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SEMI-METRICS ON THE NORMAL STATES OF A W^*-ALGEBRA

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This paper is concerned with some extensions of the Bures metric d defined on the set of normal states of a W^*-algebra $\mathfrak{A}$ [**2**]. Each subgroup G of the automorphism group of $\mathfrak{A}$ leads naturally to a semi-metric d^G. (See Definition 1.1 below.) When G is the identity group $d^G = d$.

In [**2**; **3**; **11**] the metric d was used to obtain a classification of incomplete tensor products up to product isomorphism. In § 2 we indicate the significance of d^G in classifying tensor products up to weak product isomorphism, a natural weakening of the former concept. In addition we give a similar application to tensor product representations of groups [**4**].

In order to make effective use of d^G in these, and other areas, one would like to have explicit formulas for calculating its values, such as for example a result of [**2**] (Formula 6.1 below), and [**11**, Lemma 2.1], which in certain cases express d in terms of Radon-Nikodym derivatives of the states. We have succeeded in doing this in the case that $\mathfrak{A}$ is a semi-finite factor and G contains the inner automorphisms. Most of the paper (§§ 3 to 6) is devoted to this purpose and the final result is given in Theorem 6.1.

The basic question which we face can be stated intuitively as follows. Given states μ and ν on $\mathfrak{A}$, find an automorphism α of $\mathfrak{A}$ such that $d(\mu, \nu^\alpha)$ (see § 1 for notation) is as small as possible. In § 5 we introduce a key concept of *compatibility* which enables us to handle this question.

Sections 3 and 4 are concerned mainly with definitions and technical results needed to formulate and prove the calculation formula. These sections contain little that is new. (An exception is Lemma 3.2.)

In § 7 we give some examples and remarks connecting the results in § 2 and § 6. Example 7.2 in particular extends the discussion of von Neumann [**9**, Section 7.3] dealing with tensor products of I_2 factors.

1. Preliminaries. If $\mathfrak{A}$ is a W^*-algebra we let $\Sigma_{\mathfrak{A}}$ denote the set of all normal states on $\mathfrak{A}$. (We consider a state μ to be normalized so that $\mu(1) = 1$.) If $\mu \in \Sigma_{\mathfrak{A}}$ and $T \in \mathfrak{A}$ is such that $\mu(TT^*) = 1$, we define $\mu_T \in \Sigma_{\mathfrak{A}}$ by $\mu_T(A) = \mu(TAT^*)$ for all $A \in \mathfrak{A}$.

We let Aut $(\mathfrak{A})$ denote the automorphism group of a W^*-algebra $\mathfrak{A}$, and let Int $(\mathfrak{A})$ denote the subgroup of inner automorphisms. For $\mu \in \Sigma_{\mathfrak{A}}$ and

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$\alpha \in \text{Aut}(\mathfrak{A})$ we define $\mu^\alpha \in \Sigma_{\mathfrak{A}}$ by $\mu^\alpha(A) = \mu(\alpha(A))$ for all $A \in \mathfrak{A}$. We similarly define τ^α where τ is a trace on $\mathfrak{A}$.

By a *representation* Φ of a W^*-algebra $\mathfrak{A}$ on a Hilbert space H we will always mean a one-to-one, identity preserving homomorphism from $\mathfrak{A}$ into $\mathfrak{L}(H)$ such that $\Phi(\mathfrak{A})$ is a von Neumann algebra on H. For $\mu \in \Sigma_{\mathfrak{A}}$ we say that the vector $x \in H$ *induces* μ *relative to* Φ if $\mu(A) = (\Phi(A)x \mid x)$ for all $A \in \mathfrak{A}$.

The metric d on $\Sigma_{\mathfrak{A}}$ and its associated quantity ρ were defined in [**2**]. Essentially, for $\mu, \nu \in \Sigma_{\mathfrak{A}}$, $d(\mu, \nu) = \inf\{||x - y||\}$ and $\rho(\mu, \nu) = \sup\{|(x|y)|\}$, where the infimum and supremum in each case is taken over all vectors x and y inducing μ and ν respectively relative to some representation of $\mathfrak{A}$.

We generalize these definitions as follows:

Definition 1.1. Let $\mathfrak{A}$ be a W^*-algebra and let G be any subgroup of Aut $(\mathfrak{A})$. For any $\mu, \nu \in \Sigma_{\mathfrak{A}}$ we define

$$d^G(\mu, \nu) = \inf\{d(\mu^\alpha, \nu^\beta): \alpha, \beta \in G\},$$
$$\rho^G(\mu, \nu) = \sup\{\rho(\mu^\alpha, \nu^\beta): \alpha, \beta \in G\}.$$

It is easy to verify, using the corresponding fact for d and ρ [**2**, Lemma 1.4] that

$$[d^G(\mu, \nu)]^2 = 2[1 - \rho^G(\mu, \nu)]. \tag{1.1}$$

It is also easy to show that $d(\mu^\alpha, \nu^\alpha) = d(\mu, \nu)$ for all α which implies

$$d^G(\mu, \nu) = \inf\{d(\mu, \nu^\alpha: \alpha \in G\}, \tag{1.2}$$

and a similar result holds for ρ. From this it is immediate that d^G is a semi-metric on $\Sigma_{\mathfrak{A}}$.

For simplicity in notation we will let $\tilde{d}$ and $\tilde{\rho}$ denote d^G and ρ^G respectively in the case that G is all of Aut $(\mathfrak{A})$.

2. Application to infinite tensor products. Throughout this section we suppose that we have a family of W^*-algebras $(\mathfrak{A}_i)_{i \in I}$ where I is an arbitrary indexing set.

We recall some definitions introduced in [**3**]. A product for such a family is an object $(\mathfrak{A}, (\alpha_i)_{i \in I})$ where $\mathfrak{A}$ is a W^*-algebra and for each $i \in I$, α_i is an injection from $\mathfrak{A}_i$ into $\mathfrak{A}$ such that $\alpha_i(\mathfrak{A}_i)$ and $\alpha_j(\mathfrak{A}_j)$ commute pointwise for $i \neq j$, and $\{\alpha_i(\mathfrak{A}_i): i \in I\}$ generates $\mathfrak{A}$ as a W^*-algebra. Two such products $(\mathfrak{A}, (\alpha_i))$ and $(\mathfrak{B}, (\beta_i))$ are said to be *product isomorphic* if there exists an isomorphism Φ from $\mathfrak{A}$ onto $\mathfrak{B}$ such that $\Phi\alpha_i = \beta_i$ for all $i \in I$.

