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H-CONTACT UNIT TANGENT SPHERE BUNDLES OF FOUR-DIMENSIONAL RIEMANNIAN MANIFOLDS

SUN HYANG CHUN, JEONGHYEONG PARK ✉ and KOUEI SEKIGAWA

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Abstract

We study the geometric properties of a base manifold whose unit tangent sphere bundle, equipped with the standard contact metric structure, is H-contact. We prove that a necessary and sufficient condition for the unit tangent sphere bundle of a four-dimensional Riemannian manifold to be H-contact is that the base manifold is 2-stein.

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1. Introduction

The relationship between the geometric structures of Riemannian manifolds and their respective unit tangent sphere bundles is one of the interesting topics in Riemannian geometry. In this paper, we give a characterization of a 2-stein manifold in terms of the standard contact metric structure of the unit tangent sphere bundle.

A unit vector field V on M determines a map between (M, g) and $(T_1M, \bar{g})$. If the Riemannian manifold (M, g) is compact and orientable, then the energy of V is defined as the energy of the corresponding map:

$$E(V) = \frac{1}{2}\int_M |dV|^2\, dv_g = \frac{m}{2}\,\mathrm{vol}(M, g) + \frac{1}{2}\int_M |\nabla V|^2\, dv_g$$

where $m = \dim M$ (see [10]).

The vector field V is said to be a *harmonic vector field* if it is a critical point for the energy functional E in the set of all unit vector fields of M (see [10]). Following [9],

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a contact metric manifold whose characteristic vector field ξ is a harmonic vector field is called an *H-contact manifold*.

Perrone [9] proved that a contact metric manifold is an H-contact manifold if and only if the characteristic vector field ξ is an eigenvector of the Ricci operator. Boeckx and Vanhecke [2] proved that the unit tangent sphere bundle of a two-dimensional or three-dimensional Riemannian manifold is H-contact if and only if the base manifold has constant sectional curvature. Calvaruso and Perrone [4] obtained the same result in the case of an n-dimensional conformally flat manifold when $n \geq 4$. The authors [7] proved that the unit tangent sphere bundle T_1M of an n-dimensional Einstein manifold is H-contact if and only if the base manifold is 2-stein when $n \geq 3$. The result was further extended by Calvaruso and Perrone [5] in the setting of Riemannian g-natural contact metric structures defined by Kaluza–Klein type metrics.

An η-Einstein manifold is a special case of an H-contact manifold. The authors [8] have also worked on the problem of determining the base space when the unit tangent bundle of a Riemannian manifold is η-Einstein. In [7] we raised the question: 'If the unit tangent sphere bundle T_1M equipped with the standard contact metric structure on n-dimensional Riemannian manifold is H-contact, where $n \geq 3$, then is the base Riemannian manifold M Einstein?' In this paper we answer this question when $n = 4$ by proving the following theorem.

THEOREM 1.1. *Let $M = (M, g)$ be a four-dimensional Riemannian manifold. Then the unit tangent sphere bundle T_1M equipped with the standard contact metric structure $(\bar{g}, \phi, \xi, \eta)$ is H-contact if and only if the base manifold M is 2-stein.*

2. Standard contact metric structure on a unit tangent sphere bundle

All manifolds in this paper are assumed to be of class C^∞. We begin with some preliminaries on contact metric manifolds. We refer the interested reader to [1] for more details.

A differentiable $(2n-1)$-dimensional manifold $\bar{M}$ is said to be a *contact manifold* if it admits a global 1-form η such that $\eta \wedge (d\eta)^{n-1} \neq 0$ everywhere on $\bar{M}$. Here the exponent denotes the $(n-1)$th exterior power. We call such an η a *contact form* of $\bar{M}$. It is well known that, given a contact form η, there exists a unique vector field ξ, which is called the *characteristic vector field*, satisfying $\eta(\xi) = 1$ and $d\eta(\xi, \bar{X}) = 0$ for any vector field $\bar{X}$ on $\bar{M}$.

A Riemannian metric $\bar{g}$ on $\bar{M}$ is a metric associated to a contact form η if there exists a $(1, 1)$-tensor field ϕ satisfying

$$\eta(\bar{X}) = \bar{g}(\bar{X}, \xi), \quad d\eta(\bar{X}, \bar{Y}) = \bar{g}(\bar{X}, \phi\bar{Y}), \quad \phi^2\bar{X} = -\bar{X} + \eta(\bar{X})\xi \tag{2.1}$$

where $\bar{X}$ and $\bar{Y}$ are vector fields on $\bar{M}$. A Riemannian manifold $\bar{M}$ equipped with structure tensors $(\bar{g}, \phi, \xi, \eta)$ satisfying (2.1) is said to be a *contact metric manifold*.

Let (M, g) be an n-dimensional Riemannian manifold and let ∇ be the associated Levi-Civita connection. The Riemann curvature tensor R of (M, g) is defined by

$$R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

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for all vector fields X, Y and Z on M. The tangent bundle of (M, g), denoted by TM, consists of pairs (p, u) where p is a point in M and u is a tangent vector to M at p. The mapping $\pi : TM \to M$ given by $\pi(p, u) = p$ is the natural projection from TM onto M.

For a vector field X on M, the *vertical lift* X^v on TM is the vector field defined by $X^v\omega = \omega(X) \circ \pi$ where ω is a 1-form on M. For a Levi-Civita connection ∇ on M, the *horizontal lift* X^h of X is defined by $X^h\omega = \nabla_X\omega$.

The tangent bundle TM can be endowed in a natural way with a Riemannian metric $\tilde{g}$ which is the so-called *Sasaki metric*. This metric depends only on the Riemannian metric g on M. It is determined by

$$\tilde{g}(X^h, Y^h) = \tilde{g}(X^v, Y^v) = g(X, Y) \circ \pi, \quad \tilde{g}(X^h, Y^v) = 0$$

for all vector fields X and Y on M. The tangent bundle TM also admits an almost complex structure tensor J defined by $JX^h = X^v$ and $JX^v = -X^h$. The metric $\tilde{g}$ is a Hermitian metric for the almost complex structure J.

