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RESEARCH ARTICLE

The homology of moduli spaces of 4-manifolds may be infinitely generated

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Abstract
For a simply-connected closed manifold X of $\dim X \neq 4$, the mapping class group $\pi_0(\mathrm{Diff}(X))$ is known to be finitely generated. We prove that analogous finite generation fails in dimension 4. Namely, we show that there exist simply-connected closed smooth 4-manifolds whose mapping class groups are not finitely generated. More generally, for each $k > 0$, we prove that there are simply-connected closed smooth 4-manifolds X for which $H_k(B\mathrm{Diff}(X);\mathbb{Z})$ are not finitely generated. The infinitely generated subgroup of $H_k(B\mathrm{Diff}(X);\mathbb{Z})$ which we detect are topologically trivial, and unstable under the connected sum of $S^2 \times S^2$. These results are proven by constructing and computing an infinite family of characteristic classes using Seiberg–Witten theory.

Contents

1. Introduction

1.1. Main results

The purpose of this paper is to present a new special phenomenon in dimension 4 in terms diffeomorphism groups. To describe our result, let $\mathrm{Diff}(X)$ denote the diffeomorphism group equipped with the C^∞-topology for a given a smooth manifold X. It is known that the mapping class group $\pi_0(\mathrm{Diff}(X))$ is finitely generated, if X is simply-connected closed and $\dim X \neq 4$. For $\dim X \geq 5$, this is due to Sullivan [33, Theorem (13.3)]. For $\dim X \leq 3$, finite generation holds even dropping the simple-connectivity: In fact, even stronger finiteness is known in all dimensions ≠ 4 (see Subsection 5.1, including a remark for dim=5).

We prove that analogous finite generation fails in dimension 4. Namely, we show that there exist simply-connected closed smooth 4-manifolds whose mapping class groups are infinitely generated:

Theorem 1.1. *For $n \geq 2$, set $X = E(n)\#S^2 \times S^2$. Then $\pi_0(\mathrm{Diff}(X))$ is not finitely generated.*

Here, $E(n)$ denotes the simply-connected elliptic surface of degree n without multiple fiber. As is well-known, $E(n)\#S^2 \times S^2$ can be written in terms of further basic 4-manifolds (e.g., [15, Corollary 8]):

$$E(n)\#S^2 \times S^2 \cong \begin{cases} 2n\mathbb{CP}^2\#10n\overline{\mathbb{CP}}^2 & \text{for } n \text{ odd,} \\ m(K3\#S^2 \times S^2) & \text{for } n = 2m \text{ even.} \end{cases}$$

Remark 1.2. After completing a preprint version of this paper, the author was informed that David Baraglia [4] also proved that the mapping class groups of simply-connected 4-manifolds can be infinitely generated. Baraglia's proof is based on essentially the same method as ours; however, we obtained our proofs completely independently.

Remark 1.3 (Topological mapping class group). Let $\mathrm{Homeo}(X)$ denote the homeomorphism group of X. If X is a simply-connected closed topological 4-manifold, then $\pi_0(\mathrm{Homeo}(X))$ is finitely generated. This follows from a result by Quinn [29] and Perron [28]. Thus, infinite generation exhibited in Theorem 1.1 is special to the 4-dimensional smooth category.

Theorem 1.1 is a consequence of a more general result on the (co)homology of the moduli spaces $B\mathrm{Diff}(X)$ of 4-manifolds X. The (co)homology of $B\mathrm{Diff}(X)$ is a fundamental object, since it corresponds to the set of characteristic classes of fiber bundles with fiber X. We shall prove that, for each $k \geq 0$, there exist simply-connected closed smooth 4-manifolds X where $H_k(B\mathrm{Diff}(X);\mathbb{Z})$ are infinitely generated. More strongly, we shall see that the 'topologically trivial parts' of $H_k(B\mathrm{Diff}(X);\mathbb{Z})$ can be infinitely generated. To state this, let $i : \mathrm{Diff}(X) \hookrightarrow \mathrm{Homeo}(X)$ denote the inclusion map into the homeomorphism group. We shall prove the following:

Theorem 1.4. *For $n \geq 2$ and $k \geq 1$, set $X = E(n)\#kS^2 \times S^2$. Then*

$$\ker(i_* : H_k(B\mathrm{Diff}(X);\mathbb{Z}) \to H_k(B\mathrm{Homeo}(X);\mathbb{Z}))$$

contains a direct summand isomorphic to $(\mathbb{Z}/2)^\infty$. In particular, $H_k(B\mathrm{Diff}(X);\mathbb{Z})$ is not finitely generated.

Here, $(\mathbb{Z}/2)^\infty$ denotes the countably infinite direct sum $\bigoplus_{\mathbb{N}} \mathbb{Z}/2$. Rephrasing Theorem 1.4 for $k = 1$, we have the following result, which immediately implies Theorem 1.1:

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Corollary 1.5. *For $n \geq 2$, set $X = E(n)\#S^2 \times S^2$. Then*

$$\ker(i_* : \pi_0(\mathrm{Diff}(X))_{\mathrm{ab}} \to \pi_0(\mathrm{Homeo}(X))_{\mathrm{ab}})$$

contains a direct summand isomorphic to $(\mathbb{Z}/2)^\infty$. Here the subscript ab *indicates the abelianization.*

To our knowledge, Theorem 1.4 gives the first examples of simply-connected closed manifolds X where $H_k(B\mathrm{Diff}(X);\mathbb{Z})$ are confirmed to be infinitely generated for given $k \geq 1$ (for $k = 1$, this follows also from the aforementioned result by Baraglia [4]). It is worth noting that there are several established and expected finiteness in dim $\neq 4$ (Remark 1.8). Thus, infiniteness given in Theorem 1.4 reflects a specialty of dimension 4, described in terms of characteristic classes of fiber bundles (see Remark 1.6 below).

Remark 1.6. In terms of cohomology, Theorem 1.4 deduces that non-topological characteristic classes may form a group isomorphic to $(\mathbb{Z}/2)^\infty$ for some 4-manifolds. Here, we call an element of

$$\mathrm{coker}(i^* : H^k(B\mathrm{Homeo}(X);\mathbb{Z}/2) \to H^k(B\mathrm{Diff}(X);\mathbb{Z}/2))$$

a *non-topological characteristic class* (over $\mathbb{Z}/2$). This $(\mathbb{Z}/2)^\infty$-subgroup is generated by gauge-theoretic characteristic classes we shall introduce (Subsection 1.3). In contrast, the Mumford–Morita–Miller classes, the most basic characteristic class of manifold bundles, are topological over a field of characteristic 2 or 0 [11].

It is worth noting a consequence about stabilization. Recently, Lin and the author [20] proved that the moduli spaces $B\mathrm{Diff}(X)$ of 4-manifolds X do not satisfy homological stability with respect to connected sums of $S^2 \times S^2$, unlike what happens in dimension $\neq 4$ [16, 14]. The proof of Theorem 1.4 shows also that the unstable part of $H_*(B\mathrm{Diff}(X))$ may be infinitely generated. To state this precisely, given a closed 4-manifold X, take a smoothly embedded 4-disk D^4 in X, and set $\mathring{X} = X \setminus \mathrm{Int}(D^4)$. Let $\mathrm{Diff}_\partial(\mathring{X})$ denote the group of diffeomorphisms that are the identity near $\partial\mathring{X}$. Form the (inner) connected sum $\mathring{X}\#S^2 \times S^2$ by $\mathring{X} \cup_{S^3} ((S^3 \times [0,1])\#S^2 \times S^2)$. Then one can define the stabilization map

$$s : \mathrm{Diff}_\partial(\mathring{X}) \to \mathrm{Diff}_\partial(\mathring{X}\#S^2 \times S^2)$$

by extending by the identity on $(S^3 \times [0,1])\#S^2 \times S^2$. We shall prove the following:

Theorem 1.7. *For $n \geq 2$ and $k \geq 1$, set $X = E(n)\#kS^2 \times S^2$. Then the kernel of the induced map*

$$s_* : H_k(B\mathrm{Diff}_\partial(\mathring{X});\mathbb{Z}) \to H_k(B\mathrm{Diff}_\partial(\mathring{X}\#S^2 \times S^2);\mathbb{Z})$$

contains a direct summand isomorphic to $(\mathbb{Z}/2)^\infty$.

1.2. Related results

The following remarks list related results in more detail:

Remark 1.8 (Other finiteness in dim $\neq 4$). Let us compare infinite generation of $H_k(B\mathrm{Diff}(X);\mathbb{Z})$ in Theorem 1.4 with other dimensions. For a manifold X of even dim ≥ 6 and with finite $\pi_1(X)$, Bustamante–Krannich–Kupers proved that $H_k(B\mathrm{Diff}(X);\mathbb{Z})$ is finitely generated for each k [9, Corollary B]. Also, in his earlier paper, Kupers [22, Corollary C] has proved an analogous statement for a 2-connected manifold X of dim $\neq 4, 5, 7$. As mentioned in [9], there is an expectation that finiteness may hold even dropping the 2-connectivity. For finiteness of mapping class groups in dimension $\neq 4$, see Subsection 5.1.