We now define another type of isomorphism (formalizing the idea contained in [**2**, Theorem 4.2]). The products $(\mathfrak{A}, (\alpha_i))$ and $(\mathfrak{B}, (\beta_i))$ are said to be *weakly product isomorphic* if there exists an isomorphism Φ from $\mathfrak{A}$ onto $\mathfrak{B}$ and $\Phi_i \in \text{Aut}(\mathfrak{A}_i)$ such that $\Phi\alpha_i = \beta_i\Phi_i$ for all $i \in I$; or equivalently, if there exists an isomorphism Φ from $\mathfrak{A}$ onto $\mathfrak{B}$ such that $\Phi\alpha_i(\mathfrak{A}_i) = \beta_i(\mathfrak{A}_i)$ for all $i \in I$.

In particular we consider the incomplete tensor products $\otimes_{i\in I}(\mathfrak{A}_i, \mu_i)$ determined by a family $\mu_i \in \Sigma_{\mathfrak{A}_i}$ [**11**, Definition 3.1]. We have as an analogous result to [**11**, Corollary 3.5):

THEOREM 2.1. *Let μ_i and $\nu_i \in \Sigma_{\mathfrak{A}_i}$ for each $i \in I$. Then $\otimes (\mathfrak{A}_i, \mu_i)$ and $\otimes (\mathfrak{A}_i, \nu_i)$ are weakly product isomorphic if and only if*

$$\sum_{i\in I} [\tilde{d}(\mu_i, \nu_i)]^2 < \infty$$

and for all but a countable number of $i \in I$, the infimum in the definition of $\tilde{d}$ is attained.

Proof. This follows directly from [**2**, Theorem 4.2] (noting that the semi-finiteness restriction can be removed by the results in [**11**]), [**11**, Corollary 3.5], and (1.2).

We can in a similar way use these semi-metrics to extend a result in [**4**] (to which the reader is referred for terminology used below). Suppose $(G_i)_{i\in I}$ is a family of groups and for each $i \in I$, U_i is a representation of G_i on the W^*-algebra $\mathfrak{A}_i$. Let G be the restricted direct product of (G_i). Then [**4**] is concerned with the equivalence of various representations of G, where representations U and V of G on the algebras $\mathfrak{A}$ and $\mathfrak{B}$ respectively are defined to be equivalent if $\Phi U = V$ for some isomorphism Φ of $\mathfrak{A}$ onto $\mathfrak{B}$.

In many instances one wants to consider other types of equivalences between representations. For example let H be the group of all automorphisms h of G which satisfy $hj_i(G_i) \subset j_i(G_i)$ where j_i denotes the canonical injection from G_i into G. Let us call the representations U and V on $\mathfrak{A}$ and $\mathfrak{B}$ respectively, *weakly equivalent* if

$$\Phi U = Vh$$

for some isomorphism Φ from $\mathfrak{A}$ onto $\mathfrak{B}$ and $h \in H$.

For each $i \in I$ let

$$K_i = \{\alpha \in \mathrm{Aut}\,(\mathfrak{A}_i) : \alpha U_i(G_i) = U_i(G_i)\}.$$

Then suitably modifying portions of the proofs in [**4**, Lemma 2.1, Theorem 2.2] we obtain:

THEOREM 2.2. *Suppose that the tensor product representations of G,*

$$\otimes_{i\in I} (U_i, \mu_i) \quad \textit{and} \quad \otimes_{i\in I} (U_i, \nu_i)$$

are weakly equivalent. Then

$$\sum_{i\in I} [d^{K_i}(\mu_i, \nu_i)]^2 < \infty.$$

3. Monotone functions. Let P denote the set of non-negative real numbers and let $P^* = P \cup \{+\infty\}$. Let M denote the set of all functions f from P^*

to itself which satisfy, $x \leqq y$ implies $f(x) \geqq f(y)$. For any $f \in M$ we let $\bar{f}$ denote the so-called "inverse" of f; that is

$$\bar{f}(a) = \inf \{x \in P^*: f(x) \leqq a\}.$$

Then $\bar{f} \in M$ and is moreover right continuous.

We summarize some facts which we need in the following lemma. These are either well-known or easily verified.

LEMMA 3.1. (a) *$\bar{f}(a) = x$ if and only if, $f(y) > a$ for all $y < x$ and $f(y) \leqq a$ for all $y > x$.*

(b) *For $f \in M$ and positive k define $f^k \in M$ by $f^k(x) = f(x/k)$. Then $(\bar{f})^k = (\overline{kf})$ and $(\overline{f^k}) = k\bar{f}$.*

(c) *If $f = \sup \{f_n\}$, a non-decreasing sequence in M, then $f \in M$ and $\bar{f} = \sup \{\bar{f}_n\}$.*

(d) *Suppose $f \in M$ is such that $f(x) < \infty$ for $x > 0$ and $f(x) \to 0$ as $x \to \infty$. Then for any function Φ from P to itself with $\Phi(0) = 0$,*

$$\int_0^\infty \Phi(x) df(x) = -\int_0^\infty \Phi(\bar{f}(t)) dt$$

if these integrals exist. (Here the integral is the limit as $n, m \to \infty$ of the integral from $1/n$ to m, and is allowed to take a value of $+\infty$.)

LEMMA 3.2. *Let Φ be a continuous function from P^n to P, n a positive integer. Let $(f_i)_{i=1,2\ldots n}$ and g be functions in M such that for all $v = (v_1, v_2 \ldots v_n) \in P^n$, $g[\Phi(v)] \in$ convex hull $\{f_i(v_i): i = 1, 2, \ldots n\}$. Then for all $a \in P^*$ such that $\bar{f}_i(a)$ is positive for all i,*

$$\bar{g}(a) = \Phi(\bar{f}_1(a), \ldots, \bar{f}_n(a)).$$

Proof. Denote $\bar{f}_i(a)$ by x_i. Given any $\epsilon > 0$ choose $\delta > 0$ so that

$$\Phi(x_1 - \delta, x_2 - \delta, \ldots, x_n - \delta) \geqq \Phi(x_1, x_2, \ldots, x_n) - \epsilon, \tag{3.1}$$

and

$$\Phi(x_1 + \delta, x_2 + \delta, \ldots, x_n + \delta) \leqq \Phi(x_1, x_2, \ldots, x_n) + \epsilon. \tag{3.2}$$

From (3.1) $g[\Phi(x_1, x_2, \ldots, x_n) - \epsilon] \geqq g[\Phi(x_1 - \delta, \ldots, x_n - \delta)]$ which is $>a$ by our hypothesis and Lemma 3.1(a). A similar argument using (3.2) shows that $g[\Phi(x_1, x_2 \ldots x_n) + \epsilon] \leqq a$, and the other direction of Lemma 3.1(a) completes the proof.