The unit tangent sphere bundle $\bar{\pi} : T_1M \to M$ is a hypersurface of TM given by $g_p(u, u) = 1$. Note that $\bar{\pi} = \pi \circ i$ where i is the immersion. A unit normal vector field $N = u^v$ to T_1M is given by the vertical lift of u for (p, u). The horizontal lift of a vector is tangent to T_1M, but the vertical lift of vector is not tangent to T_1M in general and so we define the *tangential lift* of X to $(p, u) \in T_1M$ by

$$X^t_{(p,u)} = (X - g(X, u)u)^v.$$

Clearly the tangent space $T_{(p,u)}T_1M$ is spanned by vectors of the form X^h and X^t where $X \in T_pM$.

We now define the standard contact metric structure on the unit tangent sphere bundle T_1M of a Riemannian manifold (M, g). The metric g' on T_1M is induced from the Sasaki metric $\tilde{g}$ on TM. Using the almost complex structure J on TM, we can define a unit vector field ξ', a 1-form η' and a $(1, 1)$-tensor field ϕ' on T_1M by

$$\xi' = -JN, \quad \phi' = J - \eta' \otimes N.$$

Since

$$g'(\bar{X}, \phi'\bar{Y}) = 2\, d\eta'(\bar{X}, \bar{Y}),$$

the quadruple (g', ϕ', ξ', η') is not a contact metric structure. If we rescale:

$$\xi = 2\xi', \quad \eta = \tfrac{1}{2}\eta', \quad \phi = \phi', \quad \bar{g} = \tfrac{1}{4}g',$$

then we get the standard contact metric structure $(\bar{g}, \phi, \xi, \eta)$. From now on we endow $T_1M = (T_1M, \bar{g}, \phi, \xi, \eta)$ with the standard contact metric structure.

Let $e_1, \ldots, e_n = u$ be an orthonormal basis of T_pM. Then

$$2e^t_1, \ldots, 2e^t_{n-1}, 2e^h_1, \ldots, 2e^h_n = \xi$$

is an orthonormal basis for $T_{(p,u)}T_1M$. The Ricci tensor $\bar{\rho}$ of T_1M is given by

$$\begin{aligned}\bar{\rho}(X^t, Y^t) &= (n-2)(g(X, Y) - g(X, u)g(Y, u)) \\ &\qquad + \frac{1}{4}\sum_{i=1}^{n} g(R(u, X)e_i, R(u, Y)e_i), \\ \bar{\rho}(X^t, Y^h) &= \frac{1}{2}((\nabla_u\rho)(X, Y) - (\nabla_X\rho)(u, Y)), \\ \bar{\rho}(X^h, Y^h) &= \rho(X, Y) - \frac{1}{2}\sum_{i=1}^{n} g(R(u, e_i)X, R(u, e_i)Y)\end{aligned}$$

where ρ denotes the Ricci curvature tensor of M (see [3, 8]).

We now recall the definition of the 2-stein manifold. An n-dimensional Einstein manifold $M = (M, g)$ is said to be 2-stein if

$$\sum_{i,j=1}^{n} (R_{uiuj})^2 = \mu(p)|u|^4$$

for all $u \in T_pM$ and $p \in M$ where μ is a real-valued function on M (see [6, p. 47]).

3. *H*-contact unit tangent sphere bundles

Let $M = (M, g)$ be an n-dimensional Riemannian manifold where $n \geq 3$, and let $\{e_i\}_{i=1}^n$ be a local orthonormal frame field around an arbitrary point $p \in M$. We assume that T_1M is H-contact with respect to the standard contact metric structure $(\bar{g}, \phi, \xi, \eta)$. Then the base manifold M satisfies the following conditions (see [4]):

$$\nabla_i\rho_{jk} - \nabla_j\rho_{ik} = 0, \tag{3.1}$$

$$2\rho_{ab} = \sum_{i,j=1}^{n} R_{aibj}R_{aiaj} \tag{3.2}$$

where $a \neq b$. From (3.1) we may easily see that the scalar curvature τ of M is constant.

We now deduce several easy consequences of formula (3.2) for later use. We set

$$\begin{cases} u = \cos\theta e_a + \sin\theta e_b, \\ x = -\sin\theta e_a + \cos\theta e_b \end{cases} \tag{3.3}$$

where $a \neq b$. Substituting (3.3) into the left-hand side of (3.2) and using some standard trigonometric identities, we obtain

$$2\rho(\cos\theta e_a + \sin\theta e_b, -\sin\theta e_a + \cos\theta e_b) = 2\rho_{ab}\cos(2\theta) + (\rho_{bb} - \rho_{aa})\sin(2\theta). \tag{3.4}$$

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Similarly, substituting (3.3) into the right-hand side of (3.2), we get

$$\begin{aligned}\sum_{i,j=1}^{n} & R(\cos\theta e_a + \sin\theta e_b, e_i, -\sin\theta e_a + \cos\theta e_b, e_j)\\ &\times R(\cos\theta e_a + \sin\theta e_b, e_i, \cos\theta e_a + \sin\theta e_b, e_j)\\ &= 2\rho_{ab}\cos(2\theta) + \frac{1}{4}\Big\{\sum_{i,j=1}^{n}(R_{bibj})^2 - \sum_{i,j=1}^{n}(R_{aiaj})^2\Big\}\sin(2\theta)\\ &\quad + \frac{1}{4}\Big\{\sum_{i,j=1}^{n}(R_{aibj})^2 + \sum_{i,j=1}^{n}R_{aibj}R_{biaj} + \sum_{i,j=1}^{n}R_{aiaj}R_{bibj}\\ &\quad - \frac{1}{2}\sum_{i,j=1}^{n}(R_{aiaj})^2 - \frac{1}{2}\sum_{i,j=1}^{n}(R_{bibj})^2\Big\}\sin(4\theta).\end{aligned} \tag{3.5}$$