Remark 1.9 (Infiniteness of the Torelli group). Given a smooth closed oriented 4-manifold X, let $\mathrm{TDiff}(X)$ denote the *Torelli diffeomorphism group* (i.e., the group of diffeomorphisms acting trivially on $H_*(X;\mathbb{Z})$). Ruberman [31, Theorem A] proved that $\pi_0(\mathrm{TDiff}(X))$ are infinitely generated for

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$X = E(n)\#\mathbb{CP}^2\#k\overline{\mathbb{CP}}^2$ with $k \geq 2$. Note that infinite generation of $\pi_0(\mathrm{Diff}(X))$ does not necessarily follow from that of $\pi_0(\mathrm{TDiff}(X))$. For example, $\pi_0(\mathrm{TDiff}(X))$ is infinitely generated if one takes X to be the genus 2 surface, whereas $\pi_0(\mathrm{Diff}(X))$ is finitely generated [26]. This phenomenon may occur since the index of $\pi_0(\mathrm{TDiff}(X))$ in $\pi_0(\mathrm{Diff}(X))$ is infinite (also for X in Theorem 1.4), and an infinite index subgroup of a finitely generated group is not necessarily finitely generated.

Remark 1.10 (Other infiniteness in dim = 4). Baraglia [3] and Lin [24] proved that $\pi_1(\mathrm{Diff}(X))$ have infinite-rank summands for some simply-connected (irreducible) 4-manifolds X. Further, Auckly–Ruberman [1] announced that, for each $k > 0$, there are simply-connected 4-manifolds X such that $\pi_k(\mathrm{Diff}(X))$ have infinite-rank summands. They prove an analogous result also for $H_k(B\mathrm{TDiff}(X); \mathbb{Z})$.

Remark 1.11 (Non-simply-connected manifolds). For non-simply-connected manifolds of dim ≥ 4, it has been known that the mapping class group may be infinitely generated. For instance, Hatcher [18, Theorem 4.1] proved that the mapping class groups of the tori T^n for $n \geq 5$ are infinitely generated. In dimension 4, Budney–Gabai [7] and Watanabe [36] gave examples of non-simply-connected 4-manifolds whose mapping class groups are infinitely generated. Budney–Gabai [8] also proved that their infinitely generated subgroups of mapping class groups are nontrivial also in the topological category.

1.3. Scheme of the proof

Now we describe the idea of proofs of our results given in Subsection 1.1. We shall introduce an infinite family of characteristic classes

$$\mathbb{SW}^k_{\text{half-tot}}(X, \mathcal{S}) \in H^k(B\mathrm{Diff}^+(X); \mathbb{Z}/2) \tag{1}$$

using Seiberg–Witten theory for families. Here, $\mathcal{S}$ are $\mathrm{Diff}^+(X)$-invariant subsets of the set of spin^c structures $\mathrm{Spin}^c(X, k)$ on X with Seiberg–Witten formal dimension $-k$, divided by the charge conjugation (see Subsection 3.1 for the precise definition).

A characteristic class for families of 4-manifolds using Seiberg–Witten theory was introduced by the author [19], under the assumption that the monodromies of families preserve a given spin^c structure. Later, Lin and the author [20] defined a version without the assumption on monodromy. The classes (1) are refinements of the characteristic class defined in [20].

Using the characteristic classes (1), we can define a homomorphism

$$\bigoplus_{\mathcal{S}} \langle \mathbb{SW}^k_{\text{half-tot}}(X, \mathcal{S}), - \rangle : H_k(B\mathrm{Diff}^+(X); \mathbb{Z}) \to \bigoplus_{\mathcal{S}} \mathbb{Z}/2.$$

The above results follow by seeing that this homomorphism has infinitely generated image in $\bigoplus_{\mathcal{S}} \mathbb{Z}/2$ for some class of 4-manifolds X, including $X = E(n)\#kS^2 \times S^2$. More precisely, we shall see that $\langle \mathbb{SW}^k_{\text{half-tot}}(X, \mathcal{S}), - \rangle$ are nontrivial for infinitely many orbits $\mathcal{S}$ for the action of $\mathrm{Diff}^+(X)$ on $\mathrm{Spin}^c(X, k)$, which are distinguished by divisibilities of the first Chern classes.

1.4. Structure of the paper

The following is an outline of the sections of the paper. In Section 2, we construct infinitely many homology classes of $B\mathrm{Diff}(X)$ for some class of 4-manifolds X, which will be shown to be linearly independent over $\mathbb{Z}/2$. The most general statement is given as Theorem 2.7, which implies all results explained above. In Section 3, we construct characteristic classes (1) and compute them in Section 4 to prove Theorem 2.7.

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2. Construction of homology classes

In this section, we construct infinitely many homology classes of $B\mathrm{Diff}(X)$ for 4-manifolds X with certain conditions, which will be shown to be linearly independent over $\mathbb{Z}/2$.

2.1. Mod 2 basic classes in $H^2(M;\mathbb{Z})/\mathrm{Aut}(H^2(M;\mathbb{Z}))$

A building block of the construction of homology classes of $B\mathrm{Diff}^+(X)$ is a 4-manifold M that admits infinitely many exotic structures. This is inspired by Ruberman's argument [31] in his work on Torelli groups. (See also Auckly's recent work [2] for one version of Ruberman's argument in a Seiberg–Witten context.) Compared with [31, 2], what we newly need to require is that those exotic structures are distinguished by mod 2 basic classes that are distinct in $H^2(M;\mathbb{Z})/\mathrm{Aut}(H^2(M;\mathbb{Z}))$, the quotient of $H^2(M;\mathbb{Z})$ by the automorphism group of the intersection form. This is a reflection that we shall consider the whole diffeomorphism group, rather than the Torelli diffeomorphism group.

To formulate this precisely, let us introduce some notation. Let M be a smooth simply-connected closed oriented 4-manifold with $b^+(M) \geq 2$. Since $H^2(M;\mathbb{Z})$ has no torsion, we can identify a spin^c structure on M with a characteristic element in $H^2(M;\mathbb{Z})$. Recall that a characteristic element $c \in H^2(M;\mathbb{Z})$ is called a basic class if $SW(M,c)$, the Seiberg–Witten invariant, is nonzero. If $SW(M,c) \neq 0$ mod 2, we say that c is a *mod 2 basic class*. For simplicity, whenever we say that c is a (mod 2) basic class, we further impose that the formal dimension of c is zero (see (5)). We denote by $\mathcal{B}_2(M)$ the set of mod 2 basic classes of M. Note that $\mathcal{B}_2(M)$ is preserved under the $\mathbb{Z}/2$-action on $H^2(M;\mathbb{Z})$ via multiplication by -1. For a nonzero cohomology class $x \in H^2(M;\mathbb{Z})$, let $\mathrm{div}(x)$ denote the divisibility of x – namely,

$$\mathrm{div}(x) = \max\left\{n \in \mathbb{Z} \mid \exists y \in H^2(M;\mathbb{Z}) \text{ such that } ny = x\right\}.$$

For the zero element, we formally set $\mathrm{div}(0) = 0$ in this paper. For a characteristic element $c \in H^2(M;\mathbb{Z})$, define

$$N(M;c) = \#\{[x] \in \mathcal{B}_2(M)/(\mathbb{Z}/2) \mid \mathrm{div}(x) = \mathrm{div}(c),\ x^2 = c^2\},$$

where x^2 denotes the self-intersection of x. In this section, we consider a 4-manifold M to satisfy the following assumption:

Assumption 2.1. Let M be an indefinite smooth simply-connected closed oriented 4-manifold with $b^+(M) \geq 2$. Assume that there exist smooth 4-manifolds $\{M_i\}_{i=1}^{\infty}$ that satisfy the following three properties:

(i) Each M_i is homeomorphic to M.
(ii) For every i, $M_i \# S^2 \times S^2$ is diffeomorphic to $M \# S^2 \times S^2$.
(iii) For every i, there exists a mod 2 basic class c_i on M_i with $N(M_i;c_i)$ odd, and the sequence $\{c_i\}_{i=1}^{\infty}$ satisfies that $\mathrm{div}(c_i) \longrightarrow +\infty$ as $i \longrightarrow +\infty$.

It is worth adding notes on the last property (iii) of Assumption 2.1. The principal intention of (iii) is to ensure that the mod 2 basic classes are distinct even in the quotient $H^2(M;\mathbb{Z})/\mathrm{Aut}(H^2(M;\mathbb{Z}))$ (after passing to a subsequence, if necessary). For most 4-manifolds, increasing either divisibilities or self-intersections is the only possible way to get infinitely many characteristics distinct in $H^2(M;\mathbb{Z})/\mathrm{Aut}(H^2(M;\mathbb{Z}))$ (*cf.* Proposition 4.5). The reason why we suppose $N(M_i;c_i)$ is odd is that we want to control sums of mod 2 Seiberg–Witten invariants over some class of spin^c structures.

As a series of examples of M satisfying Assumption 2.1, we have the following:

Lemma 2.2. *For $n \geq 1$, $M = E(n)$ satisfies Assumption 2.1.*

To see Lemma 2.2, let us consider logarithmic transformations. For $n \geq 2$ and $i \geq 1$, let $E(n;i)$ denote the logarithmic transformation of order $i > 0$ performed on $E(n)$ (i.e., $E(n;i)$ is the elliptic

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surface of degree n with a single multiple fiber of order i). (Note that $E(n;1) = E(n)$.) The Seiberg–Witten invariants of $E(n;i)$ were computed by Fintushel–Stern [12]. For readers' convenience, we recall the result following their survey [13, Lecture 2].

In general, let Z be an oriented closed smooth 4-manifold with $b^+(Z) \geq 2$, without torsion in $H^2(Z;\mathbb{Z})$. Consider the Laurent polynomial

$$\mathcal{SW}_Z = \sum_c SW(Z,c)t_c.$$

Here, $c \in H^2(Z;\mathbb{Z})$ are characteristic elements and t_c are formal variables in $\mathbb{Z}[H^2(Z;\mathbb{Z})]$ corresponding to c. Note that $t_c t_{c'} = t_{c+c'}$ – in particular, $t_c^m = t_{mc}$ for $m \in \mathbb{Z}$.