4. Distribution functions and σ. Let $\mathfrak{A}$ be a von Neumann algebra with a semi-finite, faithful normal trace τ and let $\mu \in \Sigma_{\mathfrak{A}}$. By the well-known Radon-Nikodym theorem of Segal, there exists a unique positive self-adjoint operator S affiliated with $\mathfrak{A}$, which is square integrable with respect to τ and which satisfies

$$\mu(A) = \tau_0(S^2 A) \quad \text{for all } A \in \mathfrak{A}.$$

Here τ_0 denotes the extension of τ to the class of integrable operators. S^2 is usually known as the Radon-Nikodym derivative of μ with respect to τ. We will denote this by writing

$$\mu = \tau_S$$

which agrees with our notation of § 1 in the case that τ is finite and $S \in \mathfrak{A}$.

We will make use of the following properties of positive, self-adjoint operators which are square integrable or integrable with respect to τ.

Any such operator S has a spectral resolution $\{E(\lambda), -\infty < \lambda < \infty\}$ where each $E(\lambda) \in \mathfrak{A}$. Moreover,

$$\tau(1 - E(\lambda)) < \infty, \quad \text{for all } \lambda > 0. \tag{4.1}$$

If S is square integrable

$$\tau_0(S^2) = -\int_0^\infty \lambda^2 d\tau(1 - E(\lambda)) < \infty. \tag{4.2}$$

If S is integrable

$$\tau_0(S) = -\int_0^\infty \lambda d\tau(1 - E(\lambda)) < \infty. \tag{4.3}$$

See [**13**, particularly Theorem 14.1] for details concerning these concepts.

Note that the above properties are all algebraic so we can apply them in what follows to an abstract W^*-algebra without regard to any particular representation.

In the rest of this section we will consider a W^*-aglebra $\mathfrak{A}$ with a semi-finite, faithful, normal trace τ.

Definition 4.1. (a) For any positive self-adjoint operator S affiliated with $\mathfrak{A}$ we define a function f_S on P^* by

$$f_S(\lambda) = \tau(1 - E(\lambda)), \quad 0 \leqq \lambda < \infty$$
$$f_S(\infty) = 0,$$

where $\{E(\lambda)\}$ is the spectral resolution of S.

(b) For any two such operators S and T we define,

$$\sigma(S, T) = \int_0^\infty \bar{f}_S(x)\bar{f}_T(x)dx.$$

(c) For any $\mu, \nu \in \Sigma_{\mathfrak{A}}$ we define

$$\sigma(\mu, \nu) = \sigma(S, T), \quad \text{where } \mu = \tau_S, \nu = \tau_T.$$

To avoid ambiguity in the above definitions a spectral resolution will always be right continuous, i.e., $\bigcap_{\lambda>\lambda_0} E(\lambda) = E(\lambda_0)$, for all real λ_0.

Remark 4.2. The function f_S will be called the *distribution function* of the operator S with respect to τ by analogy wtih probability theory. Note that the above definition is preferable to considering the increasing function, $\lambda \to \tau(E(\lambda))$, since for square integrable operators $\bar{f}_S$ is finite valued except perhaps at 0 and $\sigma(S, T)$ is finite (see (4.1) and (4.2)). For S integrable we have from (4.3) and Lemma 3.1(d) that

$$\tau_0(S) = \int_0^\infty \bar{f}_S(t)dt. \tag{4.4}$$

We are particularly concerned with the case where $\mathfrak{A}$ is a factor. The main significance of this is:

LEMMA 4.3. *If $\mathfrak{A}$ is a factor, $\sigma(\mu, \nu)$ is independent of the particular trace τ.*

Proof. Let τ' be any other semi-finite trace on $\mathfrak{A}$. Then $\tau' = k^2\tau$ for some $k > 0$. If $\mu = \tau_S$ where S has spectral resolution $\{E(\lambda)\}$, $\mu = \tau'_{S/k}$ and S/k has spectral resolution $\{E(k\lambda)\}$. Letting g denote distribution functions with respect to τ', we can use Lemma 3.1(b) to verify that

$$\bar{g}_{S/k}(\lambda) = \frac{1}{k}[\bar{f}_S(\lambda/k^2)]$$

and the result follows by a direct calculation.

LEMMA 4.4. *Suppose that μ and $\nu \in \Sigma_{\mathfrak{A}}$ where $\mathfrak{A}$ is a semi-finite factor and that P is a projection in $\mathfrak{A}$ with $\mu(P) = \nu(P) = 1$. Let μ' and ν' denote the restrictions of μ and ν respectively to $\mathfrak{A}_p$. Then $\sigma(\mu, \nu) = \sigma(\mu', \nu')$.*

Proof. Suppose $\mu = \tau_S$ where S has spectral resolution $\{E(\lambda)\}$. Since $\mu(P) = 1$ it is clear that $1 - E(\lambda) \leqq P$ for all $\lambda > 0$. Moreover, if τ' is the restriction of τ to $\mathfrak{A}_p$, $\mu' = \tau'_{S'}$ where S' has spectral resolution $\{E(\lambda)P\}$ as an operator on the Hilbert space P. Now if g denotes the distribution function of S' with respect to τ', we have for all $\lambda > 0$,

$$g(\lambda) = \tau'(P - E(\lambda)P) = \tau'(P[1 - E(\lambda)]) = \tau(1 - E(\lambda) = f_S(\lambda).$$

The result follows after a similar calculation for ν.

Now suppose that $\mathfrak{A}$ is a factor. Let Ψ be the homomorphism from Aut $(\mathfrak{A})$ onto the multiplicative group of positive real numbers defined by $\Psi(\alpha) = k$ if $\tau^\alpha = k\tau$.

Definition 4.5. Let $\mu = \tau_S$ and $\nu = \tau_T$ be any element of $\Sigma_{\mathfrak{A}}$. We define

(a) $\sigma_k(\mu, \nu) = \sqrt{k}\int_0^\infty \bar{f}_S(x)\bar{f}_T(kx)dx,$ for any $k > 0$.

(b) $\sigma^G(\mu, \nu) = \sup\{\sigma_k(\mu, \nu): k \in \Psi(G)\}$.

It follows by a direct calculation similar to that used in Lemma 4.3 that for all $\alpha \in$ Aut $(\mathfrak{A})$,

$$\sigma(\mu, \nu^\alpha) = \sigma_{\Psi(\alpha)}(\mu, \nu)$$

and therefore that

$$\sigma^G(\mu, \nu) = \sup\{\sigma(\mu, \nu^\alpha): \alpha \in G\}. \tag{4.5}$$

Remark 4.6. For a factor of Type I or Type II_1, we have $\Psi(G) = \{1\}$ and hence $\sigma^G = \sigma$ for all G. The purpose of the definition is to deal with II_∞ factors where this is not the case. The same reasoning as employed in an example of Suzuki [**12**, p. 188] shows that if $\mathfrak{A} = \mathfrak{B} \otimes I_\infty$ for a II_1 factor $\mathfrak{B}$, then $\Psi(\mathrm{Aut}(\mathfrak{A}))$ is the fundamental group of $\mathfrak{B}$ (as defined in [**8**, Theorem 8]). We can then only state in general that $\sigma^G = \sigma$ for G contained in Int $(\mathfrak{A})$.

