# AN EXAMPLE OF A FUNCTION WITH NON-ANALYTIC ITERATES

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## 1. Preliminaries

Let $\Omega$ be the set of the analytic functions $F(z)$, regular in some neighbourhood of the origin with the expansion

$$F(z) = z + f_2 z^2 + f_3 z^3 + \cdots. \tag{1.1}$$

There may exist a function $F(s, z)$ analytic in $s$ and satisfying the following conditions ($s$ and $s'$ are any complex numbers):

$$F(1, z) = F(z), \tag{1.2}$$

$$F(s, z) \in \Omega, \tag{1.3}$$

$$F[s, F(s', z)] = F[(s+s'), z], \tag{1.4}$$

$$F(s, z) = \sum_{k=1}^{\infty} f_k(s) z^k \quad \text{for} \quad |z| < \rho_s, \qquad \rho_s > 0, \tag{1.5}$$

and the $f_k(s)$ are polynomials in $s$.

If $F(s, z)$ exists, it will be called the analytic iterate of $F(z)$. (The necessity and independence of these four conditions are discussed in [4]).

It is easily seen that $F(z) = z/(1-z)$ possesses the analytic iterate $F(s, z) = z/(1-sz)$. Functions which possess an analytic iterate will be called functions of type $A$. Functions for which (1.5) does not hold for every $s$, will be called functions of type $B$. In [2] I. N. Baker shows that for type $B$ the set $S$ of points $s$ for which (1.5) converges in some neighbourhood of $z = 0$ is a discreet lattice (one- or two-dimensional). In [7] G. Szekeres shows that the class of entire functions of $\Omega$ belongs to $B$, also the class of rational functions of $\Omega$ unless $F(z) = z/(1-az)$. Baker [3] extends the $B$-property to the class of meromorphic functions.

It is the purpose of this paper to prove the $B$-property for a certain function using much more elementary considerations. The function $e^z - 1$ has been chosen to illustrate the method used. (By essentially the same method the functions $z+z^2$ and $z/(1-z)^2$ had been dealt with.)

## 2. Jabotinsky's $L$-functions

Let $F(z) \in A$. It then has an analytic iterate $F(s, z)$. Define $L(z)$ by

$$L(z) = \left.\frac{\partial F(s, z)}{\partial s}\right|_{s=0}. \tag{2.1}$$

It is shown in [5] that the expansion of $L(z)$ is of the form

$$L(z) = \sum_{k=l}^{\infty} l_k z^{k+l}, \quad \text{for} \quad |z| < \rho_L, \qquad \rho_L > 0, \tag{2.2}$$

and that $L(z)$ satisfies the functional equation

$$L[F(z)] = F'(z)L(z), \tag{2.3}$$

which may however have other solutions that are not $L$-functions of the function $F$. Clearly equation (2.3) has the particular solution:

$$L(z) = 0, \tag{2.4}$$

whatever the given function $F$. The only function of $\Omega$ for which $L(z) = 0$, is $F(z) = z$.

It is shown in [5] that the sequences $f_n$ in (1.1) and $l_n$ in (2.2) determine each other uniquely (though the series in (2.2) corresponding to a given function $F$ may converge only for $z = 0$). To show that $F$ of $\Omega$ belongs to $B$ it is thus sufficient to show that the series (2.2) corresponding to this $F$ converges only for $z = 0$.

## 3. $F(z) = e^z - 1$

Put $e^z - 1 = e^{x+iy} - 1 = u + iv$ ($x, y, u, v$ real), so that:

$$u = e^x \cos y - 1; \quad v = e^x \sin y, \tag{3.1}$$

and

$$u^2 + v^2 = e^{2x} - 2e^x \cos y + 1. \tag{3.2}$$

It is easily seen that for $0 < |y| < \pi/2$:

$$\frac{2(1 - \cos y)}{\cos y} > y^2, \tag{3.3}$$

and for $y \neq 0$:

$$2(1 - \cos y) < y^2. \tag{3.4}$$

LEMMA 3.1. *The function $e^z - 1$ maps each point of the right half-strip* $\operatorname{Re} z > 0$, $-\pi < \operatorname{Im} z \leqq \pi$ *either into the left half-plane (including the imaginary axis) or else farther away from the origin.*