Now consider $Z = E(n;i)$. Let $F \in H_2(E(n;i);\mathbb{Z})$ be the class that represents a generic fiber of the elliptic fibration. The multiple fiber of $E(n;i)$ represents a primitive homology class, which is given by F/i. Let F_i denote the Poincaré dual of F/i and set $t = t_{F_i}$. Then the Seiberg–Witten polynomial for $E(n;i)$ is given by

$$\mathcal{SW}_{E(n;i)} = (t^i - t^{-i})^{n-2}(t^{i-1} + t^{i-3} + \cdots + t^{1-i}). \tag{2}$$

Lemma 2.3. *The classes* $\pm(ni-i-1)F_i \in H^2(E(n;r);\mathbb{Z})$ *are mod 2 basic classes of* $E(n;i)$. *Further, we have* $\operatorname{div}((ni-i-1)F_i) = ni-i-1$, *and there is no mod 2 basic class of* $\operatorname{div} = ni-i-1$ *other than* $\pm(ni-i-1)F_i$.

Proof. Since the right-hand side of (2) is a polynomial only in t, all basic classes of $E(n;i)$ lie in the set $\{kF_i \in H^2(E(n;i);\mathbb{Z}) \mid k \in \mathbb{Z}\}$. Thus, for each $d \geq 1$, we have at most two basic classes of div $= d$, related by multiplication by -1 if exist. However, the top degree term of the right-hand side of (2) is given by $t^{(n-2)i}t^{i-1} = t_{(ni-i-1)F_i}$. Thus, $\pm(ni-i-1)F_i$ are mod 2 basic classes. The assertion of the lemma follows from this by recalling that F_i is a primitive class. □

Proof of Lemma 2.2. Set $M_i = E(n;i)$. Here, i runs over the natural numbers, but we restrict i to be odd if n is spin, so that the spinness of M_i is the same as that of M. Then M_i satisfies the properties (i) and (ii) of Assumption 2.1 by [15]. To check the property (iii), set $c_i = (ni-i-1)F_i$. Then it follows from Lemma 2.3 that $\operatorname{div}(c_i) \to +\infty$ as $i \to +\infty$, and $N(M_i;c_i) = 1$ for all $i \geq 1$. Hence, the property (iii) is satisfied. This completes the proof. □

2.2. Families over the torus

Fix $k > 0$, and let us take a 4-manifold M satisfying Assumption 2.1. Fix a diffeomorphism $M_i \# S^2 \times S^2 \to M \# S^2 \times S^2$ and identify $M_i \# kS^2 \times S^2$ with X for every i. Set $X = M \# kS^2 \times S^2$.

We recall a construction of a smooth fiber bundle over T^k with fiber X considered in [19, 20]. Define an orientation-preserving diffeomorphism $f_0 : S^2 \times S^2 \to S^2 \times S^2$ by $f_0(x,y) = (r(x), r(y))$, where $r : S^2 \to S^2$ is the reflection about the equator. By isotoping f_0, we can obtain a diffeomorphism $f : S^2 \times S^2 \to S^2 \times S^2$ that fixes a disk $D^4 \subset S^2 \times S^2$ pointwise. Take copies $f_1, \dots, f_k$ of f, and implant them into $M_i \# kS^2 \times S^2$ for each i, by extending by the identity. Thus, we obtain diffeomorphisms $f_1, \dots, f_k : M_i \# kS^2 \times S^2 \to M_i \# kS^2 \times S^2$. Since the supports of $f_1, \dots, f_k$ are mutually disjoint, and $f_1, \dots, f_k$ commute each other. Using these commuting diffeomorphisms, we can form the multiple mapping torus $E_i \to T^k$, which is a smooth fiber bundle with fiber $M_i \# kS^2 \times S^2$. Using the fixed identification between $M_i \# kS^2 \times S^2$ and X, we obtain smooth fiber bundles (denoted by the same notation) $X \to E_i \to T^k$ with fiber X. Since f is orientation-preserving, the resulting fiber bundles E_i are oriented (i.e., the structure group reduces to $\mathrm{Diff}^+(X)$, the orientation-preserving diffeomorphism group).

For each $i \geq 1$, regard E_i as a continuous map $E_i : T^k \to B\mathrm{Diff}^+(X)$. Now we set

$$\alpha_i := (E_1)_*([T^k]) - (E_i)_*([T^k]) \in H_k(B\mathrm{Diff}^+(X);\mathbb{Z}). \tag{3}$$

This construction of the homology class α_i is the same as the one in [20, Proof of Theorem 3.10], except only for a condition on the Seiberg–Witten invariants of 4-manifolds. The origin of this construction is the first examples of exotic diffeomorphisms of 4-manifolds due to Ruberman [30].

Let us observe a few properties of α_i:

Lemma 2.4. *The homology class α_i lies in*

$$\ker(i_* : H_k(B\mathrm{Diff}^+(X);\mathbb{Z}) \to H_k(B\mathrm{Homeo}^+(X);\mathbb{Z})).$$

Proof. Since M_1 and M_i are homeomorphic, E_1 and E_i are isomorphic as topological bundles. The assertion follows from this. □

For each i, fix a smoothly embedded 4-disk $D_i^4 \subset M_i$ and similarly take $D^4 \subset M$. Set $\mathring{M}_i = M_i \setminus \mathrm{Int}(D_i^4)$ and $\mathring{M} = M \setminus \mathrm{Int}(D^4)$. We can find a diffeomorphism $\psi_i : M_i \# S^2 \times S^2 \to M \# S^2 \times S^2$ with $\psi_i(D_i^4) = D^4$ that respect parametrizations of D_i^4 and D^4. Thus, we can identify $\mathring{M}_i \# S^2 \times S^2$ with $\mathring{M} \# S^2 \times S^2$. Using these identifications, the construction of E_i can be carried out with fixing the 4-disks, so we also have a homology class $\mathring{\alpha}_i \in H_k(B\mathrm{Diff}_\partial(\mathring{X});\mathbb{Z})$ defined similarly to α_i – namely,

$$\mathring{\alpha}_i := (E_1)_*([T^k]) - (E_i)_*([T^k]) \in H_k(B\mathrm{Diff}_\partial(\mathring{X});\mathbb{Z}),$$

where E_i are regarded as maps $E_i : T^k \to B\mathrm{Diff}_\partial(\mathring{X})$. Letting $\rho : \mathrm{Diff}_\partial(\mathring{X}) \to \mathrm{Diff}^+(X)$ be the extension map by the identity, we have that α_i is the image of $\mathring{\alpha}_i$ under the induced map

$$\rho_* : H_k(B\mathrm{Diff}_\partial(\mathring{X});\mathbb{Z}) \to H_k(B\mathrm{Diff}^+(X)\mathbb{Z}).$$

Lemma 2.5. *The homology class $\mathring{\alpha}_i$ lies in the kernel of*

$$s_* : H_k(B\mathrm{Diff}_\partial(\mathring{X});\mathbb{Z}) \to H_k(B\mathrm{Diff}_\partial(\mathring{X} \# S^2 \times S^2);\mathbb{Z}).$$

Proof. Let $E_S \to T^k$ denote the trivialized bundle with fiber $(S^2 \times S^2) \setminus \mathrm{Int}(D^4)$. Since $M_1 \# S^2 \times S^2$ and $M_i \# S^2 \times S^2$ are diffeomorphic, the stabilized bundles $E_1 \#_{\mathrm{fib}} E_S$ and $E_i \#_{\mathrm{fib}} E_S$ are smoothly isomorphic; here, $\#_{\mathrm{fib}}$ denotes the fiberwise connected sum along the trivial sphere bundle $T^k \times S^3 \to T^k$. This implies the assertion of the lemma. □

Lemma 2.6. *We have $2\mathring{\alpha}_i = 0$ and $2\alpha_i = 0$.*

Proof. It follows from [20, Lemma 3.6] that $(E_i)_*([T^k])$ is 2-torsion for every i. Thus, $\mathring{\alpha}_i$ is also 2-torsion. Since $\alpha_i = \rho_*(\mathring{\alpha}_i)$, we have α_i is 2-torsion as well. □

The following is the most general statement of this paper:

Theorem 2.7. *Let $k > 0$ and let M be a smooth simply-connected closed oriented 4-manifold that satisfies Assumption 2.1. Set $X = M \# kS^2 \times S^2$. Then we have the following:*

(i) *The set $\{\alpha_i \mid i \geq 2\}$ generates a direct summand of*

$$\ker(i_* : H_k(B\mathrm{Diff}^+(X);\mathbb{Z}) \to H_k(B\mathrm{Homeo}^+(X);\mathbb{Z}))$$

isomorphic to $(\mathbb{Z}/2)^\infty$.

(ii) *The set $\{\mathring{\alpha}_i \mid i \geq 2\}$ generates a direct summand of*

$$\ker(s_* : H_k(B\mathrm{Diff}_\partial(\mathring{X});\mathbb{Z}) \to H_k(B\mathrm{Diff}_\partial(\mathring{X} \# S^2 \times S^2);\mathbb{Z}))$$

isomorphic to $(\mathbb{Z}/2)^\infty$.

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To prove Theorem 2.7, due to Lemmas 2.4 to 2.6, it suffices to show that there is a homomorphism

$$H_k(B\mathrm{Diff}_\partial(\mathring{X});\mathbb{Z}) \to (\mathbb{Z}/2)^\infty$$

that restricts to a surjection

$$\langle \mathring{\alpha}_i \mid i \geq 2\rangle \twoheadrightarrow (\mathbb{Z}/2)^\infty.$$

We shall prove this remaining part in the subsequent sections.

