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MINIMAX INEQUALITIES IN G-CONVEX SPACES

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In this paper we establish two minimax theorems of Sion-type in G-convex spaces. As applications we obtain generalisations of some theorems concerning compatibility of some systems of inequalities.

1. INTRODUCTION AND PRELIMINARIES

Motivated by Nash equilibrium and the theory of non-cooperative games, Fan [4] generalised Sion's minimax theorem obtaining the following two-function minimax inequality:

THEOREM 1. *Let X and Y be compact convex subsets of topological vector spaces and $f, g : X \times Y \to \mathbb{R}$. Suppose that f is lower semicontinuous on Y and quasiconcave on X, g is upper semicontinuous on X and quasiconvex on Y, and $f \leqslant g$ on $X \times Y$. Then $\min_{y \in Y} \sup_{x \in X} f(x,y) \leqslant \max_{x \in X} \inf_{y \in Y} g(x,y)$.*

Granas and Liu [6, 7] obtained generalisations and versions of Theorem 1 involving three real functions f, g, h. On the other hand Park [14] extended Ky Fan's result to G-convex spaces. In this paper we obtain a unified generalisation of all these results. Also we give a version of our main result for the case when X is a convex subset of a topological vector space. As applications we obtain generalisations of some theorems of Granas and Liu [6, 7] and Liu [11] concerning compatibility of some systems of inequalities.

Let us recall some notions necessary in our paper.

A *generalised convex space* or a *G-convex space* $(X, D; \Gamma)$ consists of a topological space X and a nonempty set D such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n+1$, there exist a subset $\Gamma(A)$ of X and a continuous function $\Phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\Phi_A(\Delta_J) \subset \Gamma(J)$.

Here $\langle D \rangle$ denotes the set of all nonempty finite subsets of D, Δ_n any n-simplex with vertices $\{e_i\}_{i=0}^n$ and Δ_J the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{u_0, u_1, \dots, u_n\}$ and $J = \{u_{i_0}, u_{i_1}, \dots, u_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1} \dots, e_{i_k}\}$.

In case $D \subset X$ then $(X, D; \Gamma)$ will be denoted by $(X \supset D; \Gamma)$. For $(X \supset D; \Gamma)$, a subset C of X is said to be *G-convex* if $\Gamma(A) \subset C$ whenever $A \in \langle C \cap D \rangle$.

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The main example of G-convex space corresponds to the case when $X = D$ is a convex subset of a Hausdorff topological vector space and for each $A \in \langle X \rangle$, $\Gamma(A)$ is the convex hull of A. For other major examples of G-convex spaces see [**15, 16**].

Let $(X \supset D; \Gamma)$ be a G-convex space. A function $f : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ is said to be *G-quasiconcave* (respectively, *G-quasiconvex*) if for any finite set $\{u_1, \dots, u_n\} \subset D$ and for each $x \in \Gamma(\{, u_1, \dots, u_n\})$ we have $f(x) \geqslant \min_{1 \leqslant i \leqslant n} f(u_i)$ (respectively, $f(x) \leqslant \max_{1 \leqslant i \leqslant n} f(u_i)$). We note that f is G-quasiconcave (respectively, G-quasiconvex) if and only if, for each $\lambda \in \mathbb{R}$ the set $\{x \in X : f(x) > \lambda\}$ (respectively, $\{x \in X : f(x) < \lambda\}$) is G-convex. A function $f : X \times Y \to \overline{\mathbb{R}}$ (Y nonempty set) is said to be *G-quasiconcave* (respectively, *G-quasiconvex*) on X if for each $y \in Y$ the function $x \to f(x, y)$ is G-quasiconcave (respectively, G-quasiconvex). Inspirated by [**1**] and [**9**] we shall introduce two more general concepts.

Let $(X, D; \Gamma)$ be a G-convex space, Y be a nonempty set and $f : D \times Y \to \overline{\mathbb{R}}$, $g : X \times Y \to \overline{\mathbb{R}}$. We say that g is *G-f-quasiconcave on X* if for any finite set $\{u_1, \dots, u_n\} \subset D$ and for each $y \in Y$ we have

$$g(x, y) \geqslant \min_{1 \leqslant i \leqslant n} f(u_i, y) \text{ for all } x \in \Gamma(\{u_1, \dots, u_n\}).$$

Note that the notion introduced above coincides with the corresponding notion in [**9**, Definition 2] only when $D = X$.