PROOF. If $u \leqq 0$, there is nothing to prove. If

$$u = e^x \cos y - 1 > 0, \tag{3.5}$$

then

$$\cos y > 0 \text{ and } |y| < \frac{\pi}{2}. \tag{3.6}$$

For $x > 0$:

$$e^x - 1 > x, \tag{3.7}$$

or, by squaring and subtracting $2e^x \cos y$ from both sides:

$$e^{2x} - 2e^x \cos y + 1 > x^2 + 2e^x(1 - \cos y), \tag{3.8}$$

But (3.5) implies:

$$e^x > \frac{1}{\cos y}, \tag{3.9}$$

so that by (3.2), (3.8) and (3.9):

$$u^2 + v^2 > x^2 + \frac{2(1 - \cos y)}{\cos y} > x^2 + y^2,$$

the right inequality following from (3.3) and (3.6). This proves the lemma.

LEMMA 3.2. *The function* $e^z - 1$ *maps all the points of the left half-strip* $\operatorname{Re} z \leqq 0$, $-\pi < \operatorname{Im} z \leqq \pi$ *nearer to the origin.*

PROOF. For $x = 0$ this follows directly from (3.2) and (3.4). For $x < 0$:

$$e^x < 1, \tag{3.10}$$

and also (since (3.7) holds generally for $x \neq 0$)

$$|e^x - 1| < |x|, \tag{3.11}$$

or

$$e^{2x} + 1 < x^2 + 2e^x. \tag{3.12}$$

By squaring and subtracting as before we obtain $u^2 + v^2 < x^2 + y^2$.

We now show the divergence of the $L(z)$ series. Equation (2.3) becomes in this case

$$L(e^z - 1) = e^z L(z). \tag{3.13}$$

It is sufficient to prove that this functional equation has no solution of the form (2.2) with a positive radius of convergence $\rho$, and which is not identically zero.

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It is obviously sufficient to consider the singular points of $L(z)$ in the strip $-\pi < \operatorname{Im} z \leqq \pi$ since $L(z)$ is periodic with period $2\pi i$, if it is defined in a large enough region.

A. First suppose that $\rho$ is finite. $e^z-1$ maps the left half of the strip onto the circle $|z+1| < 1$. If $\rho > 0$ let $\zeta$ be a singular point of $L(z)$ on the circle of convergence. If $\zeta$ belongs to the left half-strip or is purely imaginary, then $w$ for which $w = e^{\zeta}-1$ is also a singular point and is nearer to the origin than $\zeta$ by lemma 3.2 which is a contradiction.

If $\zeta$ is a point of the right half-strip, then $\omega$ for which $e^{\omega}-1 = \zeta$ is also a singular point. If $\zeta$ belongs to the right half-strip, then, by lemma 3.1, so does $\omega$ and also $|\omega| < |\zeta|$. Again there is a singular point inside the circle of convergence — a contradiction.

B. Suppose now that $L(z)$ is an entire function. It is then periodic and cannot therefore be a polynomial.

Equation (3.13) can be written in the form

$$L(z) = (1+z)L[\log (1+z)]. \tag{3.14}$$

This implies that for all large enough $r$ the function $M(r) = \max_{|z|=r}|L(z)|$ satisfies

$$M(r) < 2rM(2 \log r), \tag{3.15}$$

and putting, $V(e^s) = \log M(r)$, $r = e^s$, the increasing function $V$ of $s$ satisfies $V(4s) < V(e^s) < V(2s)+s+\log 2$. Hence $(V(4s)-V(2s))/2s < 1$ for all large $s$ and thus it easily follows that $V(s) < s+K$ for some constant. Hence $V(e^s) < 3s+K$ so that

$$M(r) < e^K r^3, \tag{3.16}$$

which implies $L(z)$ to be a constant $C$ which can only be zero because of (3.13). But $L(z) = 0$ corresponds only to $F(z) = z$ so that our proof is complete.

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