Assuming Theorem 2.7, we obtain the proofs of results exhibited in Section 1:

Proofs of Theorems 1.4 and 1.7. These follow from Lemma 2.2 and Theorem 2.7. (Note that $\mathrm{Diff}^+(X) = \mathrm{Diff}(X)$ and $\mathrm{Homeo}^+(X) = \mathrm{Homeo}(X)$, since X in the assertions of Theorems 1.4 and 1.7 have nonzero signature.) □

3. Characteristic classes from Seiberg–Witten theory

3.1. Output

The proof of Theorem 2.7 uses characteristic classes defined by using Seiberg–Witten theory. Fix $k \geq 0$, and let X be a smooth closed oriented 4-manifold with $b^+(X) \geq k+2$. Lin and the author [20] defined a characteristic class

$$\mathbb{SW}^k_{\text{half-tot}}(X) \in H^k(B\mathrm{Diff}^+(X);\mathbb{Z}/2), \tag{4}$$

which we called the half-total Seiberg–Witten characteristic class. This was inspired by Ruberman's total Seiberg–Witten invariant of diffeomorphisms [32], together with a gauge-theoretic construction of characteristic classes by the author [19]. We introduce generalizations of the characteristic class (4) to prove Theorem 2.7 as follows.

Let $\mathrm{Spin}^c(X,k)$ denote the set of isomorphism classes of spin^c structures $\mathfrak{s}$ with $d(\mathfrak{s}) = -k$, where $d(\mathfrak{s})$ is the formal dimension of the Seiberg–Witten moduli space:

$$d(\mathfrak{s}) = \frac{1}{4}(c_1(\mathfrak{s})^2 - 2\chi(X) - 3\sigma(X)). \tag{5}$$

The group $\mathbb{Z}/2$ acts on $\mathrm{Spin}^c(X,k)$ by the charge conjugation, which flips the sign of the first Chern class of a spin^c structure. Let $\mathrm{Spin}^c(X,k)/\mathrm{Conj}$ denote the quotient of $\mathrm{Spin}^c(X,k)$ under this $\mathbb{Z}/2$-action. However, $\mathrm{Diff}^+(X)$ acts on $\mathrm{Spin}^c(X,k)$ via pull-back. Since the charge conjugation commutes with and the action of $\mathrm{Diff}^+(X)$ on $\mathrm{Spin}^c(X,k)$, we have an action of $\mathrm{Diff}^+(X)$ on $\mathrm{Spin}^c(X,k)/\mathrm{Conj}$.

Let $\mathcal{S}$ be a subset of $\mathrm{Spin}^c(X,k)/\mathrm{Conj}$ which is setwise preserved under the action of $\mathrm{Diff}^+(X)$. We suppose that $\mathcal{S}$ does not contain the coset of a self-conjugate spin^c structure, which is needed to ensure a families perturbation can be taken to be nonzero and transverse. We shall define a cohomology class

$$\mathbb{SW}^k_{\text{half-tot}}(X,\mathcal{S}) \in H^k(B\mathrm{Diff}^+(X);\mathbb{Z}/2) \tag{6}$$

by repeating the construction in [20] only for spin^c structures $\mathfrak{s}$ whose cosets $[\mathfrak{s}]$ under the $\mathbb{Z}/2$-action lie in $\mathcal{S}$.

Before explaning the construction of $\mathbb{SW}^k_{\text{half-tot}}(X,\mathcal{S})$, it is worth looking at the lowest degree case to see which spin^c structures involve: suppose $k = 0$, and take a section $\tau : \mathrm{Spin}^c(X,0)/\mathrm{Conj} \to \mathrm{Spin}^c(X,k)$ of the quotient map $\mathrm{Spin}^c(X,0) \to \mathrm{Spin}^c(X,0)/\mathrm{Conj}$. Then $\mathbb{SW}^0_{\text{half-tot}}(X,\mathcal{S})$ is given by

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$$\mathbb{SW}^0_{\text{half-tot}}(X, \mathcal{S}) = \sum_{\mathfrak{s} \in \tau(\mathcal{S})} SW(X, \mathfrak{s}) \in \mathbb{Z}/2.$$

Note that this number in $\mathbb{Z}/2$ is independent of τ, determined only by $\mathcal{S}$.

The characteristic class (6) is a generalization of the characteristic class (4) given in [20]: by setting $\mathcal{S} = \mathrm{Spin}^c(X, k)/\mathrm{Conj}$, we obtain (4) – namely,

$$\mathbb{SW}^k_{\text{half-tot}}(X, \mathrm{Spin}^c(X, k)/\mathrm{Conj}) = \mathbb{SW}^k_{\text{half-tot}}(X).$$

3.2. Construction of the characteristic classes

We explain the construction of $\mathbb{SW}^k_{\text{half-tot}}(X, \mathcal{S})$ below. We omit some details which are completely analogous to arguments in [20]: see [20, Section 2] for the full treatment. First, let us recall the basics of the Seiberg–Witten equations. To write down the (perturbed) Seiberg–Witten equations, we need to fix a spin^c structure $\mathfrak{s}$ on X, a Riemannian metric g on X and an imaginary-valed self-dual 2-form $\mu \in i\Omega^+_g(X)$ on X. Here, $\Omega^+_g(X)$ denotes the set of self-dual 2-forms for the metric g. The Seiberg–Witten equations perturbed by μ are of the form

$$\begin{cases} F^+_A = \sigma(\Phi, \Phi) + \mu, \\ D_A \Phi = 0. \end{cases} \tag{7}$$

Here, A is a $U(1)$-connection of the determinant line bundle for $\mathfrak{s}$, Φ is a positive spinor for $\mathfrak{s}$, $\sigma(-,-)$ is a certain quadratic form, and D_A is the spin^c Dirac operator associated with A. The Seiberg–Witten equations is $\mathrm{Map}(X, U(1))$-equivariant, and we define the moduli space of solutions to the Seiberg–Witten equations by

$$\mathcal{M}(X, \mathfrak{s}, g, \mu) = \{(A, \Phi) \mid (A, \Phi) \text{ satisfies } (7)\}/\mathrm{Map}(X, U(1)).$$

Next, let us recall the charge conjugation symmetry on the Seiberg–Witten equations. Let $\bar{\mathfrak{s}}$ denote the conjugate spin^c structure to $\mathfrak{s}$, which satisfies $c_1(\bar{\mathfrak{s}}) = -c_1(\mathfrak{s})$. Then there is a bijection

$$c : \mathcal{M}(X, \mathfrak{s}, g, \mu) \to \mathcal{M}(X, \bar{\mathfrak{s}}, g, -\mu) \tag{8}$$

called the charge conjugation symmetry, which becomes a diffeomorphism between the moduli spaces if the perturbation μ is generic so that $\mathcal{M}(X, \mathfrak{s}, g, \mu)$ is a smooth manifold (then so is $\mathcal{M}(X, \bar{\mathfrak{s}}, g, -\mu)$ automatically).

Let $\mathscr{R}(X)$ denote the space of Riemannian metrics. Set

$$\Pi(X) = \bigcup_{g \in \mathscr{R}(X)} i\Omega^+_g(X).$$

We think of $\Pi(X)$ as a vector bundle over the Frechet manifold $\mathscr{R}(X)$ and then take a fiberwise completion with respect to a suitable Sobolev norm. Let us use the same notation $\Pi(X) \to \mathscr{R}(X)$ also for the Hilbert bundle obtained by this completion. Let $\mathring{\Pi}(X)$ be the subset of $\Pi(X)$ consisting of perturbations μ such that:

- $\|\mu\| \leq 1$ for the Sobolev norm on $\Omega^+_g(X)$, and
- there is no reducible solution for μ.

The space $\mathring{\Pi}(X)$ is $(b^+(X) - 2)$ -connected, and $\mathring{\Pi}(X)$ is invariant under the fiberwise (-1)-multiplication on the Hilbert bundle $\Pi(X) \to \mathscr{R}(X)$.

What makes the construction of the half-total Seiberg–Witten characteristic class complicated is the fact that the charge conjugation acts on the space of perturbations nontrivially; the action is given as (fiberwise) multiplication by -1. Because of this, to implement a construction equivariantly under the charge conjugation, we need a 'multi-valued perturbation' when we form a collection of moduli spaces over a set of spin^c structures, not just a copy of a common families self-dual 2-form. This is formulated as follows.

Let $\varpi : \mathrm{Spin}^c(X,k) \to \mathrm{Spin}^c(X,k)/\mathrm{Conj}$ be the quotient map. Define $\tilde{\mathcal{S}} := \varpi^{-1}(\mathcal{S}) \subset \mathrm{Spin}^c(X,k)$. Since $\mathcal{S}$ is invariant under the $\mathrm{Diff}^+(X)$-action, $\tilde{\mathcal{S}}$ is also $\mathrm{Diff}^+(X)$-invariant. Define

$$\mathring{\Pi}(X,\mathcal{S})' := (\tilde{\mathcal{S}} \times \mathring{\Pi}(X))/(\mathbb{Z}/2),$$

where $\mathbb{Z}/2$ acts on $\tilde{\mathcal{S}}$ via the charge conjugation and on $\mathring{\Pi}(X)$ via the (fiberwise) (-1)-multiplication. (To make our notation consistent with that in [20], let us use the notation $\mathring{\Pi}(X,\mathcal{S})'$ with prime, not like $\mathring{\Pi}(X,\mathcal{S})$. This remark applies throughout this section.)