When X is a convex subset of a topological vector space the concept of G-f-quasiconcavity reduces to that of f-quasiconcavity introduced by Chang and Yen in [**1**]. More precisely, in this case, if $f, g : X \times Y \to \overline{\mathbb{R}}$ we say that g is *f-quasiconcave on X* if for any $\{x_1, \dots, x_n\} \in \langle X \rangle$ and each $y \in Y$ we have

$$g(x, y) \geqslant \min_{1 \leqslant i \leqslant n} f(x_i, y) \text{ for all } x \in \operatorname{co}\{x_1, \dots, x_n\}.$$

Similarly, if X is a nonempty set, $(Y, D; \Gamma)$ a G-convex space and

$$g : X \times Y \to \overline{\mathbb{R}},$$
$$h : X \times D \to \overline{\mathbb{R}}$$

two functions, we say that g is *G-h-quasiconvex on Y* if for any $\{v_1, \dots, v_n\} \in \langle D \rangle$ and each $x \in X$ we have

$$g(x, y) \leqslant \max_{1 \leqslant i \leqslant n} h(x, v_i) \text{ for all } y \in \Gamma(\{v_1, \dots, v_n\}).$$

REMARK 1. It is easy to see that if $D \subset Y$, g is G-h-quasiconvex on Y whenever there exists a function $k : X \times Y \to \overline{\mathbb{R}}$ such that:

(i) $g \leqslant k$ on $X \times Y$;

(ii) $k \leqslant h$ on $X \times D$;

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(iii) k is G-quasiconvex on Y.

Let X be a nonempty set, $(Y, D; \Gamma)$ be a G-convex space and $G : Y \multimap X$, $H : D \multimap X$ be two mappings (that is, set-valued functions). We say that H is a *generalised G-KKM mapping with respect to* G if for each $A \in \langle D \rangle$, $G(\Gamma(A)) \subset H(A)$. If X is a topological space, $G : Y \multimap X$ is said to have the *G-KKM property* if for any mapping $H : D \multimap X$ generalised G-KKM with respect to G, the family $\{\overline{H}(v) : v \in D\}$ has the finite intersection property (where $\overline{H}(v)$ denotes the closure of $H(v)$).

Let X be a topological space and Y be a nonempty set. A function $f : X \times Y \to \overline{\mathbb{R}}$ is said to be λ*-transfer upper semicontinuous* (respectively λ*-transfer lower semicontinuous*) on X for some $\lambda \in \mathbb{R}$ [2] if for all $x \in X$, $y \in Y$ with $f(x,y) < \lambda$ (respectively $f(x,y) > \lambda$) there exist a neighbourhood $V(x)$ of x and a point $y' \in Y$ such that $f(z,y') < \lambda$ (respectively $f(z,y') > \lambda$) for all $z \in V(x)$. If f is λ-transfer upper (respectively lower) semicontinuous on X for any $\lambda \in \mathbb{R}$, we say that f is transfer upper (respectively lower) semicontinous on X.

It is clear that every function upper semicontinuous (respectively, lower semicontinuous) on X is λ-transfer upper semicontinuous (respectively, λ-transfer lower semicontinuous) on X for any real λ, but the converse is not true (see [2]).

2. Main results

First we state three results from the literature which will be used in this section. The following continuous selection theorem is well-known (see [**10, 13, 17**]).

Lemma 2. *Let $(X, D; \Gamma)$ be a G-convex space and Y be a compact topological space. Let $F : Y \multimap D$, $G : Y \multimap X$ be two mappings satisfying the following conditions:*

(a) *for each $y \in Y$, $A \in \langle F(y) \rangle$ implies $\Gamma(A) \subset G(y)$;*

(b) $Y = \cup\{\operatorname{int} F^{-1}(u) : u \in D\}$.

Then G has a continuous selection; that is, there exists a continuous function $p : Y \to X$ such that $p(y) \in G(y)$ for each $y \in Y$.

The next result is a particular case of Corollary in [**12**].

Lemma 3. *Let X be a topological space and $(Y, D; \Gamma)$ be a G-convex space, Then any continuous function $p : Y \to X$ has the G-KKM property.*

Combining assertions (ii) and (iii) in Lemma 3 and assertion (ii) in Lemma 4 in [8] one obtains

Lemma 4. *Let X be a topological space and D a nonempty set. If $h : X \times D \to \overline{\mathbb{R}}$ is λ-transfer upper semicontinuous, then $\bigcap_{v \in D} H(v) = \bigcap_{v \in D} \overline{H}(v)$, where*

$$H(v) = \{x \in X : h(x,v) \geqslant \lambda\}.$$

The main result of the paper is as shown in the following theorem.