Then, comparing the finite Fourier series in (3.4) and (3.5), we obtain the two equations:

$$4(\rho_{aa} - \rho_{bb}) = \sum_{i,j=1}^{n}(R_{aiaj})^2 - \sum_{i,j=1}^{n}(R_{bibj})^2,$$

$$2\Big\{\sum_{i,j=1}^{n}(R_{aibj})^2 + \sum_{i,j=1}^{n}R_{aibj}R_{biaj} + \sum_{i,j=1}^{n}R_{aiaj}R_{bibj}\Big\} = \sum_{i,j=1}^{n}(R_{aiaj})^2 + \sum_{i,j=1}^{n}(R_{bibj})^2. \tag{3.6}$$

Next we set

$$u = \cos\theta e_a + \sin\theta e_b, \quad x = e_c, \tag{3.7}$$

where $a \neq b \neq c \neq a$. Substituting (3.7) into the left-hand side of (3.2), we get

$$2\rho(\cos\theta e_a + \sin\theta e_b, e_c) = 2(\rho_{ac}\cos\theta + \rho_{bc}\sin\theta). \tag{3.8}$$

Similarly, substituting (3.7) into the right-hand side of (3.2), we get

$$\begin{aligned}\sum_{i,j=1}^{n} & R(\cos\theta e_a + \sin\theta e_b, e_i, e_c, e_j)\\ &\times R(\cos\theta e_a + \sin\theta e_b, e_i, \cos\theta e_a + \sin\theta e_b, e_j)\\ &= \sum_{i,j=1}^{n}\{R_{aicj}\cos\theta + R_{bicj}\sin\theta\}\\ &\quad\times\{R_{aiaj}\cos^2\theta + R_{bibj}\sin^2\theta + (R_{aibj} + R_{biaj})\sin\theta\cos\theta\}\\ &= 2\rho_{ac}\cos^3\theta + 2\rho_{bc}\sin^3\theta\\ &\quad + \Big\{\sum_{i,j}R_{aicj}(R_{aibj} + R_{biaj}) + \sum_{i,j}R_{bicj}R_{aiaj}\Big\}\cos^2\theta\sin\theta\\ &\quad + \Big\{\sum_{i,j}R_{aicj}R_{bibj} + \sum_{i,j}R_{bicj}(R_{aibj} + R_{biaj})\Big\}\cos\theta\sin^2\theta.\end{aligned} \tag{3.9}$$

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Since

$$2(\rho_{ac}\cos\theta + \rho_{bc}\sin\theta) - 2\rho_{ac}\cos^3\theta - 2\rho_{bc}\sin^3\theta = 2(\rho_{ac}\sin\theta + \rho_{bc}\cos\theta)\sin\theta\cos\theta,$$

applying (3.8) and (3.9) enables us to deduce that

$$\begin{aligned}2(\rho_{ac}\sin\theta + \rho_{bc}\cos\theta) = &\Big\{\sum_{i,j} R_{aicj}(R_{aibj} + R_{biaj}) + \sum_{i,j} R_{bicj}R_{aiaj}\Big\}\cos\theta \\ &+ \Big\{\sum_{i,j} R_{aicj}R_{bibj} + \sum_{i,j} R_{bicj}(R_{aibj} + R_{biaj})\Big\}\sin\theta\end{aligned}$$

for all θ, and hence

$$2\rho_{ac} = \sum_{i,j} R_{aicj}R_{bibj} + \sum_{i,j} R_{bicj}(R_{aibj} + R_{biaj}). \tag{3.10}$$

4. Proof of the main theorem

We begin by recalling some elementary facts from planar geometry. Let $\mathbb{R}^2$ be the Euclidean two-plane equipped with the canonical inner product $\langle\ ,\ \rangle$.

For any $\mathbb{x} = (x_1, x_2) \in \mathbb{R}^2$, we set

$$\mathbb{x}' = (x_1, -x_2), \quad \mathbb{x}^\perp = (-x_2, x_1), \quad |\mathbb{x}| = \sqrt{\langle \mathbb{x}, \mathbb{x}\rangle}.$$

Then the following identities hold:

$$(\mathbb{x}')' = \mathbb{x}, \quad (\mathbb{x}^\perp)^\perp = -\mathbb{x}, \quad |\mathbb{x}| = |\mathbb{x}'| = |\mathbb{x}^\perp| \quad \forall \mathbb{x} \in \mathbb{R}^2.$$

Also, we see that if $\mathbb{x} \perp \mathbb{y}$ (that is, $\langle \mathbb{x}, \mathbb{y}\rangle = 0$), then $\mathbb{x}' \perp \mathbb{y}'$ and $\mathbb{x}^\perp \perp \mathbb{y}^\perp$.

Suppose now that M is a four-dimensional Riemannian manifold and let $\{e_i\}_{i=1}^4$ be an orthonormal basis of eigenvectors of the Ricci operator Q_p at a point $p \in M$, that is,

$$Qe_i = \lambda_i e_i.$$

Then the Ricci tensor of type $(0, 2)$ is given by a diagonal matrix. Substituting the equalities

$$R_{4142} = -R_{1323}, \ldots, R_{2324} = -R_{1314}$$

into (3.2), we obtain, after explicit computations, the information in Table 1.