We now wish to develop a continuity property of σ in the case that $\mathfrak{A}$ is a finite factor. We consider the trace τ to be normalized so that $\tau(1) = 1$.

We make use of [**7**, Lemma 15.21], which adapted to our terminology says

LEMMA 4.7. *Consider any representation of* $\mathfrak{A}$. *For any* $S \in \mathfrak{A}^+$ *and* $0 \leqq a \leqq \infty$,

$$\bar{f}_S(a) = \inf_E(\sup\{(Ax|x): ||x|| = 1, Ex = x\})$$

where E *runs over all projections of* $\mathfrak{A}$ *with* $\tau(1 - E) \leqq a$.

LEMMA 4.8. *Let* $\mathfrak{A}$ *be a finite factor. Suppose* S, S', T *and* $T' \in \mathfrak{A}^+$ *and satisfy* $||S - S'|| \leqq \delta$, $||T - T'|| \leqq \delta$ *for some* $0 \leqq \delta \leqq 1$. *Then*

$$|\sigma(S, T) - \sigma(S', T')| \leqq \delta(||S|| + ||T|| + 1).$$

Proof. Since $\bar{f}_S(||S||) = \bar{f}_T(||T||) = 0$ we have

$$f_S(a) \leqq ||S||, \quad f_T(a) \leqq ||T||, \quad \text{for } 0 \leqq a \leqq \infty. \tag{4.6}$$

Now consider any representation of $\mathfrak{A}$ and any unit vector x. We have $|(Sx|x) - (S'x|x)| \leqq \delta$, so from Lemma 4.7

$$|\bar{f}_S(a) - \bar{f}_{S'}(a)| \leqq \delta \text{ and similarly } |\bar{f}_T(a) - \bar{f}_{T'}(a)| \leqq \delta \text{ for } 0 \leqq a \leqq \infty. \tag{4.7}$$

Since $\tau(1) = 1$, $\bar{f}_S(a) = \bar{f}_T(a) = 0$ for all $a > 1$ and

$$|\sigma(S, T) - \sigma(S', T')| \leqq \int_0^1 |\bar{f}_S(a)\bar{f}_T(a) - \bar{f}_{S'}(a)\bar{f}_{T'}(a)|da.$$

Now a straightforward estimation using (4.6) and (4.7) completes the proof.

5. Compatibility.

Definition 5.1. Let S and T be self-adjoint operators on a Hilbert space H with spectral resolutions $\{E(\lambda)\}$ and $\{F(\lambda)\}$ rexpectively. We say that S and T are *compatible* if given any ordered pair (α, β) of real numbers either (1) or (2) holds:

$$(1)\ E(\alpha) \leqq F(\beta),$$
$$(2)\ F(\beta) \leqq E(\alpha).$$

THEOREM 5.2. *Let* $\mathfrak{A}$ *be a W*-algebra with a normal semi-finite trace* τ. *Let* S *and* $T \in \mathfrak{A}^+$ *be compatible and satisfy* $\tau(S^2) < \infty$, $\tau(T^2) < \infty$. *Then*

$$\bar{f}_{ST}(a) = \bar{f}_S(a)\bar{f}_T(a) \quad \textit{for } 0 < a \leqq \infty.$$

Proof. We apply Lemma 3.2 with $n = 2$, $\Phi(\alpha, \beta) = \alpha\beta$, $g = f_{ST}$, $f_1 = f_S$, $f_2 = f_T$. It reamins to verify the conditions of that theorem. Note first that $\tau(S^2)$ and $\tau(T^2) < \infty$ imply that $\bar{f}_S$ and $\bar{f}_T$ are finite valued except perhaps at 0.

Let $\{E(\lambda)\}$, $\{F(\lambda)\}$, and $\{G(\lambda)\}$ be the spectral resolutions of S, T and ST respectively. By the compatibility of S and T these operators all commute with each other so that any two of these spectral projections will commute. Note also that by right continuity of the spectral resolutions, if a unit vector $x \in E(\lambda) - E(\mu)$ for example then $\mu < (Sx \mid x) \leqq \lambda$.

Now let (α, β) be any ordered pair of positive numbers. By compatibility we can assume that $E(\alpha) \leqq F(\beta)$. Then for any unit vector $x \in E(\alpha)$, $S^{\frac{1}{2}}x \in E(\alpha) \leqq F(\beta)$ and

$$(STx \mid x) = (TS^{\frac{1}{2}}x \mid S^{\frac{1}{2}}x) \leqq \beta(Sx \mid x) \leqq \alpha\beta.$$

Our remark above shows that $E(\alpha)(1 - G(\alpha\beta)) = 0$. A similar calculation shows $G(\alpha\beta)(1 - F(\beta)) = 0$. It is then clear that

$$f_T(\beta) \leqq f_{ST}(\alpha\beta) \leqq f_S(\alpha),$$

and we have the required conditions to apply Lemma 3.2.

COROLLARY 5.3. *For S, T satisfying the conditions of Theorem* 5.2,

$$\sigma(S, T) = \tau(ST).$$

Proof. This is immediate from Theorem 5.2 and (4.4).

We now consider the question of compatibility as applied to the class of simple operators.

If $\mathfrak{A}$ is a W^*-algebra, a finite set $\{E_i: i = 1, 2, \ldots, n\}$ of mutually orthogonal non-zero projections $\in \mathfrak{A}$ such that $\sum_{i=1}^n E_i = 1$ will be called a *partition* in $\mathfrak{A}$. A self-adjoint element $S \in \mathfrak{A}$ is called *simple* if $S = \sum_{i=1}^n c_i E_i$, for some partition $\{E_i\}$ and real numbers $\{c_i\}$. The spectral resolution $\{E(\lambda)\}$ of S is obviously given by

$$E(\lambda) = \sum_j E_j, \quad \text{where } c_j \leqq \lambda. \tag{5.1}$$

THEOREM 5.4. *If S and T are simple self-adjoint elements of a W^*-algebra $\mathfrak{A}$, they are compatible if and only if there exists a partition in $\mathfrak{A}$, $\{G_i, i = 1, \ldots, m\}$, and non-increasing sequences of real numbers $(c_1, c_2, \ldots, c_m)$ and $(d_1, d_2, \ldots, d_m)$, such that*

$$S = \sum_{i=1}^n c_i G_i \quad \textit{and} \quad T = \sum_{i=1}^n d_i G_i.$$

Proof. If S and T are of the above form they are obviously compatible.