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Theorem 5. *Let $(X, D; \Gamma_1)$ and $(Y, D; \Gamma_2)$ be two compact G-convex spaces and let $f : D_1 \times Y \to \overline{\mathbb{R}}$, $g : X \times Y \to \overline{\mathbb{R}}$, $h : X \times D_2 \to \overline{\mathbb{R}}$ be three functions such that:*

(i) *g is G-f-quasiconcave on X;*

(ii) *g is G-h-quasiconvex on Y;*

(iii) *f is transfer lower semicontinuous on Y;*

(iv) *h is transfer upper semicontinuous on X;*

Then $\inf\limits_{y\in Y} \sup\limits_{u\in D_1} f(u,y) \leqslant \sup\limits_{x\in X} \inf\limits_{v\in D_2} h(x,v)$.

Proof: We may suppose that $\inf\limits_{y\in Y} \sup\limits_{u\in D_1} f(u,y) > -\infty$. It suffices to prove that for any real $\lambda < \inf\limits_{y\in Y} \sup\limits_{u\in D_1} f(u,y)$ we have $\lambda \leqslant \sup\limits_{x\in X} \inf\limits_{v\in D_2} h(x,v)$. Fix such a λ and define the mappings $F : Y \multimap D_1$, $G : Y \multimap X$, $H : D_2 \multimap X$ by

$$F(y) = \{u \in D_1 : f(u,y) \geqslant \lambda\}, \quad G(y) = \{x \in X : g(x,y) \geqslant \lambda\} \quad \text{and}$$
$$H(v) = \{x \in X : h(x,v) \geqslant \lambda\}.$$

First we show that G and F satisfy the conditions of Lemma 2. Let $y \in Y$, $\{u_1, \dots, u_n\} \subset F(y)$ and $x \in \Gamma_1(\{u_1, \dots, u_n\})$. Since g is f-quasiconcave on X, $g(x,y) \geqslant \min\limits_{1\leqslant i\leqslant n} f(u_i, y) \geqslant \lambda$, hence $x \in G(y)$. Thus $\Gamma_1(\{u_1, \dots, u_n\}) \subset G(y)$.

For each $y \in Y$ there exists $u \in D_1$ such that $f(u,y) > \lambda$ (as consequence of $\lambda < \inf\limits_{y\in Y} \sup\limits_{u\in D_1} f(u,y)$). By (iii) there exist $u' \in D_1$ and a neighbourhood $V(y)$ of y such that

$$u' \in \bigcap_{z\in V(y)} \{u \in D_1 : f(u,z) > \lambda\} \subset \bigcap_{z\in V(y)} F(z),$$

hence $y \in \operatorname{int} F^{-1}(u')$. Thus condition (b) in Lemma 2 is satisfied. By Lemma 2, there exists a continuous function $p : Y \to X$ such that $p(y) \in G(y)$ for every $y \in Y$.

Next we prove that H is a generalised G-KKM mapping with respect to G. Suppose that there exist a nonempty finite set $\{v_1, \dots, v_n\} \subset D_2$ and a point $x \in X$ such that

$$x \in G\Big(\Gamma_2(\{v_1, \dots, v_n\})\Big) \Big\backslash \bigcup_{i=1}^{n} H(v_i).$$

Since $x \in G\Big(\Gamma_2(\{v_1, \dots, v_n\})\Big)$, there exists $y \in \Gamma_2(\{v_1, \dots, v_n\})$ such that $g(x,y) \geqslant \lambda$. By $x \notin \bigcup\limits_{i=1}^{n} H(v_i)$ we get $h(x, v_i) < \lambda$ for each $i \in \{1, \dots, n\}$. Taking into account (ii) we obtain the following contradiction

$$\lambda \leqslant g(x,y) \leqslant \max_{1\leqslant i\leqslant n} h(x, v_i) < \lambda.$$

Thus H is a generalised G-KKM mappings with respect to G, and consequently it is generalised G-KKM mapping with respect to p, too. By Lemma 3, the family of sets