Performing direct calculation on the information in Table 1, we obtain

$$\begin{aligned}
&(R_{1213}^2 - R_{1224}^2)(R_{1212} + R_{3434} - R_{1313} - R_{2424}) = 0,\\
&(R_{1213}^2 - R_{1224}^2)(R_{1234} + R_{1324}) = 0,\\
&(R_{1214}^2 - R_{1223}^2)(R_{1212} + R_{3434} - R_{1414} - R_{2323}) = 0,\\
&(R_{1214}^2 - R_{1223}^2)(R_{1234} - R_{1423}) = 0,\\
&(R_{1314}^2 - R_{1323}^2)(R_{1313} + R_{2424} - R_{1414} - R_{2323}) = 0,\\
&(R_{1314}^2 - R_{1323}^2)(R_{1324} + R_{1423}) = 0.
\end{aligned} \tag{4.1}$$

We now apply (3.6) to obtain Table 2. We now obtain Table 3 from (3.10).

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TABLE 1. Calculations of (3.2).

a	b	$2\rho_{ab} = \sum_{i,j=1}^{4} R_{aibj}R_{aiaj}$
1	2	$R_{1323}(R_{1313} - R_{1414}) + R_{1314}(R_{1324} + R_{1423}) + R_{1213}R_{1223} + R_{1214}R_{1224} = 0$
2	1	$R_{1323}(R_{2323} - R_{2424}) - R_{1314}(R_{1324} + R_{1423}) + R_{1213}R_{1223} + R_{1214}R_{1224} = 0$
1	3	$R_{1223}(R_{1414} - R_{1212}) + R_{1214}(R_{1234} + R_{1432}) - R_{1213}R_{1323} - R_{1224}R_{1314} = 0$
3	1	$R_{1223}(R_{3434} - R_{2323}) - R_{1214}(R_{1234} + R_{1432}) - R_{1213}R_{1323} - R_{1224}R_{1314} = 0$
1	4	$R_{1224}(R_{1313} - R_{1212}) + R_{1213}(R_{1342} - R_{1234}) - R_{1223}R_{1314} + R_{1323}R_{1214} = 0$
4	1	$R_{1224}(R_{3434} - R_{2424}) - R_{1213}(R_{1342} - R_{1234}) - R_{1223}R_{1314} + R_{1323}R_{1214} = 0$
2	3	$R_{1213}(R_{1212} - R_{2424}) + R_{1224}(R_{1234} + R_{1324}) + R_{1223}R_{1323} - R_{1214}R_{1314} = 0$
3	2	$R_{1213}(R_{1313} - R_{3434}) - R_{1224}(R_{1234} + R_{1324}) + R_{1223}R_{1323} - R_{1214}R_{1314} = 0$
2	4	$R_{1214}(R_{1212} - R_{2323}) + R_{1223}(R_{1423} - R_{1234}) - R_{1224}R_{1323} - R_{1213}R_{1314} = 0$
4	2	$R_{1214}(R_{1414} - R_{3434}) - R_{1223}(R_{1423} - R_{1234}) - R_{1224}R_{1323} - R_{1213}R_{1314} = 0$
3	4	$R_{1314}(R_{1313} - R_{2323}) + R_{1323}(R_{1324} + R_{1423}) - R_{1223}R_{1224} - R_{1213}R_{1214} = 0$
4	3	$R_{1314}(R_{1414} - R_{2424}) - R_{1323}(R_{1324} + R_{1423}) - R_{1223}R_{1224} - R_{1213}R_{1214} = 0$

TABLE 2. Calculations of (3.6).

a	b	$2\{\sum_{i,j=1}^{4}(R_{aibj})^2 + \sum_{i,j=1}^{4} R_{aibj}R_{biaj} + \sum_{i,j=1}^{4} R_{aiaj}R_{bibj}\}$ $= \sum_{i,j=1}^{4}(R_{aiaj})^2 + \sum_{i,j=1}^{4}(R_{bibj})^2$
1	2	$8(R_{1323}^2 - R_{1314}^2) + 2(R_{1423} + R_{1324})^2 = (R_{1313} - R_{2323})^2 + (R_{1414} - R_{2424})^2$
3	4	$8(R_{1314}^2 - R_{1323}^2) + 2(R_{1324} + R_{1423})^2 = (R_{1313} - R_{1414})^2 + (R_{2323} - R_{2424})^2$
1	3	$8(R_{1223}^2 - R_{1214}^2) + 2(R_{1234} - R_{1423})^2 = (R_{1212} - R_{2323})^2 + (R_{1414} - R_{3434})^2$
2	4	$8(R_{1214}^2 - R_{1223}^2) + 2(R_{1234} - R_{1423})^2 = (R_{1212} - R_{1414})^2 + (R_{2323} - R_{3434})^2$
1	4	$8(R_{1224}^2 - R_{1213}^2) + 2(R_{1234} + R_{1324})^2 = (R_{1212} - R_{2424})^2 + (R_{1313} - R_{3434})^2$
2	3	$8(R_{1213}^2 - R_{1224}^2) + 2(R_{1234} + R_{1324})^2 = (R_{1212} - R_{1313})^2 + (R_{2424} - R_{3434})^2$

From the first and second equations in Table 1 we obtain

$$R_{1323}(R_{1313} - R_{1414} + R_{2323} - R_{2424}) + 2R_{1213}R_{1223} + 2R_{1214}R_{1224} = 0. \tag{4.2}$$

From the fifth and sixth equations in Table 3 we get

$$R_{1323}(R_{1313} - R_{1414} + R_{2323} - R_{2424}) - 6R_{1213}R_{1223} - 6R_{1214}R_{1224} = 0. \tag{4.3}$$

Thus from (4.2) and (4.3) we may deduce that

$$\begin{aligned} &R_{1323}(R_{1313} - R_{1414} + R_{2323} - R_{2424}) = 0, \\ &R_{1213}R_{1223} + R_{1214}R_{1224} = 0. \end{aligned} \tag{4.4}$$

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TABLE 3. Calculations of (3.10).