Conversely, suppose that S and T are compatible. Since S and T commute it is easy to find a partition $\{G_i, i = 1, 2, \ldots, n\}$ such that $S = \sum_{i=1}^n c_i G_i$, $T = \sum_{i=1}^n d_i G_i$, where $(c_1 \geqq c_2 \ldots \geqq c_n)$ and if $c_i = c_{i+1}$ then $d_i \leqq d_{i+1}$. Now

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assume that for some i, $d_i < d_{i+1}$. Then necessarily $c_{i+1} < c_i$ and we find c and d such that $d_i < d < d_{i+1}$ and $c_{i+1} < c < c_i$. Letting $\{E(\lambda)\}$ and $\{F(\lambda)\}$ denote the spectral resolutions of S and T respectively we have from (5.1) that $G_i \leqq F(d)(1 - E(c))$ and $G_{i+1} \leqq E(c)(1 - F(d))$, which since G_i and G_{i+1} are both non-zero is a contradiction to the assumed compatibility of S and T.

Our main result concerning simple operators is:

THEOREM 5.5. *Let S and T be self-adjoint simple elements of a factor $\mathfrak{A}$. Then there exists a unitary $U \in \mathfrak{A}$ such that S and U^*TU are compatible.*

We apply this theorem later only to semi-finite $\mathfrak{A}$ and the proof in that case is quite easy. However the result appears to be of sufficient interest to warrant a proof in the general case. We first need some preliminary lemmas concerning the comparison of projections in a factor. See [**5**, Chapter III, § 1] for the standard terminology and results.

We recall the usual notation. For two projections E and F, $E \sim F$ means that they are equivalent, $E \precsim F$ means that E is equivalent to a subprojection of F. The standard comparison lemma [**5**, Chapter III, § 1, Corollary 1] states that for any projections E and F in a factor,

$$\text{either} \quad E \precsim F \quad \text{or} \quad F \precsim E. \tag{5.2}$$

Definition 5.6. For two projections E and F in a W^*-algebra $\mathfrak{A}$ we write

$$E \precsim^u F$$

to mean that there exist projections P and R in $\mathfrak{A}$ such that $E \leqq P$, $Q \leqq F$, and P and Q are unitarily equivalent (in other words, $P \sim Q$ and $1 - P \sim 1 - Q$).

It is obvious that $E \precsim^u F$ implies $E \precsim F$ and for a finite algebra the two notions coincide.

LEMMA 5.7. *Let E and F be projections in a factor $\mathfrak{A}$. Then either $E \precsim^u F$ or $F \precsim^u E$.*

Proof. By (5.2) and the above remark we can assume that $\mathfrak{A}$ is infinite and that there exists a projection H with $E \sim H \leqq F$.

Suppose first that E is not equivalent to 1. Since 1 is infinite it is easy to see that $1 - E \sim 1$ and similarly $1 - H \sim 1$. Then E and H are unitarily equivalent which implies $E \precsim^u F$.

Now suppose that $E \sim F \sim 1$. We can assume that there exists a projection G with $1 - F \sim G \leqq 1 - E$. Then $E \leqq 1 - G$ so that $1 - G \sim 1 \sim F$. Hence F and $1 - G$ are unitarily equivalent which implies $E \precsim^u F$.

LEMMA 5.8 (cf. [**5**, Chapter III, § 2, Proposition 6). *Let E, F, and H be projections in a factor $\mathfrak{A}$ with $E \leqq H$ and $F \leqq H$. Then there exists a unitary $U \in \mathfrak{A}$ such that*

(a) $U^*GU = G$ *for all projections* $G \leqq 1 - H$;
(b) *either* $U^*FU \leqq E$ *or* $U^*FU \geqq E$.

Proof. Apply Lemma 5.7 to the factor $\mathfrak{A}_H$. Suppose that $E \lesssim^u F$ in $\mathfrak{A}_H$. Then there exist partial isometries V and W in $\mathfrak{A}$ with $E \leqq V^*V$, $VV^* \leqq F$, $W^*W = H - V^*V$, and $WW^* = H - VV^*$. We can then take

$$U = V + W + 1 - H$$

as the desired unitary.

Proof of Theorem 5.5. Let $S = \sum_{k=1}^{n} a_k E_k$ and $T = \sum_{l=1}^{p} b_l F_l$ where $\{E_k\}$ and $\{F_l\}$ are partitions in $\mathfrak{A}$ and, $a_1 < a_2 \ldots < a_m$, $b_1 < b_2 \ldots < b_p$. We define the projections

$$E^{(i)} = \begin{cases} \sum_{k=1}^{i} E_k, & \text{for } i = 1, 2, \ldots, m, \\ 0, & \text{for } i = 0, \end{cases}$$

and similarly define projections $F^{(j)}$, $j = 0, 1, \ldots, p$ corresponding to T. Let $A = \{0, 1, \ldots, m\}$, $B = \{0, 1, \ldots, p\}$ and order the set $A \times B$ lexicographically starting with the second coordinate. Consider the set of all pairs $(i, j) \in A \times B$ such that

$$(*) \qquad E^{(i)} \nleqq F^{(j)} \quad \text{and} \quad F^{(j)} \nleqq E^{(i)}.$$

Let (i_0, j_0) be the smallest such element. (If none exists then S and T are already compatible and we are done.) Using the minimality of (i_0, j_0) and symmetry we can conclude that

$$E^{(i_0-1)} \leqq F^{(j_0-1)} \leqq E^{(i_0)}.$$

We now apply Lemma 5.8 to the projections $H = 1 - F^{(j_0-1)}$, $E = E^{(i_0)} - F^{(j_0-1)}$, and $F = F_{j_0}$. Let U be the unitary satisfying the conclusions of this lemma, and let (*′) denote statement (*) above with $F^{(j)}$ replaced by $F'^{(j)} = U^*F^{(j)}U$. Then $F^{(j)} = F'^{(j)}$ for $j < j_0$ and either

(a) $U^*FU \leqq E$ in which case

$$F'^{(j_0)} = (F^{(j_0-1)} + U^*FU) \leqq (F^{(j_0-1)} + E) = E^{(i_0)},$$

or

(b) $E \leqq U^*FU$, in which case $E^{(i_0)} \leqq F'^{(j_0)}$.

Therefore (*′) does not hold for (i_0, j_0).

Now consider any $(i, j) < (i_0, j_0)$. If $j < j_0$, (*′) does not hold by the minimality of (i_0, j_0) and if $j = j_0$, $i \leqq i_0 - 1$ and $E^{(i)} \leqq E^{(i_0-1)} \leqq F^{(j_0-1)} \leqq F'^{(j_0)}$. So again (*′) does not hold.