Now we consider a family of 4-manifolds. Let $X \to E \to B$ be a fiber bundle with structure group $\mathrm{Diff}^+(X)$ over a CW complex B. For $b \in B$, we denote by E_b the fiber of E over b. Associated with E, we have several natural fiber bundles. For instance, since $\mathrm{Diff}^+(X)$ acts on $\mathcal{S}$ via pull-back, we obtain an associated fiber bundle over B with fiber $\mathcal{S}$. We denote it by

$$\mathcal{S} \to \mathcal{S}(E) \to B.$$

Similarly, we get a fiber bundle with fiber $\mathring{\Pi}(X,\mathcal{S})'$, denoted by

$$\mathring{\Pi}(X,\mathcal{S})' \to \mathring{\Pi}(E,\mathcal{S})' \to B.$$

This has underlying families of spaces of metrics, denoted by

$$\mathscr{R}(X) \to \mathscr{R}(E) \to B.$$

A section of $\mathscr{R}(E) \to B$ is a fiberwise metric on E. Note that the forgetful map $\mathring{\Pi}(X,\mathcal{S})' \to \mathcal{S}$ induces a surjection

$$\mathring{\Pi}(E,\mathcal{S})' \to \mathcal{S}(E),$$

which commutes with the projections onto B.

It could be worth unpackaging the data $\mathring{\Pi}(E,\mathcal{S})'$. Let $\vec{\mu}$ be a point in $\mathring{\Pi}(E,\mathcal{S})'$. Let $b \in B$ and $g \in \mathscr{R}(E_b)$ be the images of $\vec{\mu}$ under the projections $\mathring{\Pi}(E,\mathcal{S})' \to B$ and $\mathring{\Pi}(E,\mathcal{S})' \to \mathscr{R}(E)$. Let $\mathcal{S}(E)_b$ be the fiber of $\mathcal{S}(E) \to B$ over b. Picking a representative $\mathfrak{s}$ of each coset $[\mathfrak{s}] \in \mathcal{S}(E)_b$, we can express $\vec{\mu}$ as a collection of a self-dual 2-forms $\{\mu_{\mathfrak{s}} \in \Omega^+_g(E_b)\}_{[\mathfrak{s}]\in\mathcal{S}(E)_b}$. We set

$$\mathcal{M}(E_b,\mathcal{S},\vec{\mu}) = \bigsqcup_{[\mathfrak{s}]\in\mathcal{S}(E)_b} \mathcal{M}(E_b,\mathfrak{s},g,\mu_{\mathfrak{s}}).$$

If all $\mu_{[\mathfrak{s}]}$ are generic, $\mathcal{M}(E_b,\mathcal{S},\vec{\mu})$ is a smooth manifold. Further, as an unoriented manifold, $\mathcal{M}(E_b,\mathcal{S},\vec{\mu})$ is independent of choice of representatives $\mathfrak{s}$ of $[\mathfrak{s}]$. Indeed, if we choose the other representative $\bar{\mathfrak{s}}$ of $[\mathfrak{s}]$, the chosen perturbation becomes $\mu_{\bar{\mathfrak{s}}} = -\mu_{\mathfrak{s}}$, so we can use the diffeomorphism (8).

Now let us take a fiberwise metric $\tilde{g} : B \to \mathscr{R}(E)$, and pick a section $\sigma' : \mathcal{S} \to \mathring{\Pi}(E,\mathcal{S})'$ that makes the following diagram commutative:

$$\begin{array}{ccc} \mathcal{S}(E) & \xrightarrow{\sigma'} & \mathring{\Pi}(E,\mathcal{S})' \\ \downarrow & & \downarrow \\ B & \xrightarrow{\tilde{g}} & \mathscr{R}(E). \end{array}$$

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Define the *half-total moduli space for σ'* by

$$\mathcal{M}_{\mathrm{half}}(E,\mathcal{S},\sigma') = \bigcup_{b\in B} \mathcal{M}(E_b,\mathcal{S},\sigma'(b)).$$

(This was denoted by $\mathcal{M}_{\sigma',\mathrm{half}}$ in [20, Definition 2.11], but let us use the notation $\mathcal{M}_{\mathrm{half}}(E,\mathcal{S},\sigma')$ to keep track of E and $\mathcal{S}$.) If B is a comapct manifold, by choosing generic σ', $\mathcal{M}_{\mathrm{half}}(E,\mathcal{S},\sigma')$ becomes a compact manifold too (*cf.* Lemma 3.1), and the dimension of $\mathcal{M}_{\mathrm{half}}(E,\mathcal{S},\sigma')$ is given by $\dim B - k$. In particular, for $\dim B = k$, we can define a $\mathbb{Z}/2$-valued invariant by counting the zero dimensional compact manifold $\mathcal{M}_{\mathrm{half}}(E,\mathcal{S},\sigma')$.

For a general case where B is neither compact nor a manifold, we define a cochain

$$\mathcal{SW}^k_{\text{half-tot}}(E,\mathcal{S},\sigma') \in C^k(B)$$

as follows, where $C^k(B)$ denotes the $\mathbb{Z}/2$-coefficient cellular cochain group. Loosely speaking, for each k-cell e of B with a characteristic map $\varphi_e : D^k \to B$, we define

$$\mathcal{SW}^k_{\text{half-tot}}(E,\mathcal{S},\sigma')(e) = \#\mathcal{M}_{\mathrm{half}}(\varphi_e^* E,\mathcal{S},\varphi_e^*\sigma') \in \mathbb{Z}/2.$$

Here, the right-hand side is a finite sum (*cf.* Lemma 3.1), and we can justify the necessary transversality by using a virtual neighborhood technique, just as in [19, 20]. Using the assumption that $b^+(X) \geq k+2$, we can prove that $\mathcal{SW}^k_{\text{half-tot}}(E,\mathcal{S},\sigma')$ is a cocycle, and that the cohomology class

$$\mathbb{SW}^k_{\text{half-tot}}(E,\mathcal{S}) := [\mathcal{SW}^k_{\text{half-tot}}(E,\mathcal{S},\sigma')] \in H^k(B;\mathbb{Z}/2)$$

is independent of the choice of σ' ([20, Propositions 2.22, 2.23]). We set

$$\mathbb{SW}^k_{\text{half-tot}}(X,\mathcal{S}) := \mathbb{SW}^k_{\text{half-tot}}(E\mathrm{Diff}^+(X),\mathcal{S}) \in H^k(B\mathrm{Diff}^+(X);\mathbb{Z}/2).$$

3.3. Finiteness

Here, we record some finiteness result, which was used in Subsection 3.2 and is necessary in a subsequent argument too.

First, let us recall the following well-known finiteness of Seiberg–Witten moduli spaces (see, for example, [27]). Fix a metric g and $k \in \mathbb{Z}$. Then there are only finitely many spin^c structures $\mathfrak{s}$ with $d(\mathfrak{s}) = k$ for which the moduli space $\mathcal{M}(X,\mathfrak{s},g,\mu)$ for the perturbed equations (7) are nonempty for some $\mu \in \Omega^+_g(X)$ with $\|\mu\| \leq 1$. Here, $\|-\|$ denotes a suitable Sobolev norm. Moreover, for a fixed pair (g,μ), the moduli space $\mathcal{M}(X,\mathfrak{s},g,\mu)$ is compact. A families generalization of this fact in our context is as follows.

As in Subsection 3.2, fix $k \geq 0$, let X be a smooth closed oriented 4-manifold with $b^+(X) \geq k+2$, and let $X \to E \to B$ be a fiber bundle with structure group $\mathrm{Diff}^+(X)$ over a CW complex B.

Lemma 3.1. *Suppose that B is compact. If we pick a section σ' as in Subsection 3.2, then the half-total moduli space*

$$\mathcal{M}_{\mathrm{half}}(E,\mathrm{Spin}^c(X,k)/\mathrm{Conj},\sigma')$$

is compact.

Proof. This follows from that we used perturbations with $\|\mu\| \leq 1$ in the definition of $\mathring{\Pi}(X)$. □

For our purpose, an important case is that $\mathcal{S}$ is an orbit of the action of $\mathrm{Diff}^+(X)$ on $\mathrm{Spin}^c(X,k)/\mathrm{Conj}$. Set

$$\mathrm{Spin}^c(X,k)^{\vee} := \{\mathfrak{s} \in \mathrm{Spin}^c(X,k) \mid \mathfrak{s} \not\cong \bar{\mathfrak{s}}\}$$

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and let $\mathbb{S}(X,k)$ denote the orbit space for the $\mathrm{Diff}^+(X)$-action on $\mathrm{Spin}^c(X,k)^\vee/\mathrm{Conj}$,

$$\mathbb{S}(X,k) = (\mathrm{Spin}^c(X,k)^\vee/\mathrm{Conj})/\mathrm{Diff}^+(X).$$

As an analog of the notion of a basic class, we call $\mathcal{S} \in \mathbb{S}(X,k)$ a *basic orbit of E* if $\mathbb{SW}^k_{\text{half-tot}}(E,\mathcal{S}) \neq 0$. Let $\mathcal{B}_{\text{half}}(E,k)$ denote the set of basic orbits:

$$\mathcal{B}_{\text{half}}(E,k) = \{\mathcal{S} \in \mathbb{S}(X,k) \mid \mathbb{SW}^k_{\text{half-tot}}(E,\mathcal{S}) \neq 0\}.$$

Then we have the following:

Lemma 3.2. *Suppose that B is compact. Then $\mathcal{B}_{\text{half}}(E,k)$ is a finite set.*

Proof. Fix a section σ'. Lemma 3.1 implies that there are only finitely many $\mathfrak{s} \in \mathrm{Spin}^c(X,k)$ such that there is $b \in B$ with $\mathcal{M}(E_b, \mathfrak{s}, g_b, \sigma'(b)) \neq \emptyset$, where g_b is the underlying metric of $\sigma'(b)$ on E_b. Since $\#\mathcal{B}_{\text{half}}(E,k)$ is bounded above by the number of such $\mathfrak{s}$, the assertion follows. □

4. Computing the invariant

In this section, we prove Theorem 2.7 by evaluating the Seiberg–Witten characteristic classes $\mathbb{SW}^k_{\text{half-tot}}(X,\mathcal{S})$ introduced in Section 3 at homology classes α_i defined in (3).