a	b	c	$2\rho_{ac} = \sum_{i,j}^{4} R_{aicj}R_{bibj} + \sum_{i,j}^{4} R_{bicj}(R_{aibj} + R_{biaj})$
1	2	3	$R_{1223}(R_{1313} + R_{2424} - R_{1212} - R_{2323}) + 3R_{1213}R_{1323} + 3R_{1224}R_{1314} + 3R_{1214}R_{1324} = 0$
1	4	3	$R_{1223}(R_{1414} + R_{3434} - R_{1313} - R_{2424}) + 3R_{1213}R_{1323} + 3R_{1224}R_{1314} - 3R_{1214}R_{1324} = 0$
1	2	4	$R_{1224}(R_{2323} + R_{1414} - R_{1212} - R_{2424}) + 3R_{1223}R_{1314} - 3R_{1214}R_{1323} + 3R_{1213}R_{1423} = 0$
1	3	4	$R_{1224}(R_{1313} + R_{3434} - R_{1414} - R_{2323}) - 3R_{1214}R_{1323} + 3R_{1223}R_{1314} - 3R_{1213}R_{1423} = 0$
1	3	2	$R_{1323}(R_{1313} + R_{2323} - R_{1212} - R_{3434}) - 3R_{1214}R_{1224} - 3R_{1213}R_{1223} + 3R_{1314}R_{1234} = 0$
1	4	2	$R_{1323}(R_{1212} + R_{3434} - R_{1414} - R_{2424}) - 3R_{1213}R_{1223} - 3R_{1214}R_{1224} - 3R_{1314}R_{1234} = 0$
2	1	3	$R_{1213}(R_{1212} + R_{1313} - R_{1414} - R_{2323}) + 3R_{1214}R_{1314} - 3R_{1223}R_{1323} - 3R_{1224}R_{1423} = 0$
2	4	3	$R_{1213}(R_{1414} + R_{2323} - R_{2424} - R_{3434}) - 3R_{1223}R_{1323} + 3R_{1214}R_{1314} + 3R_{1224}R_{1423} = 0$
2	1	4	$R_{1214}(R_{1212} + R_{1414} - R_{1313} - R_{2424}) + 3R_{1213}R_{1314} + 3R_{1224}R_{1323} - 3R_{1223}R_{1324} = 0$
2	3	4	$R_{1214}(R_{1313} + R_{2424} - R_{2323} - R_{3434}) + 3R_{1224}R_{1323} + 3R_{1213}R_{1314} + 3R_{1223}R_{1324} = 0$
3	1	4	$R_{1314}(R_{1313} + R_{1414} - R_{1212} - R_{3434}) + 3R_{1213}R_{1214} + 3R_{1223}R_{1224} + 3R_{1323}R_{1234} = 0$
3	2	4	$R_{1314}(R_{1212} + R_{3434} - R_{2323} - R_{2424}) + 3R_{1223}R_{1224} + 3R_{1213}R_{1214} - 3R_{1323}R_{1234} = 0$

Similarly,

$$\begin{aligned} &R_{1223}(R_{1414} - R_{1212} + R_{3434} - R_{2323}) = 0, \\ &R_{1213}R_{1323} + R_{1224}R_{1314} = 0, \end{aligned} \tag{4.5}$$

$$\begin{aligned} &R_{1314}(R_{1313} - R_{2323} + R_{1414} - R_{2424}) = 0, \\ &R_{1213}R_{1214} + R_{1223}R_{1224} = 0, \end{aligned} \tag{4.6}$$

$$\begin{aligned} &R_{1224}(R_{1313} - R_{1212} + R_{3434} - R_{2424}) = 0, \\ &R_{1214}R_{1323} - R_{1223}R_{1314} = 0, \end{aligned} \tag{4.7}$$

$$\begin{aligned} &R_{1214}(R_{1212} - R_{2323} + R_{1414} - R_{3434}) = 0, \\ &R_{1213}R_{1314} + R_{1224}R_{1323} = 0 \end{aligned} \tag{4.8}$$

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and

$$R_{1213}(R_{1313} - R_{2424} + R_{1212} - R_{3434}) = 0,$$
$$R_{1223}R_{1323} - R_{1214}R_{1314} = 0. \tag{4.9}$$

From the seventh and eighth equations in Table 3 and (4.9), we get

$$R_{1213}(R_{1212} + R_{1313} - R_{1414} - R_{2323}) - 3R_{1224}R_{1423} = 0,$$
$$R_{1213}(R_{1414} + R_{2323} - R_{2424} - R_{3434}) + 3R_{1224}R_{1423} = 0. \tag{4.10}$$

Similarly,

$$R_{1224}(R_{1414} + R_{2323} - R_{1212} - R_{2424}) + 3R_{1213}R_{1423} = 0,$$
$$R_{1224}(R_{1313} + R_{3434} - R_{1414} - R_{2323}) - 3R_{1213}R_{1423} = 0. \tag{4.11}$$

In addition,

$$R_{1214}(R_{1212} + R_{1414} - R_{1313} - R_{2424}) - 3R_{1223}R_{1324} = 0,$$
$$R_{1214}(R_{1313} + R_{2424} - R_{2323} - R_{3434}) + 3R_{1223}R_{1324} = 0$$

as well as

$$R_{1223}(R_{1313} + R_{2424} - R_{1212} - R_{2323}) + 3R_{1214}R_{1324} = 0,$$
$$R_{1223}(R_{1414} + R_{3434} - R_{1313} - R_{2424}) - 3R_{1214}R_{1324} = 0.$$

We also obtain similarly

$$R_{1314}(R_{1313} + R_{1414} - R_{1212} - R_{3434}) + 3R_{1323}R_{1234} = 0,$$
$$R_{1314}(R_{1212} + R_{3434} - R_{2323} - R_{2424}) - 3R_{1323}R_{1234} = 0$$

and

$$R_{1323}(R_{1313} + R_{2323} - R_{1212} - R_{3434}) + 3R_{1314}R_{1234} = 0,$$
$$R_{1323}(R_{1212} + R_{1313} - R_{1414} - R_{2323}) - 3R_{1314}R_{1234} = 0.$$