$\{\overline{H}(v) : v \in D_2\}$ has the finite intersection property. Since for each $v \in D_2$, $\overline{H}(v)$ is a closed subset of compact space Y, by Lemma 4 we infer that $\bigcap_{v \in D_2} H(v) = \bigcap_{v \in D_2} \overline{H}(v) \neq \emptyset$, that is, $\sup_{x \in X} \inf_{v \in D_2} h(x, v) \geqslant \lambda$. $\square$

REMARK 2. Following the proof of Theorem 5 it seems that if $\inf_{y \in Y} \sup_{u \in D_1} f(u, y) > -\infty$, instead of conditions (iii) and (iv) it would be sufficient to put the following conditions:

(iii′) f is λ-transfer lower semicontinuous on Y for any $\lambda < \inf_{y \in Y} \sup_{u \in D_1} f(u, y)$;

(iv′) h is λ-transfer upper semicontinuous on X for any $\lambda < \inf_{y \in Y} \sup_{u \in D_1} f(u, y)$.

But this clearly less demanding conditions make really no difference. In fact, assume

$$a = \inf_{y \in Y} \sup_{u \in D_1} f(u, y) > -\infty$$

and define the functions

$$f'(u, y) = \min(f(u, y), a),$$
$$g'(x, y) = \min(g(x, y), a),$$
$$h'(x, v) = \min(h(x, v), a).$$

We observe that:

(a) if conditions (i), (ii) in Theorem 5 hold for f, g, h, then they hold also for f', g', h';

(b) if f is λ-transfer lower semicontinuous on Y (respectively, h is λ-transfer upper semicontinuous on X) whenever $\lambda < a$, then f' is transfer lower semicontinuous on Y (respectively, h' is transfer upper semicontinuous on X);

(c) $\inf_{y \in Y} \sup_{u \in D_1} f'(u, y) \leqslant \sup_{x \in X} \inf_{v \in D_2} h'(x, v)$ implies $\inf_{y \in Y} \sup_{u \in D_1} f(u, y) \leqslant \sup_{x \in X} \inf_{v \in D_2} h(x, v)$.

A mapping $F : Y \multimap X$ (X nonempty set, Y topological space) is said to have the *local intersection property* (see [**18**]) if for each $y \in Y$ with $F(y) \neq \emptyset$, there exists an open neighbourhood $V(y)$ of y such that $\bigcap_{z \in V(y)} F(z) \neq \emptyset$.

The following continuous selection theorem is [**18**, Theorem 1].

LEMMA 6. *Let X be a nonempty subset of a topological vector space and Y be a paracompact topological space. Suppose that $F, G : Y \multimap X$ are two mappings satisfying the following conditions:*

(a) *for each $y \in Y$, $F(y)$ is nonempty and $\operatorname{co} F(y) \subset G(y)$;*

(b) *F has local intersection property.*

Then G has a continuous selection.

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It can be easily prove that if $D = X$ and F is a mapping with nonempty values, then conditions (b) in Lemmas 2 and 6 are equivalent (see [8, Proposition 1]).

The following version of Theorem 5 shows that in the case when X is a convex subset of a topological vector space the conclusion holds if the G-convex space $(Y, D; \Gamma)$ is only paracompact. The proof is similar to that of Theorem 5 using as argument Lemma 6 instead of Lemma 2.

Theorem 7. *Let X be a compact convex subset of a topological vector space and $(Y, D; \Gamma)$ be a paracompact G-convex space. Let $f, g : X \times Y \to \overline{\mathbb{R}}$ and $h : X \times D \to \overline{\mathbb{R}}$ be three functions such that:*

(i) *g is f-quasiconcave on X;*

(ii) *g is G-h-quasiconvex on Y;*

(iii) *f is transfer lower semicontinuous on Y;*

(iv) *h is transfer upper semicontinuous on X.*

Then $\inf_{y \in Y} \sup_{x \in X} f(x, y) \leqslant \sup_{x \in X} \inf_{v \in D} h(x, v)$.

Let Y be an arbitrary set and D a nonempty subset of Y. Given two families of functions $\mathcal{G} = \{g : Y \to \overline{\mathbb{R}}\}$ and $\mathcal{H} = \{h : D \to \overline{\mathbb{R}}\}$ we write $\mathcal{G} \leqslant \mathcal{H}$ *on* D if for every $g \in \mathcal{G}$ there is $h \in \mathcal{H}$ such that $g(v) \leqslant h(v)$ for all $v \in D$. Following Ky Fan [3] a family of functions $\mathcal{H} = \{h : D \to \overline{\mathbb{R}}\}$ is said to be *concave* provided given any $h_1, \dots, h_n \in \mathcal{H}$ and $x_1, \dots, x_n \in \mathbb{R}$ such that $x_i \geqslant 0$ and $\sum_{i=1}^{n} x_i = 1$ there is an $h \in \mathcal{H}$ satisfying $h(v) \geqslant \sum_{i=1}^{n} x_i h_i(v)$ for all $v \in D$.