Precisely, we shall consider the homomorphism

$$\bigoplus_{\mathcal{S}\in\mathbb{S}(X,k)} \langle \mathbb{SW}^k_{\text{half-tot}}(X,\mathcal{S}), -\rangle : H_k(B\mathrm{Diff}^+(X);\mathbb{Z}) \to \bigoplus_{\mathcal{S}\in\mathbb{S}(X,k)} \mathbb{Z}/2.$$

We shall show that this homomorphism has infinitely generated image in $\bigoplus_{\mathcal{S}\in\mathbb{S}(X,k)} \mathbb{Z}/2$ for 4-manifolds X considered in Theorem 2.7.

4.1. Reducing to the monodromy invariant part

The characteristic class $\mathbb{SW}^k_{\text{half-tot}}(X,\mathcal{S})$ involves spin^c structures that are not invariant under the monodromies of the families that we consider. Adapting an argument in [20, Section 3.1] to our setup, we shall see that such spin^c structures do not contribute to the final computation.

To describe it, let us recall the numerical families Seiberg–Witten invariant. Let B be a closed smooth manifold of dimension $k \geq 0$, X be a smooth oriented closed 4-manifold of $b^+(X) \geq k+2$, and $X \to E \to B$ be a fiber bundle with structure group $\mathrm{Diff}^+(X)$ over B. Given a spin^c structure $\mathfrak{s}$ on X of formal dimension $-k$, suppose that the monodromy of E fixes the isomorphism class of $\mathfrak{s}$. Then the numerical families Seiberg–Witten invariant

$$SW(E,\mathfrak{s}) \in \mathbb{Z}/2$$

can be defined. If the structure of E lifts to the automorphism group of the spin^c 4-manifold $(X,\mathfrak{s})$, this is the invariant defined by Li–Liu [23]. However, even if E does not admit such a lift, one can still define $SW(E,\mathfrak{s})$ [19, 5].

Pick an orbit $\mathcal{S} \in \mathbb{S}(X,k)$. We regard $\mathcal{S}$ also as a subset of $\mathrm{Spin}^c(X,k)/\mathrm{Conj}$. Let τ : $\mathrm{Spin}^c(X,k)/\mathrm{Conj} \to \mathrm{Spin}^c(X,k)$ be a section of the quotient map $\mathrm{Spin}^c(X,k) \to \mathrm{Spin}^c(X,k)/\mathrm{Conj}$. For mutually commuting diffeomorphisms $f_1,\dots,f_k$ of X, we denote by $X_{f_1,\dots,f_k} \to T^k$ the multiple mapping torus of $f_1,\dots,f_k$.

Proposition 4.1 (*cf.* [20, Corollary 3.4]). *Let $f_1,\dots,f_k : X \to X$ be mutually commuting orientation-preserving diffeomorphisms. Suppose that they satisfy the following conditions:*

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(i) *For each* $i = 1, \dots, k$, f_i *preserves* $\tau(\mathcal{S})$ *setwise.*
(ii) *For each* i, *there exists a smooth isotopy* $(F_i^t)_{t\in[0,1]}$ *from* f_i^2 *to* id_X. *For* $i \neq j$, F_i^t *commutes with* f_j *for any* $t \in [0, 1]$.

Then we have

$$\langle \mathbb{SW}^k_{\text{half-tot}}(X_{f_1,\dots,f_k}, \mathcal{S}), [T^k]\rangle = \sum_{\substack{\mathfrak{s}\in\tau(\mathcal{S}),\\ {f_i}^*\mathfrak{s}=\mathfrak{s}\ (1\le i\le k)}} SW(X_{f_1,\dots,f_k}, \mathfrak{s})$$

in $\mathbb{Z}/2$.

Proof. The proof is obtained by repeating the proof of [20, Corollary 3.4] with replacing $\mathrm{Spin}^c(X, k)/\mathrm{Conj}$ with $\mathcal{S}$. We just give a slight comment on how to do the modification.

If the actions of all f_i on $\tau(\mathcal{S})$ are trivial, there is nothing to prove. To treat the other case, first note that we have a modification of [20, Lemma 3.3] obtained by replacing $\mathrm{Spin}^c(X, k)/\mathrm{Conj}$ with $\mathcal{S}$. Let us consider a $(\mathbb{Z}/2)^k$-action on $\tau(\mathcal{S})$ generated by $f_1, \dots, f_k$. For $\mathfrak{s} \in \tau(\mathcal{S})$, if there is i with $f_i^*\mathfrak{s} \neq \mathfrak{s}$, we may use the modified [20, Lemma 3.3] to conclude that the sum of the counts of the moduli spaces for the $(\mathbb{Z}/2)^k$-orbit of $\mathfrak{s}$ is zero over $\mathbb{Z}/2$. Thus, in any case, $\langle \mathbb{SW}^k_{\text{half-tot}}(X_{f_1,\dots,f_k}, \mathcal{S}), [T^k]\rangle$ is computed from the counts of the moduli spaces only for the monodromy invariant spin^c structures, and it ends up with the assertion of Proposition 4.1. □

4.2. Gluing result

Another thing we need is a gluing result proven by Baraglia and the author [5]. We recall the statement for readers' convenience. In general, let Z be an oriented smooth closed 4-manifold, and $Z \to E \to B$ be an oriented smooth fiber bundle with fiber Z. Then we get an associated vector bundle

$$\mathbb{R}^{b^+(Z)} \to H^+(E) \to B$$

by considering maximal-dimensional positive-definite subspaces of the second cohomology fiberwise. The isomorphism class of $H^+(E)$ is determined only by E.

The gluing result we need is formulated as follows. Let $k > 0$, and let M, N be closed oriented smooth 4-manifolds with $b^+(M) \geq 2$ and $b^+(N) = k$, and with $b_1(M) = b_1(N) = 0$. Set $X = M\#N$. Let $\mathfrak{t} \in \mathrm{Spin}^c(M, 0)$ and $\mathfrak{t}' \in \mathrm{Spin}^c(N, k + 1)$. Then we have $d(\mathfrak{t}\#\mathfrak{t}') = -k$. Let B be a closed smooth manifold of dimension k, and $M \to E_M \to B$ and $N \to E_N \to B$ be oriented smooth fiber bundles. Fix sections $\iota_M : B \to E_M$, $\iota_N : B \to E_N$ whose normal bundles are isomorphic via a fiberwise orientation-reversing isomorphism, so that we can form the fiberwise connected sum $X \to E_X \to B$ of E_M and E_N along ι_M, ι_N. Then we have the following:

Theorem 4.2 [5, Theorem 1.1]. *If* $w_{b^+(N)}(H^+(E_N)) \neq 0$, *then we have*

$$SW(E_X, \mathfrak{t}\#\mathfrak{t}') = SW(M, \mathfrak{t})$$

in $\mathbb{Z}/2$.

Now we apply Theorem 4.2 to the multiple mapping torus $E_i \to T^k$ constructed in Subsection 2.2 for $i \geq 1$. For each $j = 1, \dots, k$, recall that f_j acts on the j-th copy of $H^+(S^2 \times S^2) \subset H^2(S^2 \times S^2)$ via multiplication by -1. We can see that the vector bundle

$$H^+((kS^2 \times S^2)_{f_1,\dots,f_k}) \to T^k$$

associated to the multiple mapping torus $(kS^2 \times S^2)_{f_1,\dots,f_k} \to T^k$ satisfies

$$w_k(H^+((kS^2 \times S^2)_{f_1,\dots,f_k})) \neq 0. \tag{9}$$

Let $\mathfrak{s}_S$ denote the unique spin structure on $kS^2 \times S^2$. Then we have $\mathfrak{s}_S \in \mathrm{Spin}^c(kS^2 \times S^2, k + 1)$.

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Lemma 4.3. *Let* $\mathfrak{t} \in \mathrm{Spin}^c(M_i, 0)$. *Then we have*

$$SW(E_i, \mathfrak{t}\#\mathfrak{s}_S) = SW(M_i, \mathfrak{t})$$

in $\mathbb{Z}/2$.

Proof. This follows from (9) and Theorem 4.2. □

4.3. Completion of the proof

As in Section 2, fix $k > 0$, take a 4-manifold M satisfying Assumption 2.1. We shall use M_i and c_i that appear in Assumption 2.1, and we shall use the notation E_i and α_i for $i \geq 1$ introduced in Subsection 2.2. Set $X = M\#kS^2 \times S^2$ and $X_i = M_i\#kS^2 \times S^2$.