Now we set

$$\mathbb{a} = (R_{1213}, R_{1224}), \quad \mathbb{b} = (R_{1214}, -R_{1223}), \quad \mathbb{c} = (R_{1314}, -R_{1323}).$$

Then, from the second equations of (4.4)–(4.9), we obtain

$$\begin{aligned}
\langle \mathbb{a}, \mathbb{b}^{\perp} \rangle = 0 &\Longrightarrow \langle \mathbb{a}^{\perp}, \mathbb{b} \rangle = 0, \\
\langle \mathbb{a}, \mathbb{c}^{\perp} \rangle = 0 &\Longrightarrow \langle \mathbb{a}^{\perp}, \mathbb{c} \rangle = 0, \\
\langle \mathbb{a}, \mathbb{b}' \rangle = 0 &\Longrightarrow \langle \mathbb{a}', \mathbb{b} \rangle = 0, \\
\langle \mathbb{b}, \mathbb{c}^{\perp} \rangle = 0 &\Longrightarrow \langle \mathbb{b}^{\perp}, \mathbb{c} \rangle = 0, \\
\langle \mathbb{a}, \mathbb{c}' \rangle = 0 &\Longrightarrow \langle \mathbb{a}', \mathbb{c} \rangle = 0, \\
\langle \mathbb{b}, \mathbb{c}' \rangle = 0 &\Longrightarrow \langle \mathbb{b}, \mathbb{c} \rangle = 0.
\end{aligned} \tag{4.12}$$

The following is the key lemma required for our proof of Theorem 1.1.

LEMMA 4.1. *At each point of M one of the following conditions is satisfied:*

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(1) $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$;
(2) $\lambda_1 = \lambda_2 \neq \lambda_3 = \lambda_4$;
(3) $\lambda_1 = \lambda_3 \neq \lambda_2 = \lambda_4$;
(4) $\lambda_1 = \lambda_4 \neq \lambda_2 = \lambda_3$.

Proof. To prove Lemma 4.1, we proceed case by case.

Case I. Suppose that $\mathbb{a} \neq 0$, $\mathbb{b} \neq 0$, $\mathbb{c} \neq 0$. Then $\mathbb{b}^{\perp} /\!/ \mathbb{b}'$, from the first and third equations of (4.12). Since $|\mathbb{b}^{\perp}| = |\mathbb{b}'|$, either $\mathbb{b}^{\perp} = \mathbb{b}'$ or $\mathbb{b}^{\perp} = -\mathbb{b}'$. Therefore

$$R^2_{1223} - R^2_{1214} = 0. \tag{4.13}$$

Next, $\mathbb{c}^{\perp} = \mathbb{c}'$ or $\mathbb{c}^{\perp} = -\mathbb{c}'$, from the second and fifth equations of (4.12). Hence

$$R^2_{1314} - R^2_{1323} = 0. \tag{4.14}$$

Similarly, $\mathbb{a}^{\perp} = \mathbb{a}'$ or $\mathbb{a}^{\perp} = -\mathbb{a}'$, that is,

$$R^2_{1213} - R^2_{1224} = 0. \tag{4.15}$$

From Table 2 and (4.13)–(4.15), the following equations hold:

$$\begin{aligned}
2(R_{1423} + R_{1324})^2 &= (R_{1313} - R_{2323})^2 + (R_{1414} - R_{2424})^2,\\
2(R_{1423} + R_{1324})^2 &= (R_{1313} - R_{1414})^2 + (R_{2323} - R_{2424})^2,\\
2(R_{1234} - R_{1423})^2 &= (R_{1212} - R_{2323})^2 + (R_{1414} - R_{3434})^2,\\
2(R_{1234} - R_{1423})^2 &= (R_{1212} - R_{1414})^2 + (R_{2323} - R_{3434})^2,\\
2(R_{1234} + R_{1324})^2 &= (R_{1212} - R_{2424})^2 + (R_{1313} - R_{3434})^2,\\
2(R_{1234} + R_{1324})^2 &= (R_{1212} - R_{1313})^2 + (R_{2424} - R_{3434})^2.
\end{aligned} \tag{4.16}$$

Now from the first and second equations of (4.16), we may deduce that

$$(R_{1313} - R_{2424})(R_{1414} - R_{2323}) = 0. \tag{4.17}$$

Similarly, from the third and fourth equations of (4.16), we deduce that

$$(R_{1212} - R_{3434})(R_{1414} - R_{2323}) = 0. \tag{4.18}$$

From the fifth and sixth equations of (4.16), we obtain that

$$(R_{1212} - R_{3434})(R_{1313} - R_{2424}) = 0. \tag{4.19}$$

Subcase I(i). We assume that $R_{1212} - R_{3434} \neq 0$. We deduce from (4.18) and (4.19) that

$$\begin{aligned}
R_{1414} - R_{2323} &= 0,\\
R_{1313} - R_{2424} &= 0.
\end{aligned}$$

Since $R_{1214} \neq 0$ and $R_{1213} \neq 0$, it follows from (4.8) and (4.9) that $R_{1212} - R_{3434} = 0$. But this is a contradiction.

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Subcase I(ii). We assume that $R_{1212} - R_{3434} = 0$. Then, from (4.8) and (4.9),

$$\begin{aligned} R_{1414} - R_{2323} &= 0, \\ R_{1313} - R_{2424} &= 0. \end{aligned}$$

Thus, in this case,

$$\begin{aligned} \lambda_1 &= R_{2112} + R_{3113} + R_{4114} \\ &= R_{1221} + R_{4224} + R_{3223} \quad (=\lambda_2) \\ &= R_{4334} + R_{1331} + R_{2332} \quad (=\lambda_3) \\ &= R_{3443} + R_{2442} + R_{1441} \quad (=\lambda_4) \end{aligned}$$

and hence we see that condition (1) of Lemma 4.1 holds at p.