In any event, we can always find a unitary $U \in \mathfrak{A}$ such that when we replace T by U^*TU the number of pairs (i, j) for which (*) holds is strictly decreased. The theorem now follows easily by induction, making use of (5.1).

Example 5.8. Theorem 5.5 will not hold in general if S and T are not simple. Let M be the von Neumann algebra consisting of $L^\infty[0, 1]$ acting on $L^2[0, 1]$ by multiplication. Let G be any free, ergodic, measure preserving group of transformations of $[0, 1]$ and let $\mathfrak{A}$ be the factor constructed from M and G in the standard manner (see [**7**, Chapter XII]).

For any $h \in L^\infty[0, 1]$ we let T_h denote the corresponding element of $\mathfrak{A}$. There is a faithful trace τ on $\mathfrak{A}$ which satisfies

$$\tau(T_h) = \int_0^1 h(x)\,dx.$$

In particular, consider the functions f and g defined by $f(x) = x$, $0 \leqq x \leqq 1$; $g(x) = 2x$, $0 \leqq x \leqq \frac{1}{2}$, $g(x) = 2 - 2x$, $\frac{1}{2} \leqq x \leqq 1$. If $\{E(\lambda)\}$ and $\{F(\lambda)\}$ are the spectral resolutions of T_f and T_g respectively it is easy to verify that for any unitary $U \in \mathfrak{A}$

$$\tau(U^*F(\lambda)U) = \tau(\mathrm{E}(\lambda)), \quad -\infty < \lambda < \infty.$$

Therefore, if there did in fact exist a unitary U such that U^*T_gU and T_f were compatible we would have from the fact that τ is faithful $U^*F(\lambda)U = E(\lambda)$ for all λ. Then U^*T_gU would equal T_f and the von Neumann subalgebras generated by T_f and T_g respectively would be unitarily equivalent. But this is impossible since T_f generates $\{T_h: h \in L^\infty[0, 1]\}$ which is maximal abelian, while T_g generates $\{T_h: h \in L^\infty[0, 1], h(x) = h(1 - x)\}$ which is not.

6. Calculation of ρ^G. In [**2**, Proposition 2.3], Bures proved that for any W^*-algebra $\mathfrak{A}$ with a normal finite trace τ and any $\mu = \tau_S$ and $\nu = \tau_T$ in $\Sigma_{\mathfrak{A}}$ with S and $T \in \mathfrak{A}^+$,

$$\rho(\mu, \nu) = \tau\,|ST|. \tag{6.1}$$

Using this and the results of the preceding sections we now can prove an analogous formula for ρ^G under certain restrictions on G and $\mathfrak{A}$. The result is:

THEOREM 6.1. *Let $\mathfrak{A}$ be a semi-finite factor and let G be any subgroup of* Aut $(\mathfrak{A})$ *which contains* Int $(\mathfrak{A})$. *Then for any $\mu, \nu \in \Sigma_{\mathfrak{A}}$*

$$\rho^G(\mu, \nu) = \sigma^G(\mu, \nu).$$

We prove this by means of some preliminary lemmas, first handling the situation where $\mathfrak{A}$ is finite and the operators involved are simple, and then passing to the general case by a series of approximations. We begin with a matrix inequality which has independent interest.

LEMMA 6.2. *Let (k_{ij}) be a n by n complex matrix for some positive integer n and let $(c_1, c_2, \ldots, c_n)$ be a sequence of non-negative numbers such that for any integer r, $1 \leqq r \leqq n$.*

$$\left|\sum_{j=1}^{r} k_{ij}\right| \leqq c_i, \quad 1 \leqq i \leqq n; \quad \textit{and} \quad \left|\sum_{i=1}^{r} k_{ij}\right| \leqq c_j, \quad i \leqq j \leqq n. \tag{6.2}$$

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Then for any two non-increasing sequences of non-negative numbers,

$$a = (a_1 \geqq a_2, \ldots, \geqq a_n) \quad \text{and} \quad b = (b_1 \geqq b_2, \ldots, \geqq b_n),$$

$$\left| \sum_{i,j=1}^{n} a_i b_j k_{ij} \right| \leqq \sum_{i,j=1}^{n} a_i b_i c_i. \tag{6.3}$$

Proof. Let $T(a, b)$ denote $\sum_{i,j=1}^{n} a_i b_j k_{ij}$. It is clear from the bilinearity of T that it is sufficient to verify (6.3) for sequences which consist only of the numbers 0 and 1. Accordingly, suppose $a_i = 1$, $1 \leqq i \leqq r$, $a_i = 0$, $r < i \leqq n$; and $b_j = 1$, $1 \leqq j \leqq s$, $b_j = 0$, $s < j \leqq n$. We can obviously assume that $r \leqq s$. Then using (6.2),

$$|T(a, b)| = \left| \sum_{i=1}^{r} \sum_{j=1}^{s} k_{ij} \right| \leqq \sum_{i=1}^{r} \left| \sum_{j=1}^{s} k_{ij} \right| \leqq \sum_{i=1}^{r} c_i = \sum_{i=1}^{n} a_i b_i c_i,$$

which completes the proof.

Remark 6.3. Suppose in the preceding situation that (k_{ij}) is a non-negative matrix and that $(c_1, c_2, \ldots, c_n)$ is a non-negative sequence such that for any $1 \leqq r \leqq n$, $|\sum_{j=r}^{n} k_{ij}| \geqq c_i$, and $|\sum_{i=r}^{n} k_{ij}| \geqq c_j$. Then if we have two sequences a and b, one of which is non-increasing and the other is non-decreasing, we may show in a similar way that

$$\sum_{i=1}^{n} a_i b_i c_i \leqq \sum_{i,j=1}^{n} a_i b_j k_{ij}. \tag{6.4}$$

These results can be viewed as a generalization of a classical rearrangement inequality [**6**, Formula 10.21], which we obtain from (6.3) and (6.4) by taking (k_{ij}) to be a suitable permutation matrix and letting $c_i = 1$ for all i.