For each $i \geq 1$, we fix a section

$$\tau_i^0 : \mathrm{Spin}^c(M_i)/\mathrm{Conj} \to \mathrm{Spin}^c(M_i)$$

of the quotient map $\mathrm{Spin}^c(M_i) \to \mathrm{Spin}^c(M_i)/\mathrm{Conj}$. Using τ_i^0, we define a section

$$\tau_i : \mathrm{Spin}^c(X_i)/\mathrm{Conj} \to \mathrm{Spin}^c(X_i)$$

as follows: for $\mathfrak{s} \in \mathrm{Spin}^c(X_i)$, we define $\tau([\mathfrak{s}])$ to be the spin^c structure $\mathfrak{s}'$ with $[\mathfrak{s}] = [\mathfrak{s}']$ in $\mathrm{Spin}^c(X_i)/\mathrm{Conj}$ such that $\mathfrak{s}'|_{M_i} = \tau_0([\mathfrak{s}|_{M_i}])$. Restricting this, we obtain a section (denoted by the same notation)

$$\tau_i : \mathrm{Spin}^c(X_i, k)/\mathrm{Conj} \to \mathrm{Spin}^c(X_i, k).$$

As in Subsection 4.2, let $\mathfrak{s}_S$ denote the unique spin structure on $kS^2 \times S^2$. For each $i \geq 1$, we define $\mathcal{S}_i \in \mathbb{S}(X_i, k)$ to be the $\mathrm{Diff}^+(X_i)$-orbit that contains $[c_i\#\mathfrak{s}_S] \in \mathrm{Spin}^c(X_i, k)/\mathrm{Conj}$.

Proposition 4.4. *For* $E_i \to T^k$ *constructed in Subsection 2.2, we have*

$$\langle \mathbb{SW}^k_{\text{half-tot}}(X, \mathcal{S}_i), (E_i)_*([T^k])\rangle \neq 0$$

in $\mathbb{Z}/2$.

Proof. First, the naturality of the characteristic class implies that

$$\langle \mathbb{SW}^k_{\text{half-tot}}(X, \mathcal{S}_i), (E_i)_*([T^k])\rangle = \langle \mathbb{SW}^k_{\text{half-tot}}(E_i, \mathcal{S}_i), [T^k]\rangle. \tag{10}$$

To compute the right-hand side of (10), we shall apply Proposition 4.1 to the families E_i. Recall that E_i was constructed by using a diffeomorphism $f \in \mathrm{Diff}_\partial(S^2 \times S^2 \setminus \mathrm{Int}(D^4))$. This diffeomorphism f is order 2 in $\pi_0(\mathrm{Diff}_\partial(S^2 \times S^2 \setminus \mathrm{Int}(D^4)))$. Thus, for the diffeomorphisms $f_1, \ldots, f_k$ on X_i, of which the multiples mapping torus is E_i, we can find isotopies $(F_i^t)_{t\in[0,1]}$ that satisfy the assumption (ii) of Proposition 4.1. Since f_j act trivially on M_i, by the construction of τ_i, it follows that $\tau_i(\mathcal{S}_i)$ is setwise preserved under the actions of f_j. Thus, we may apply Proposition 4.1 to the families E_i and obtain the equality

$$\langle \mathbb{SW}^k_{\text{half-tot}}(E_i, \mathcal{S}_i), [T^k]\rangle = \sum_{\substack{\mathfrak{s}\in\tau_i(\mathcal{S}_i),\\ f_j^*\mathfrak{s}=\mathfrak{s}\ (1\leq j\leq k)}} SW(E_i, \mathfrak{s}) \tag{11}$$

in $\mathbb{Z}/2$.

We shall compute the right-hand side of (11). Since f_j acts on the j-th copy of $H^2(S^2 \times S^2)$ via multiplication by -1, a spin^c structure $\mathfrak{s} \in \mathrm{Spin}^c(X_i)$ is f_j-invariant for all j if and only if $\mathfrak{s}$ is of the form $\mathfrak{t}\#\mathfrak{s}_S$, where $\mathfrak{t} \in \mathrm{Spin}^c(M_i)$. It is easy to see that, if $d(\mathfrak{t}\#\mathfrak{s}_S) = -k$, then $d(\mathfrak{t}) = 0$. Thus, we get from Lemma 4.3 that

$$\sum_{\substack{\mathfrak{s}\in\tau_i(\mathcal{S}_i),\\ f_j^*\mathfrak{s}=\mathfrak{s}\ (1\leq j\leq k)}} SW(E_i,\mathfrak{s}) = \sum_{\substack{\mathfrak{t}\#\mathfrak{s}_S\in\tau_i(\mathcal{S}_i),\\ \mathfrak{t}\in\mathrm{Spin}^c(M_i,0)}} SW(M_i,\mathfrak{t}) \tag{12}$$

in $\mathbb{Z}/2$.

To compute the right-hand side of (12), let $\mathfrak{t} \in \mathrm{Spin}^c(M_i, 0)$ be a spin^c structure on M_i. We claim that $\mathfrak{t}\#\mathfrak{s}_S$ lies in $\tau_i(\mathcal{S}_i)$ if and only if all of the following three conditions (i)–(iii) are satisfied: (i) $\mathrm{div}(c_1(\mathfrak{t})) = \mathrm{div}(c_i)$, (ii) $c_1(\mathfrak{t})^2 = c_i^2$, and (iii) $\mathfrak{t} \in \tau_i^0(\mathrm{Spin}^c(M_i, 0)/\mathrm{Conj})$. Noting $c_1(\mathfrak{t}) = c_1(\mathfrak{t}\#\mathfrak{s}_S)$ in $H^2(X_i;\mathbb{Z})$, this claim is a direct consequence of Proposition 4.5, which we shall see later.

By the claim of the last paragraph, we have

$$\sum_{\substack{\mathfrak{t}\#\mathfrak{s}_S\in\tau_i(\mathcal{S}_i),\\ \mathfrak{t}\in\mathrm{Spin}^c(M_i,0)}} SW(M_i,\mathfrak{t}) = N(M_i;c_i) \tag{13}$$

in $\mathbb{Z}/2$. Here the right-hand side of (13) was assumed to be nonzero in $\mathbb{Z}/2$ in Assumption 2.1. Thus, the assertion of the proposition follows from (10), (11), (12), (13). □

Here we record a proposition that we have used above:

Proposition 4.5 (Wall [34, 35]). *Let Z be a smooth closed oriented simply-connected 4-manifold. Suppose that $b_2(Z) - \sigma(Z) \geq 2$ and that Z is either indefinite or $b_2(Z) \leq 8$. Set $Z' = Z\#S^2 \times S^2$. Then, given $x, y \in H^2(Z';\mathbb{Z})$, there exists $f \in \mathrm{Diff}^+(Z')$ with $f^*x = y$ if and only if x, y have the same divisibility, self-intersection and type (i.e., characteristic or not).*

Proof. For a unimodular lattice Q with $\mathrm{rank}(Q) - \sigma(Q) \geq 4$, Wall [34, page 337] proved that $\mathrm{Aut}(Q)$ acts transitively on elements of given divisibility, self-intersection and type. However, each of divisibility, self-intersection and type is invariant under the action of $\mathrm{Aut}(Q)$. Thus, orbits in $Q/\mathrm{Aut}(Q)$ one-to-one correspond to triples consisting of divisibility, self-intersection and type. The assertion of the proposition follows from this applied to the intersection form of Z', together with another theorem by Wall [35, Theorem 2] on the realizability of an automorphism of the intersection form by a diffeomorphism. □

Now we can complete the proof of the most general result in this paper:

Proof of Theorem 2.7. As in the construction of E_i, we fix diffeomorphisms $\psi_i : \mathring{M}_i\#kS^2 \times S^2 \to \mathring{X}$ and its extensions $\psi_i : M_i\#kS^2 \times S^2 \to X$. Considering the pull-back of the orbits $\mathcal{S}_i \in \mathbb{S}(X_i, k)$ under ψ_i, we obtain orbits (denoted by the same notation) $\mathcal{S}_i \in \mathbb{S}(X, k)$.

Passing to a subsequence if necessary, we may suppose that all $\mathrm{div}(c_i)$ are distinct by (iii) of Assumption 2.1. Thus, we may suppose that all $\mathcal{S}_i$ are distinct elements in $\mathbb{S}(X, k)$. From this together with Lemma 3.2, by passing to a subsequence again, we may suppose that

$$\mathcal{S}_i \notin \mathcal{B}_{\mathrm{half}}(E_1, k) \cup \cdots \cup \mathcal{B}_{\mathrm{half}}(E_{i-1}, k) \tag{14}$$

for all $i \geq 2$.

Now it follows from Proposition 4.4 together with (3), (14) that the homomorphism

$$\bigoplus_{i\geq 2} \langle \mathbb{SW}^k_{\text{half-tot}}(X, \mathcal{S}_i), -\rangle : H_k(B\mathrm{Diff}^+(X);\mathbb{Z}) \to \bigoplus_{i\geq 2} \mathbb{Z}/2$$

restricts to a surjection

$$\langle \mathring{\alpha}_i \mid i \geq 2\rangle \twoheadrightarrow \bigoplus_{i\geq 2} \mathbb{Z}/2.$$

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This combined with Lemma 2.6 implies that the subgroup $\langle \mathring{\alpha}_i \mid i \geq 2 \rangle$ is a $(\mathbb{Z}/2)^\infty$-summand of $H_k(B\mathrm{Diff}^+(X);\mathbb{Z})$, which together with Lemma 2.4 completes the proof of (i) of Theorem 2.7.

Since $\rho_*(\mathring{\alpha}_i) = \alpha_i$, we obtain (ii) of Theorem 2.7 from (i) of Theorem 2.7 together with Lemma 2.5. □

5. Addenda

5.1. Finiteness of mapping class groups in dimension ≠ 4

In dimension ≠ 4, not only finite generation, but stronger finiteness on mapping class groups is known.