Case II. Suppose that $\mathbb{a} \neq 0$, $\mathbb{b} \neq 0$, $\mathbb{c} = 0$. From the first and third equations of (4.16), we see that $\mathbb{b}^{\perp} \,/\!/\, \mathbb{b}'$, and hence $\mathbb{b}^{\perp} = \pm \mathbb{b}'$. Therefore

$$R_{1223}^2 - R_{1214}^2 = 0.$$

Similarly,

$$R_{1213}^2 - R_{1224}^2 = 0.$$

Based on our assumption, it also follows that

$$R_{1314}^2 = R_{1323}^2 = 0.$$

Thus, by similar arguments to those for case I, we also see that condition (1) of Lemma 4.1 holds at p.

By applying similar arguments in case III ($\mathbb{a} \neq 0$, $\mathbb{b} = 0$, $\mathbb{c} \neq 0$) and case IV ($\mathbb{a} = 0$, $\mathbb{b} \neq 0$, $\mathbb{c} \neq 0$), we see that condition (1) of Lemma 4.1 holds at p.

Case V. Suppose that $\mathbb{a} \neq 0$, $\mathbb{b} = 0$, $\mathbb{c} = 0$. Then

$$R_{1214}^2 = R_{1223}^2 = 0, \quad R_{1314}^2 = R_{1323}^2 = 0. \tag{4.20}$$

From the first four equations of Table 2 and (4.20),

$$\begin{aligned} 2(R_{1423} + R_{1324})^2 &= (R_{1313} - R_{2323})^2 + (R_{1414} - R_{2424})^2, \\ 2(R_{1234} - R_{1423})^2 &= (R_{1212} - R_{2323})^2 + (R_{1414} - R_{3434})^2, \\ 2(R_{1234} - R_{1423})^2 &= (R_{1212} - R_{1414})^2 + (R_{2323} - R_{3434})^2, \\ 2(R_{1324} + R_{1423})^2 &= (R_{1313} - R_{1414})^2 + (R_{2323} - R_{2424})^2. \end{aligned} \tag{4.21}$$

Thus, from (4.21),

$$\begin{aligned} (R_{1313} - R_{2424})(R_{1414} - R_{2323}) &= 0, \\ (R_{1212} - R_{3434})(R_{1414} - R_{2323}) &= 0. \end{aligned} \tag{4.22}$$

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From (4.10) and (4.11), since $\mathbb{a} = (R_{1213}, R_{1224}) \neq 0$, we may deduce that

$$\begin{aligned} R_{1423}(R_{1313} - R_{2424} + R_{1212} - R_{3434}) = 0, \\ R_{1423}(R_{1313} + R_{3434} - R_{1212} - R_{2424}) = 0. \end{aligned} \tag{4.23}$$

Subcase V(i). We assume that $R_{1213}^2 - R_{1224}^2 \neq 0$. Then, from the first and second equations of (4.1), we get

$$\begin{aligned} R_{1212} - R_{1313} - R_{2424} + R_{3434} = 0, \\ R_{1234} + R_{1324} = 0. \end{aligned} \tag{4.24}$$

Further, we suppose that $R_{1414} - R_{2323} \neq 0$. Then, from (4.22),

$$R_{1212} = R_{3434} \quad \text{and} \quad R_{1313} = R_{2424}. \tag{4.25}$$

Also, applying (4.25), in this case we see that

$$\begin{aligned} \lambda_1 &= R_{2112} + R_{3113} + R_{4114} = R_{3443} + R_{4224} + R_{1441} = \lambda_4, \\ \lambda_2 &= R_{1221} + R_{3223} + R_{4224} = R_{4334} + R_{2332} + R_{1331} = \lambda_3. \end{aligned}$$

Since $R_{1414} \neq R_{2323}$ we see that condition (4) of Lemma 4.1 holds at p.

We now suppose that $R_{1414} - R_{2323} = 0$. First, we further suppose that $R_{1423} \neq 0$. Then, from (4.23),

$$R_{1212} = R_{3434} \quad \text{and} \quad R_{1313} = R_{2424}.$$

In this case, we see that condition (1) of Lemma 4.1 holds at p. Next, we further suppose that $R_{1423} = 0$. Then, from the second equation of (4.24),

$$0 = R_{1234} - R_{1342} = -2R_{1342} - R_{1423} = -2R_{1342},$$

and hence

$$R_{1342} = 0 \Longrightarrow R_{1234} = 0.$$

Thus, from the first to fourth equations of Table 2, we may deduce that

$$\begin{aligned} R_{1313} - R_{1414} = 0, &\quad R_{1414} - R_{2424} = 0, \\ R_{1212} - R_{1414} = 0, &\quad R_{1414} - R_{3434} = 0 \end{aligned}$$

and hence

$$R_{1212} = R_{3434}, \quad R_{1313} = R_{2424}.$$

Thus, in this case, condition (1) of Lemma 4.1 holds at p.

Subcase V(ii). We assume that $R_{1213}^2 - R_{1224}^2 = 0$. Then, from Table 2, we see that this case reduces to case I. More precisely, the equalities (4.17)–(4.19) hold. Therefore condition (1) of Lemma 4.1 holds at p.

We can similarly show that Lemma 4.1 is valid in case VI ($\mathbb{a} = 0$, $\mathbb{b} \neq 0$, $\mathbb{c} = 0$) and case VII ($\mathbb{a} = 0$, $\mathbb{b} = 0$, $\mathbb{c} \neq 0$).