LEMMA 6.4. *Let $\mathfrak{A}$ be a finite factor with normalized trace τ and let S and T be simple elements of $\mathfrak{A}^+$. Then*

$$\tau\, |ST| \leqq \sigma(S, T).$$

Proof. Let $V \in \mathfrak{A}$ be the partial isometry such that $|ST| = STV$. By Theorem 5.5 choose a unitary $U \in \mathfrak{A}$ so that S and U^*TU are compatible and let $S = \sum_{i=1}^{n} c_i G_i$, $U^*TU = \sum_{i=1}^{n} d_i G_i$ be the representations given by Theorem 5.4. Let $k_{ij} = \tau(G_i U G_j U^* V)$. Choose any integer r such that $1 \leqq r \leqq n$. Then $||\sum_{j=1}^{r} U G_j U^* V|| \leqq 1$ and hence (using the standard inequality, $|\tau(EA)| \leqq ||A||\tau(E)$ for any projection E) we can verify that

$$\left| \sum_{j=1}^{r} k_{ij} \right| \leqq \tau(G_i), \qquad 1 \leqq i \leqq n,$$

and a similar argument shows that

$$\left| \sum_{i=1}^{r} k_{ij} \right| \leqq \tau(G_j), \qquad 1 \leqq j \leqq n.$$

We can now apply Lemma 6.3 to conclude that

$$\tau|ST| = \tau(STV) = \sum_{i,j=1}^{n} c_i d_j k_{ij} \leqq \sum_{i=1}^{n} c_i d_i \tau(G_i). \tag{6.5}$$

From Corollary 5.3 we have

$$\sigma(S, T) = \sigma(S, U^*TU) = \tau(SU^*TU) = \sum_{i=1}^{n} c_i d_i \tau(G_i), \tag{6.6}$$

and the result follows from (6.5) and (6.6).

LEMMA 6.5. *Let $\mathfrak{A}$ be a semi-finite factor with trace τ and let $\mu = \tau_S$, $\nu = \tau_T$ be elements of $\Sigma_{\mathfrak{A}}$ where $S, T \in \mathfrak{A}^+$ and have finite range projections. Then*
(a) *$\rho(\mu, \nu) \leqq \sigma(\mu, \nu)$ and equality holds if S and T are compatible;*
(b) *given any $\epsilon > 0$, there exists $\alpha \in \operatorname{Int}(\mathfrak{A})$ such that*

$$\rho(\mu, \nu^\alpha) > \sigma(\mu, \nu) - \epsilon.$$

Proof. We can assume from the outset that $\mathfrak{A}$ is finite, since if E and F are the range projections of S and T respectively then $G = E \cup F$ is a finite projection with $\mu(G) = \nu(G) = 1$. It then suffices to prove the lemma for the restrictions of μ and ν to the finite factor $\mathfrak{A}_G$. This follows from Lemma 4.4, [**2**, Proposition 1.10] and the fact that any inner automorphism of $\mathfrak{A}_G$ extends to an inner automorphism of $\mathfrak{A}$.

Choose any ϵ such that $0 < \epsilon < 1$. Let $r = (||S|| + ||T|| + 1)$. We can choose simple elements S' and T' of $\mathfrak{A}^+$ such that $||S - S'|| \leqq \epsilon/2r$, $||T - T'|| \leqq \epsilon/2r$. From Lemma 4.8,

$$|\sigma(S, T) - \sigma(S', T')| \leqq \epsilon/2, \tag{6.7}$$

and a straightforward calculation shows that for any unitary $U \in \mathfrak{A}$,

$$|(\tau\,|SU^*TU| - \tau\,|S'U^*T'U|)| \leqq \epsilon/2. \tag{6.8}$$

In particular choose a unitary $U \in \mathfrak{A}$ so that S' and $U^*T'U$ are compatible. From Corollary 5.3,

$$\tau\,|S'U^*T'U| = \sigma(S', U^*T'U). \tag{6.9}$$

If α is the inner automorphism induced by U, then $\nu^\alpha = \tau_{U^*TU}$ and (b) follows from (6.1), (6.7), (6.8) and (6.9). Similarly, take $U = 1$ in (6.8) and use Lemma 6.4 to obtain (a).

Remark 6.8. We can of course remove the normalizing condition and define $\sigma(\mu, \nu)$ for any positive normal linear functionals. For any positive numbers r and s it is easy to verify that $\sigma(r\mu, s\nu) = (rs)^{\frac{1}{2}}\sigma(\mu, \nu)$ (use Lemma 3.1(b)). Moreover $\rho(r\mu, s\nu) = (rs)^{\frac{1}{2}}\rho(\mu, \nu)$ [**11**, p. 510]. It follows that the previous results of this section hold in this more general case and we use this fact in completing the proof of Theorem 6.1.

Proof of Theorem 6.1. We first show that we can reduce to the case where μ and ν satisfy the conditions of Lemma 6.5.

Suppose $\mu = \tau_S$ and $\nu = \tau_T$ for some semi-finite trace τ on $\mathfrak{A}$. Let $\{E(\lambda)\}$ be the spectral resolution of S. For each integer $n > 1$ let $P_n = E(n) - E(1/n)$, $S_n = SP_n$, $\mu_n = \tau_{S_n}$. Similarly define Q_n, T_n and ν_n corresponding to ν. Now (S_n) and (T_n) are increasing sequences of positive operators with S and T as their respective least upper bounds. It is easy to verify, using the normality of τ, that the sequence of distribution functions (f_{S_n}) and (f_{T_n}) increase to f_S and f_T respectively. Now we can use Lemma 3.1(c), and the monotone convergence theorem to see that the sequence $(\sigma^G(\mu_n, \nu_n))$ increases to $\sigma^G(\mu, \nu)$. Moreover it is clear that $(\mu(P_n))$ and $(\nu(Q_n))$ both increase to 1. Hence given any $\epsilon > 0$ we can choose a large enough integer m so that

$$\mu(P_m) > 1 - (\epsilon/5)^2, \quad \nu(Q_m) > 1 - (\epsilon/5)^2, \tag{6.10}$$

and

$$|\sigma^G(\mu_m, \nu_m) - \sigma^G(\mu, \nu)| \geqq \epsilon. \tag{6.11}$$

For any $\alpha \in G$, $(\mu_m)^\alpha = \mu^\alpha{}_{(\alpha^{-1}(P_m))}$, so that from (6.10) and [**2**, Proposition 1.9(c)] we can conclude that

$$|\rho^G(\mu, \nu) - \rho^G(\mu_m, \nu_m)| > \epsilon. \tag{6.12}$$

Since $\mu_m = \tau_{S_m}$ and $\nu_m = \tau_{T_m}$, where S_m and $T_m \in \mathfrak{A}^+$ and have finite range projections it is immediate from (6.11) and (6.12) that we can assume the conditions of Lemma 6.5 from the outset.

Accordingly, choose any $\epsilon > 0$. Let $\beta \in G$ be such that

$$\sigma(\mu, \nu^\beta) \geqq \sigma^G(\mu, \nu) - \epsilon/2. \tag{6.13}$$

From Lemma 5.4(b), there exists $\gamma \in \text{Int}(\mathfrak{A})$ such that

$$\rho(\mu, \nu^{\beta\gamma}) \geqq \sigma(\mu, \nu^\beta) - \epsilon/2. \tag{6.14}$$

Since G contains $\text{Int}(\mathfrak{A})$, $\beta\gamma \in G$, and from (6.13) and (6.14) we can deduce that $\rho^G(\mu, \nu) \geqq \sigma^G(\mu, \nu)$. It is immediate from Lemma 6.5(a) that

$$\rho^G(\mu, \nu) \leqq \sigma^G(\mu, \nu),$$

and the theorem is proved.