5.1.1. dimension ≥ 6

Given a simply-connected closed smooth manifold X of $\dim X \geq 6$, Sullivan [33, Theorem (13.3)] proved that $\pi_0(\mathrm{Diff}(X))$ is 'commensurable' with an arithmetic group. Krannich and Randal-Williams [21] clarified that the term 'commensurable' is used in [33] in a different way from the current common usage. In summary, given a group, we have implications:

(commensurable with an arithmetic group in the current common sense)
⇒(commensurable with an arithmetic group in the sense of [33])
⇒(finitely presented) ⇒ (finitely generated).

In particular, Theorem 1.1 implies that mapping class groups of simply-connected 4-manifolds need not be commensurable with arithmetic groups, even in Sullivan's sense.

5.1.2. dimension 5

While the above result by Sullivan [33, Theorem (13.3)] was stated in dim ≥ 6, actually his result holds also in dimension 5. We record a way to deduce this from a recent paper [9]. (The author thanks Sander Kupers for informing the author of this argument.) In the proof of [9, Theorem 2.6], the assumption that dim ≥ 6 was used only in the point (i) in the proof, but it follows from Cerf's theorem [10] that $\pi_0(C^{\mathrm{Diff}}(M)) = 0$ for a simply-connected 5-manifold M, and the assumption that dim ≥ 6 was not used in [9, Proposition 2.7], except for the part where [9, Proposition 2.6] was used.

5.1.3. dimension ≤ 3

The mapping class groups of closed orientable manifolds of dim ≤ 3 are finitely presented. See Dehn [17] for dimension 2. In dimension 3, a more general finiteness holds for the moduli space of 3-manifolds. See Boyd–Bregman–Steinebrunner [6, Theorem 6.12].

5.2. Questions: finiteness in other categories

We close this paper by posting questions on categories other than the smooth category.

As noted in Remark 1.3, for a simply-connected closed topological 4-manifold X, the topological mapping class group $\pi_0(\mathrm{Homeo}(X))$ is known to be finitely generated, and so is $H_1(B\mathrm{Homeo}(X);\mathbb{Z})$.

However, to the best of the author's knowledge, there is no known finiteness result on $H_k(B\mathrm{Homeo}(X))$ for $k > 1$ for general simply-connected 4-manifolds X. However, it may be natural to hope such finiteness results in the 4-dimensional topological category, as opposed to the smooth category:

Question 5.1. Let X be a simply-connected closed oriented topological 4-manifold. Is $H_k(B\mathrm{Homeo}(X);\mathbb{Z})$ finitely generated for each k?

Recently, Lin and Xie [25] extensively studied the *moduli space* $\mathcal{M}^{fs}(X)$ *of formally smooth 4-manifolds*, which is a middle moduli space between the smooth moduli space $\mathcal{M}^s(X) = B\mathrm{Diff}(X)$

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and the topological moduli space $\mathcal{M}^t(X) = B\mathrm{Homeo}(X)$. Lin and Xie pointed out that most exotic phenomena detected by gauge theory are relevant to the discrepancy between $\mathcal{M}^s(X)$ and $\mathcal{M}^{fs}(X)$. Since infiniteness of $\mathcal{M}^s(X)$ detected in this paper comes from gauge theory, it may be natural to expect finiteness of $\mathcal{M}^{fs}(X)$:

Question 5.2. Let X be a simply-connected closed oriented topological 4-manifold that admits a formally smooth structure. Is $H_k(\mathcal{M}^{fs}(X); \mathbb{Z})$ finitely generated for each k?

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References

[1] D. Auckly and D. Ruberman, 'Families of diffeomorphisms, embeddings, and positive scalar curvature metrics via Seiberg-Witten theory', in preparation.
[2] D. Auckly, 'Smoothly knotted surfaces that remain distinct after many internal stabilizations', Preprint, 2023, arXiv:2307.16266.
[3] D. Baraglia, 'Non-trivial smooth families of $K3$ surfaces', *Math. Ann.* **387**(3–4) (2023), 1719–1744.
[4] D. Baraglia, 'On the mapping class groups of simply-connected smooth 4-manifolds', Preprint, arXiv:2310.18819.
[5] D. Baraglia and H. Konno, 'A gluing formula for families Seiberg-Witten invariants', *Geom. Topol.* **24**(3) (2020), 1381–1456.
[6] R. Boyd, C. Bregman and J. Steinebrunner, 'Moduli spaces of 3-manifolds with boundary are finite', Preprint, 2024, arXiv:2404.12748.
[7] R. Budney and D. Gabai, 'Knotted 3-balls in s^4', Preprint, 2021, arXiv:1912.09029.
[8] R. Budney and D. Gabai, 'On the automorphism groups of hyperbolic manifolds', Preprint, 2023, arXiv:2303.05010 .
[9] M. Bustamante, M. Krannich and A. Kupers, 'Finiteness properties of automorphism spaces of manifolds with finite fundamental group', *Math. Ann.* **388**(4) (2024), 3321–3371.
[10] J. Cerf, 'La stratification naturelle des espaces de fonctions différentiables réelles et le théorème de la pseudo-isotopie', *Inst. Hautes Études Sci. Publ. Math.* **39** (1970), 5–173.
[11] J. Ebert and O. Randal-Williams, 'Generalised Miller-Morita-Mumford classes for block bundles and topological bundles', *Algebr. Geom. Topol.* **14**(2) (2014), 1181–1204.
[12] R. Fintushel and R. J. Stern, 'Rational blowdowns of smooth 4-manifolds', *J. Differential Geom.* **46**(2) (1997), 181–235.
[13] R. Fintushel and R. J. Stern, 'Six lectures on four 4-manifolds', in *Low Dimensional Topology* (IAS/Park City Math. Ser.) vol. 15 (Amer. Math. Soc., Providence, RI, 2009), 265–315.
[14] S. Galatius and O. Randal-Williams, 'Homological stability for moduli spaces of high dimensional manifolds. I', *J. Amer. Math. Soc.* **31**(1) (2018), 215–264.
[15] R. E. Gompf, 'Sums of elliptic surfaces', *J. Differential Geom.* **34**(1) (1991), 93–114.
[16] J. L. Harer, 'Stability of the homology of the mapping class groups of orientable surfaces', *Ann. of Math. (2)* **121**(2) (1985), 215–249.
[17] A. Hatcher and W. Thurston, 'A presentation for the mapping class group of a closed orientable surface', *Topology* **19**(3) (1980), 221–237.
[18] A. E. Hatcher, 'Concordance spaces, higher simple-homotopy theory, and applications', in *Algebraic and Geometric Topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 1* (Proc. Sympos. Pure Math.) vol. XXXII (Amer. Math. Soc., Providence, RI, 1978), 3–21.
[19] H. Konno, 'Characteristic classes via 4-dimensional gauge theory', *Geom. Topol.* **25**(2) (2021), 711–773.
[20] H. Konno and J. Lin, 'Homological instability for moduli spaces of smooth 4-manifolds', Preprint, 2022, arXiv:2211.03043.
[21] M. Krannich and O. Randal-Williams, 'Mapping class groups of simply connected high-dimensional manifolds need not be arithmetic', *C. R. Math. Acad. Sci. Paris* **358**(4) (2020), 469–473.
[22] A. Kupers, 'Some finiteness results for groups of automorphisms of manifolds', *Geom. Topol.* **23**(5) (2019), 2277–2333.

[23] T.-J. Li and A.-K. Liu, ‘Family Seiberg-Witten invariants and wall crossing formulas’, *Comm. Anal. Geom.* **9**(4) (2001), 777–823.
[24] J. Lin, ‘The family Seiberg-Witten invariant and nonsymplectic loops of diffeomorphisms’, Preprint, 2022, arXiv:2208.12082.
[25] J. Lin and Y. Xie, ‘Configuration space integrals and formal smooth structures’, Preprint, 2023, arXiv:2310.14156.
[26] D. McCullough and A. Miller, ‘The genus 2 Torelli group is not finitely generated’, *Topology Appl.* **22**(1) (1986), 43–49.
[27] J. W. Morgan, *The Seiberg-Witten Equations and Applications to the Topology of Smooth Four-Manifolds* (Mathematical Notes) vol. 44 (Princeton University Press, Princeton, NJ, 1996).
[28] B. Perron, ‘Pseudo-isotopies et isotopies en dimension quatre dans la catégorie topologique’, *Topology* **25**(4) (1986), 381–397.
[29] F. Quinn, ‘Isotopy of 4-manifolds’, *J. Differential Geom.* **24**(3) (1986), 343–372.
[30] D. Ruberman, ‘An obstruction to smooth isotopy in dimension 4’, *Math. Res. Lett.* **5**(6) (1998), 743–758.
[31] D. Ruberman, ‘A polynomial invariant of diffeomorphisms of 4-manifolds’, in *Proceedings of the Kirbyfest (Berkeley, CA, 1998)* (Geom. Topol. Monogr.) vol. 2 (Geom. Topol. Publ., Coventry, 1999), 473–488.
[32] D. Ruberman, ‘Positive scalar curvature, diffeomorphisms and the Seiberg-Witten invariants’, *Geom. Topol.* **5** (2001), 895–924.
[33] D. Sullivan, ‘Infinitesimal computations in topology’, *Inst. Hautes Études Sci. Publ. Math.* **47** (1977), 269–331.
[34] C. T. C. Wall, ‘On the orthogonal groups of unimodular quadratic forms’, *Math. Ann.* **147** (1962), 328–338.
[35] C. T. C. Wall, ‘Diffeomorphisms of 4-manifolds’, *J. London Math. Soc.* **39** (1964), 131–140.
[36] T. Watanabe, ‘Theta-graph and diffeomorphisms of some 4-manifolds’, Preprint, 2023, arXiv:2005.09545.