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Case VIII. Suppose that $\mathbb{a} = 0$, $\mathbb{b} = 0$, $\mathbb{c} = 0$. Then

$$R^2_{1213} = R^2_{1224} = 0, \quad R^2_{1214} = R^2_{1223} = 0, \quad R^2_{1314} = R^2_{1323} = 0.$$

By similar arguments to those for case I we obtain the following equations from Table 2:

$$\begin{aligned}(R_{1313} - R_{2424})(R_{1414} - R_{2323}) &= 0,\\ (R_{1212} - R_{3434})(R_{1414} - R_{2323}) &= 0,\\ (R_{1212} - R_{3434})(R_{1313} - R_{2424}) &= 0.\end{aligned} \tag{4.26}$$

Subcase VIII(i). We assume that $R_{1414} - R_{2323} \neq 0$. Then, from the first and second equations of (4.26),

$$R_{1212} = R_{3434}, \quad R_{1313} = R_{2424}.$$

By similar arguments to those for case V(i) we see that condition (4) of Lemma 4.1 holds at p.

Subcase VIII(ii). We assume that $R_{1414} - R_{2323} = 0$. Then, from the third equation of (4.26), we see that $R_{1212} = R_{3434}$ or $R_{1313} = R_{2424}$. By similar arguments to those for case V(i) we see that either of conditions (3) or (2) of Lemma 4.1 holds at p, respectively. □

Proof of Theorem 1.1. We now complete the proof of Theorem 1.1. We define M_1 to be the set of all points p in M such that condition (1) of Lemma 4.1 holds at p. We also define M_2 to be the set of all points p in M such that any of conditions (2), (3) or (4) of Lemma 4.1 holds at p. Then we certainly have $M = M_1 \cup M_2$ by Lemma 4.1. Further, by continuity arguments on the Ricci eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and λ_4, we see that M_2 is an open subspace of M.

We now assume that $M_2 \neq \phi$. Without loss of generality we may assume that condition (2) of Lemma 4.1 holds at some point $p_0 \in M$. We let M^0_2 denote the connected component of p_0 and set $\lambda = \lambda_1 = \lambda_2$ and $\mu = \lambda_3 = \lambda_4$. Then we may easily check that λ and μ are smooth functions on M^0_2. We denote by D_λ and D_μ the distributions on M_2 corresponding the eigenvalues λ and μ, respectively.

Let $\{e_i\} = \{e_1, e_2, e_3, e_4\}$ be a local orthonormal frame field on M_2 such that $\{e_1, e_2\}$ and $\{e_3, e_4\}$ are local bases for D_μ and D_μ, respectively. We set

$$\nabla_{e_i} e_j = \sum_k \Gamma_{ijk} e_k \tag{4.27}$$

where $i, j = 1, 2, 3, 4$. Then

$$\Gamma_{ijk} = -\Gamma_{ikj}. \tag{4.28}$$

From the equality (3.1) and (4.27) and (4.28), we deduce that λ and μ are constant on M^0_2. Thus

$$\begin{aligned}\Gamma_{ab3} &= \Gamma_{ab4} = 0,\\ \Gamma_{cd1} &= \Gamma_{cd2} = 0,\end{aligned} \tag{4.29}$$

where $1 \leq a, b \leq 2$ and $3 \leq c, d \leq 4$.

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Now, since λ and μ are constant on $M_2^0(p_0)$, we see that $M_2^0(p_0) = M$ by the continuity of M. Further, from (4.29), we see that the distributions D_λ and D_μ are both parallel on M. Therefore we see that M is locally a product of two-dimensional Riemannian manifolds M_λ and M_μ where M_λ and M_μ are the integral manifolds of the distributions D_λ and D_μ, respectively.

Since $R_{1234} = R_{1423} = 0$ it follows from the third equation in Table 2 that $R_{1212} = R_{3434}$. But this is a contradiction in the case where $\lambda = \mu$. Since this contradiction came from the assumption $M_2 \neq \phi$, it follows necessarily that $M = M_1$. Therefore M is Einstein and hence 2-stein by the main result of [7].

The converse is evident and was already proved in [7] in any dimension. This completes the proof of Theorem 1.1. □

References

[1] D. E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, Progress in Mathematics, 203 (Birkhäuser, Boston, 2002).

[2] E. Boeckx and L. Vanhecke, 'Harmonic and minimal vector fields in tangent and unit tangent bundles', *Differential Geom. Appl.* **13** (2000), 77–93.

[3] E. Boeckx and L. Vanhecke, 'Unit tangent sphere bundles with constant scalar curvature', *Czechoslovak Math. J.* **51** (2001), 523–544.

[4] G. Calvaruso and D. Perrone, 'H-contact unit tangent sphere bundles', *Rocky Mountain J. Math.* **37** (2007), 1435–1458.

[5] G. Calvaruso and D. Perrone, 'Homogeneous and H-contact unit tangent sphere bundles', *J. Aust. Math. Soc.* **88** (2010), 323–337.

[6] P. Carpenter, A. Gray and T. J. Willmore, 'The curvature of Einstein symmetric spaces', *Q. J. Math. Oxford* **33** (1982), 45–64.

[7] S. H. Chun, J. H. Park and K. Sekigawa, 'H-contact unit tangent sphere bundles of Einstein manifolds', *Q. J. Math.* **62** (2011), 59–69.

[8] J. H. Park and K. Sekigawa, 'When are the tangent sphere bundles of a Riemannian manifold η-Einstein?', *Ann. Global Anal. Geom.* **36** (2009), 275–284.

[9] D. Perrone, 'Contact metric manifolds whose characteristic vector field is a harmonic vector field', *Differential Geom. Appl.* **20** (2004), 367–378.

[10] C. M. Wood, 'On the energy of a unit vector field', *Geom. Dedicata* **64** (1997), 319–330.

SUN HYANG CHUN, Department of Mathematics, Sungkyunkwan University,
Suwon 440-746, Korea
e-mail: cshyang@skku.edu

JEONGHYEONG PARK, Department of Mathematics, Sungkyunkwan University,
Suwon 440-746, Korea
e-mail: parkj@skku.edu

KOUEI SEKIGAWA, Department of Mathematics, Faculty of Science,
Niigata University, Niigata 950-2181, Japan
e-mail: sekigawa@math.sc.niigata-u.ac.jp