Definition 6.7. We say that the states $\mu = \tau_S$ and $\nu = \tau_T$ are compatible if S and T are compatible operators.

Compatibility then minimizes distance in the following sense.

LEMMA 6.8. *Let μ and ν be compatible elements of $\Sigma_{\mathfrak{A}}$, where $\mathfrak{A}$ is a semi-finite factor. Then*

$$d^G(\mu, \nu) = d(\mu, \nu) \quad \textit{where } G = \text{Int}(\mathfrak{A});$$
$$\tilde{d}(\mu, \nu) = d(\mu, \nu) \quad \textit{if } \mathfrak{A} \textit{ is of type } \text{I or II}_1.$$

Proof. This is immediate from Theorem 6.1, Lemma 6.5(a) (which can be shown valid for arbitrary normal states; e.g. by the same type of approximation procedure as used in proving Theorem 6.1), Remark 4.6, and (1.1).

7. Examples.

Example 7.1. Let $\mathfrak{A} = \mathfrak{L}(H)$ where H is a separable or finite dimensional Hilbert space and let μ and $\nu \in \Sigma_{\mathfrak{A}}$. Then it is well-known that

$$\mu = \sum_{i=1}^{\infty} a_i \omega_{(\Phi_i)}, \qquad \nu = \sum_{i=1}^{\infty} b_i \omega_{(\Psi_i)}$$

where (a_i) and (b_i) are uniquely determined non-increasing sequences of non-negative real numbers with $\sum_{i=1}^{\infty} a_i = \sum_{i=1}^{\infty} b_i = 1$, and, (Φ_i) and (Ψ_i) are orthonormal sequences of vectors in H. Here ω_x denotes the vector state induced by x. It is easy to verify that

$$\sigma(\mu, \nu) = \sum_{i=1}^{\infty} (a_i b_i)^{\frac{1}{2}}$$

and Theorem 6.1 gives us an immediate calculation formula for $\tilde{\rho}(\mu, \nu)$ and hence $\tilde{d}(\mu, \nu)$ in the case of type I factors. In fact from (7.1), (1.1), and Remark 4.6,

$$[\tilde{d}(\mu, \nu)]^2 = 2[1 - \sigma(\mu, \nu)] = \sum_{i=1}^{\infty} (a_i^{\frac{1}{2}} - b_i^{\frac{1}{2}})^2. \tag{7.1}$$

We can use this example to obtain an explicit criteria for weak product isomorphisms of ITPFI's (infinite tensor products of finite type I factors) in terms of the eigenvalues of the states. The following example illustrates this for tensor products of I_2 factors.

Example 7.2. Let $\mathfrak{A}$ be a fixed I_2 factor represented on a two dimensional Hilbert space H. Fix any orthonormal basis (x, y) in H. For each number α in the interval $[\frac{1}{2}, 1]$, define the state $\mu_\alpha = \alpha\omega_x + (1 - \alpha)\omega_y$. Let F be the set of all sequences in the interval $[\frac{1}{2}, 1]$ and for $f = (\alpha_i) \in F$ let $\mathfrak{F}_f$ denote the tensor product $\otimes_{i=1}^{\infty}(\mathfrak{A}_i, \mu_{(\alpha_i)})$ where $\mathfrak{A}_i = \mathfrak{A}$ for all i. Then every countable tensor product of I_2 factors is weakly product isomorphic to $\mathfrak{F}_f$ for some $f \in F$. If $f = (\alpha_i)$ and $g = (\beta_i) \in F$, $\mathfrak{F}_f$ and $\mathfrak{F}_g$ are weakly product isomorphic if and only if they are product isomorphic and this occurs if and only if

$$\sum_{i=1}^{\infty} [(\alpha_i^{\frac{1}{2}} - \beta_i^{\frac{1}{2}})^2 + ((1 - \alpha_i)^{\frac{1}{2}} - (1 - \beta_i)^{\frac{1}{2}})^2] < \infty.$$

These remarks follow immediately from Theorem 2.1 [**11**, Corollary 3.5] and 7.1.

It should be noted that the results in this paper concerned with tensor products deal with weak product isomorphism and not with the more difficult question of classifying tensor products up to algebraic isomorphism (this means simply that the resulting algebras are isomorphic). This is the problem that was studied extensively in [**1**] for ITPFI's. The question arises however of whether $\tilde{d}$ can be used to extend or simplify the results in this area. We have considered this problem but as yet have made little progress. One obstacle is the lack of an effective way of calculating $\tilde{d}$ for Type III factors. At first, it seems a plausible conjecture that on a Type III factor $\tilde{d}$ is identically 0. This is however false as the following example shows.

Example 7.3. For any p in the open interval $(\frac{1}{2}, 1)$ let $\mathfrak{A}_p = \otimes_{i=1}^{\infty}(\mathfrak{A}_i, \mu_i)$ where for each i, $\mathfrak{A}_i$ is a I_2 factor and $\mu_i = \mu_p$ in our notation of Example 7.2 (i.e., $\mathfrak{A}_p$ is determined by the sequence with each term $= p$). Fix p and q in $(0, \frac{1}{2})$, with $p \neq q$. Then it is well-known that $\mathfrak{A}_p$ and $\mathfrak{A}_q$ are non-algebraically isomorphic Type III factors (see [**10**]).

On $\mathfrak{A}_p$ let ν denote the product state $\otimes_{i=1}^{\infty}(\nu_i)$ where $\nu_1 = \mu_q$, and $\nu_i = \mu_p$, $i > 1$. Let μ denote the product state with each factor equal to μ_p. Consider a tensor product $\otimes_{i=1}^{\infty}(\mathfrak{B}_i, \Psi_i)$ where for each i, $\mathfrak{B}_i = \mathfrak{A}_p$. If each $\Psi_j = \mu$ this is algebraically isomorphic to $\mathfrak{A}_p$, while if each $\Psi_i = \nu$ this is algebraically isomorphic to $\mathfrak{A}_p \otimes \mathfrak{A}_q$. These statements follow from standard results on associativity of tensor products. Clearly then it is impossible for $\tilde{d}(\mu, \nu) = 0$ regardless of the choice of p and q. If this were the case we would have from Theorem 2.1 that $\mathfrak{A}_p \otimes \mathfrak{A}_q$ was isomorphic to $\mathfrak{A}_p$, and by symmetry also isomorphic to $\mathfrak{A}_q$, a contradiction.

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