxiliary rectangles employed in the last mentioned square of side 608, and

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making A successively 136, 118, 113 and 95, squares of order 27 and reduced sides 849, 867, 872 and 890 are obtained.

2.3. *From rectangles* (x, y) *and* $(x+y, x+2y)$ *and squares* $y, x+y, x+y-A$ *and* $x+2y-A$. An example of order 28 and reduced side 577 is derived from Q (113, 127) and P(240, 353) and was first described in [4]. It is (224, 123, 129, 101), (28, 73), (117, 6), (111, 52,), (7, 66), (59), (113, 51, 21, 23, 16), (32, 337), (19, 2), (25), (11, 8), (65), (62), (240).

2.4. In the foregoing methods it has been necessary that the two auxiliary rectangles should have either no element in common, or one element of the one equal to a corner element of the other. In some cases, however, auxiliary rectangles not conforming to these conditions may serve, since certain P-rectangles (p, q) containing n elements can be transformed into Q-rectangles (p, q) of $n+4$ elements, sometimes in more than one way. It may happen, therefore, that two P-rectangles (x, y) may have one or more elements in common, but that one of them and a derived Q-rectangle may have none in common and method 2.11 may be applied to form a perfect square.

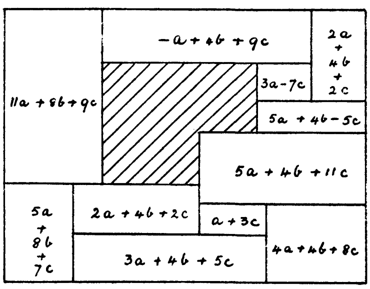

Fig. C

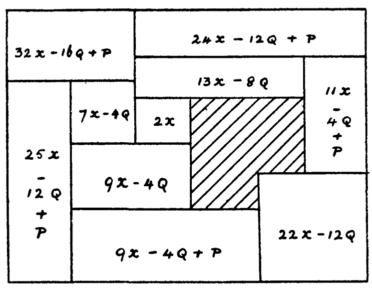

Fig. D

In the diagrams the shaded area represents a six-sided squared polygon, five angles of which are right angles. It will be seen that the three rectangles of figures A, B and C are conformal, whatever the structure of the polygon. In general A is a P-rectangle, B and C are Q-rectangles. There is a corner element $(4a + 4b + 8c)$ of A which is also a corner element of B and C, which fact allows further conformal pairs to be derived; for, if this element be removed from A and to the resulting polygon is added any arrangement of additional elements, then the same operation performed on B or C will result in a conformal figure.

Now suppose the corner square $(4a + 4b + 8c)$ of A be removed and four squares added so as to form the rectangle of Figure D, then a conformal rectangle may be formed in the following manner: the corner square $(8a + 8b + 16c)$ of A may be thought of as being composed of four equal smaller squares $(4a + 4b + 8c)$ containing a square x (of side zero). If the

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new smaller corner square be removed and four squares added as before in corresponding positions, the square x will no longer (in general) vanish, and the conformal rectangle of Figure E will be formed.

An example follows of how such considerations may lead to a perfect square. The two P-rectangles (164, 118, 197), (39, 79,), (7, 32), (122, 49), (24, 8), (284), (73), (195), and (218, 124, 137), (49, 53, 22), (9, 128), (31), (45, 4), (88), (263), (216), are conformal and have one element (49) in common. The second of these rectangles is of the form D, and it is found that the conformal Q-rectangle of form E (274, 305, 465, 872), (243, 31), (212, 124), (88, 36), (274, 227), (543), (47, 180), (321), (1052), (864) together with the first of the P-rectangles (magnified) and the addition of squares 1642 and 1650 makes a perfect square of order 33 and reduced side 3566. An alternative dissection of this square of order 37 is (960, 964, 1642), (956, 4), (304, 268, 396), (44, 96, 128), (236, 60, 8,) (52), (176, 32), (556), (412), (305, 465, 872), (1650, 243, 31), (212, 124), (88, 36), (274, 227), (543), (47, 180), (321), (1052), (864).

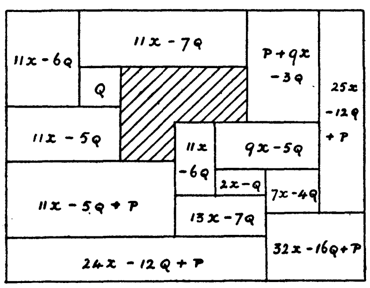

FIG. E

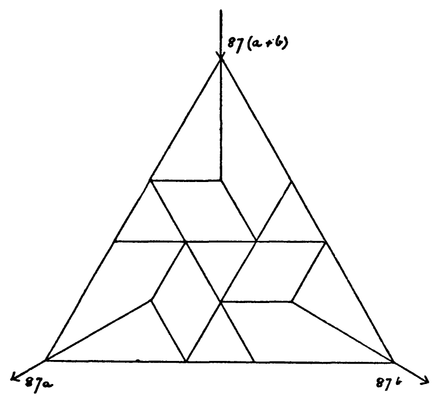

FIG. F

Now suppose two completely different squared polygons of the same size and shape are constructed by the method described in [2], and their shape is controlled so that with the addition of seven squares to one of them a P-rectangle such as Figure A above is formed, then Q-rectangles of the types B and C can be formed from the other. Since rectangle A can be combined with either B or C to form a square, the diagrams illustrated (or others derived from them) may be used to form non-trivially different squares of the same order and size (and these will, in general, be perfect.) Two such that have been evaluated are of order 73 and reduced side 1535484 and are constructed, by the methods of [2] from the diagram of Figure F, when a and b are assigned the values 506 and 185 respectively. This network was discussed in [2] and [3].

3. The construction of rectangles. The auxiliary rectangles employed in the squares described were discovered by constructing lists of isoperimetric rectangles of different sizes and selecting any pairs which were found to fulfil the required conditions. In the attempt to find P-squares of small reduced side, attention was centred mainly on semi-perimeters containing a number of small factors. It was shown in [2] that two rectangles which have the same c-net are isoperimetric.

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