In what follows we denote by Δ_{n-1} the standard $(n-1)$-simplex; that is

$$\Delta_{n-1} = \left\{ x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_i \geqslant 0,\ \sum_{i=1}^{n} x_i = 1 \right\}.$$

The next result generalises under many aspects in [7, Theorem 9.2].

Theorem 8. *Let $(Y \supset D; \Gamma)$ be a compact G-convex space and let*

$$\mathcal{F} = \{f : Y \to (-\infty, +\infty]\},$$
$$\mathcal{G} = \{g : Y \to (-\infty, +\infty]\},$$
$$\mathcal{H} = \{h : D \to (-\infty, +\infty]\}$$

be three families of functions such that:

(i) *$\mathcal{F} \leqslant \mathcal{G}$ on Y and $\mathcal{G} \leqslant \mathcal{H}$ on D;*

(ii) *for any finite subfamily $\{g_1, \dots, g_n\}$ of $\mathcal{G}$ and for each $(x_1, \dots, x_n) \in \Delta_{n-1}$ the function $y \to \sum_{i=1}^{n} x_i g_i(y)$ is G-quasiconvex on Y;*

(iii) *each $f \in \mathcal{F}$ is lower semicontinuous on Y;*

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(iv) *the family $\mathcal{H}$ is concave.*

Then $\inf_{y\in Y}\sup_{f\in\mathcal{F}} f(x) \leqslant \sup_{h\in\mathcal{H}}\inf_{v\in D} h(v)$.

PROOF: Let $\beta = \sup_{h\in\mathcal{H}}\inf_{v\in D} h(v)$. We may suppose that β is finite. For each $f \in \mathcal{F}$ let

$$S(f) = \{y \in Y : f(y) \leqslant \beta\}.$$

We have to show that $\bigcap_{f\in\mathcal{F}} S(f) \neq \emptyset$. Since Y is compact and the sets $S(f)$ are closed it suffices to prove that the family $\{S(f) : f \in \mathcal{F}\}$ has the finite intersection property.

Let $f_1, \ldots, f_n \in \mathcal{F}$; choose $g_1, \ldots, g_n \in \mathcal{G}$ and $h_1, \ldots, h_n \in \mathcal{H}$ such that

$$f_i \leqslant g_i \text{ on } Y \text{ and } g_i \leqslant h_i \text{ on } D.$$

Define the functions $f, g : \Delta_{n-1} \times Y \to (-\infty, +\infty]$, $h : \Delta_{n-1} \times D \to (-\infty, +\infty]$ by

$$f(x,y) = \sum_{i=1}^{n} x_i f_i(y), \quad g(x,y) = \sum_{i=1}^{n} x_i g_i(y) \quad \text{and}$$

$$h(x,v) = \sum_{i=1}^{n} x_i h_i(v) \text{ for } x = (x_1, \ldots, x_n) \in \Delta_{n-1},\ y \in Y,\ v \in D.$$

One readily verifies that $f.g.h$ satisfy assertions (i), (iii), (iv) in Theorem 7, for $X = \Delta_{n-1}$. Assertion (ii) of the same theorem is also proved taking into account condition (ii) in present theorem and Remark 1.

Since Δ_{n-1} and Y are compact and f is continuous on Δ_{n-1} and lower semicontinuous on Y the conclusion of Theorem 7 becomes

$$\min_{y\in Y}\max_{x\in\Delta_{n-1}} f(x,y) \leqslant \sup_{x\in\Delta_{n-1}}\inf_{v\in D}\sum_{i=1}^{n} x_i h_i(v).$$

On the other hand by (iv) we have

$$\sup_{x\in\Delta_{n-1}}\inf_{v\in D}\sum_{i=1}^{n} x_i h_i(v) \leqslant \sup_{h\in\mathcal{H}}\inf_{v\in D} h(v) = \beta.$$

Consequently, there exists $y_0 \in Y$ such that for each $x \in \Delta_{n-1}$

$$\sum_{i=1}^{n} x_i f_i(y_0) = f(x, y_0) \leqslant \beta,$$

thus we have necessarily $f_i(y_0) \leqslant \beta$ for each $i \in \{1, \ldots, n\}$, that is, $y_0 \in \bigcap_{i=1}^{n} S(f_i)$. ∎

Theorem 8 can be stated for convenience in the form of an alternative, obtaining in this way generalisations of [**5**, Theorem 1] and of [**7**, Theorem 9.1].

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THEOREM 9. *Assume that $Y, \mathcal{F}, \mathcal{G}, \mathcal{H}$ satisfy conditions of Theorem 8. Then given any $\lambda \in \mathbb{R}$ one of the following properties holds:*

(a) *there is a $h \in \mathcal{H}$ such that $h(y) > \lambda$ for all $y \in Y$;*

(b) *there is a $y_0 \in Y$ such that $f(y_0) \leqslant \lambda$ for all $f \in \mathcal{F}$.*

The following theorem generalises under many aspects a result of Liu [**11**, Theorem 3] which in turn improves a well-known theorem of Ky Fan concerning compatibility of some systems of inequalities.

THEOREM 10. *Let $(Y \supset D; \Gamma)$ be a compact G-convex space and let*

$$\{f_i : Y \to (-\infty, +\infty]\}_{i \in I}, \quad \{g_i : Y \to (-\infty, +\infty]\}_{i \in I}$$

be two families of functions such that:

(i) *$f_i \leqslant g_i$ for each $i \in I$;*

(ii) *for each $i \in I$ f_i is lower semicontinuous on Y;*

(iii) *for each $n \geqslant 1$, $\{i_1, \dots, i_n\} \subset I$ and $(x_1, \dots, x_n) \in \Delta_{n-1}$ the function $y \to \sum\limits_{i=1}^{n} x_i g_i(y)$ is G-quasiconvex on Y;*

(iv) *for each $n \geqslant 1$, $\{i_1, \dots, i_n\} \subset I$ and $(x_1, \dots, x_n) \in \Delta_{n-1}$ there is a $v \in D$ such that $\sum\limits_{i=1}^{n} x_i g_i(v) \leqslant 0$.*

Then there exists $y_0 \in Y$ such that $f_i(y_0) \leqslant 0$.

PROOF: Apply Theorem 8 when

$$\mathcal{F} = \{f_i\}_{i \in I},$$
$$\mathcal{G} = \{g_i\}_{i \in I},$$
$$\mathcal{H} = \left\{ \sum_{i=1}^{n} x_i g_i : n \geqslant 1, g_i \in \mathcal{G}, (x_1, \dots, x_n) \in \Delta_{n-1} \right\}.$$

∎

Our last result generalises [**7**, Theorem 9.3].

THEOREM 11. *Let $(Y \supset D; \Gamma)$ be a compact G-convex space, X an arbitrary set and let $f, g : X \times Y \to (-\infty, +\infty]$, $h : X \times D \to (-\infty, +\infty]$ be three functions such that*

(i) *$f(x,y) \leqslant g(x,y)$ for each $(x,y) \in X \times Y$ and $g(x,y) \leqslant h(x,y)$ for all $(x,y) \in X \times D$;*

(ii) *for any $x_1, \dots, x_n \in X$ and for each $(\alpha_1, \dots, \alpha_n) \in \Delta_{n-1}$ the function $y \to \sum\limits_{i=1}^{n} \alpha_i g(x_i, y)$ is G-quasiconvex on Y;*

(iii) *f is lower semicontinuous on Y;*

(iv) *for any $x_1, \dots, x_n \in X$ and for each $(\alpha_1, \dots, \alpha_n) \in \Delta_{n-1}$ there is an $x \in X$ such that $h(x,y) \geqslant \sum\limits_{i=1}^{n} \alpha_i h(x_i, y)$ for all $y \in Y$.*

Then

$$\inf_{y\in Y}\sup_{x\in X} f(x,y) \leqslant \sup_{x\in X}\inf_{y\in Y} h(x,y).$$

PROOF: Apply Theorem 8 when

$$\mathcal{F} = \{f(x,\cdot)\}_{x\in X},$$
$$\mathcal{G} = \{g(x,\cdot)\}_{x\in X},$$
$$\mathcal{H} = \{h(x,\cdot)\}_{x\in X}.$$

∎

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