A CLASSICAL-MODAL INTERPRETATION OF SMOOTH INFINITESIMAL ANALYSIS

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Abstract. Smooth Infinitesimal Analysis (SIA) is a remarkable late twentieth-century theory of analysis. It is based on nilsquare infinitesimals, and does not rely on limits. SIA poses a challenge of motivating its use of intuitionistic logic beyond merely avoiding inconsistency. The classical-modal account(s) provided here attempt to do just that. The key is to treat the identity of an arbitrary nilsquare, *e*, in relation to 0 or any other nilsquare, as objectually vague or indeterminate—pace a famous argument of Evans [10]. Thus, we interpret the necessity operator of classical modal logic as "determinateness" in truth-value, naturally understood to satisfy the modal system, S4 (the accessibility relation on worlds being reflexive and transitive). Then, appealing to the translation due to Gödel et al., and its proof-theoretic faithfulness ("mirroring theorem"), we obtain a core classical-modal interpretation of SIA. Next we observe a close connection with Kripke semantics for intuitionistic logic. However, to avoid contradicting SIA's non-classical treatment of identity relating nilsquares, we translate "=" with a non-logical surrogate, '*E*,' with requisite properties. We then take up the interesting challenge of adding new axioms to the core CM interpretation. Two mutually incompatible ones are considered: one being the positive stability of identity and the other being a kind of necessity of indeterminate identity (among nilsquares). Consistency of the former is immediate, but the proof of consistency of the latter is a new result. Finally, we consider moving from CM to a three-valued, semi-classical framework, SCM, based on the strong Kleene axioms. This provides a way of expressing "indeterminacy" in the semantics of the logic, arguably improving on our CM. SCM is also proof-theoretically faithful, and the extensions by either of the new axioms are consistent.

§1. Introduction and background.

1.1. Smooth infinitesimal analysis. Smooth Infinitesimal Analysis (henceforth "SIA") and its generalization, Synthetic Differential Geometry ("SDG") are remarkable developments in late twentieth-century mathematics, reviving ideas and methods predating and, to some extent, antithetical to the nineteenth-century development of the now-mainstream limit methods of classical analysis (due to Bolzano, Cauchy, Weierstrass, Dedekind, Cantor, and others).

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The mainstream account takes the real line to be literally composed of (uncountably many) points, and this structure has no infinitesimals. And, of course, the logic is classical. SIA (and SDG) requires intuitionistic logic; adding excluded middle renders the system inconsistent (and thus trivial). Instead of the machinery of limits, SIA (and SDG)[1] deploys algebraic operations with infinitesimals, specifically nilsquares, numbers e, such that $e^2 = 0$. In mainstream analysis, it follows from the field axioms, and classical logic, that 0 is the only nilsquare:[2]

$$\forall x(x^2 = 0 \rightarrow x = 0).$$

As we shall soon see, this is outright refutable in SIA:

$$\neg\forall e[e^2 = 0 \rightarrow e = 0]. \tag{1}$$

It can also be shown in SIA that there are no nilsquares that are distinct from 0:[3]

$$\neg\exists e[e^2 = 0 \wedge e \neq 0]. \tag{2}$$

These last two are, of course, contrary to classical logic, but as noted, the logic of SIA is intuitionistic. One of our main goals here is to explain and resolve this "tension" from a classical perspective.

The official background in Bell's [3] development of SIA is his version of intuitionistic set theory [4]. That is used to define or characterize sets of numbers and thus relations on numbers, and to prove that certain relations are functions. One feature of this version of intuitionistic set theory is that the separation axiom is unrestricted.[4]

An alternative would be to cast the theory in an intuitionistic higher-order language. We would then require full, unrestricted comprehension. Then one can use the various types to characterize sets, properties and relations, and to show that some relations are functions. Or one might use the language of plurals, again with full comprehension. We are officially neutral on the background theory used to formulate SIA. Any of these three, and perhaps others, will do for present purposes.

We start with a constant R, for the set (or property, or plurality) of elements of a smooth line. All of the field axioms apply to R: addition and multiplication are commutative and associative; multiplication distributes over addition; 0 is the identity

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1 We won't keep repeating "(and SDG)."

2 This also holds in intuitionistic analysis. In particular, one does not need excluded middle to show that if the square of a Cauchy sequence a converges to 0, then a itself converges to 0. Again, SIA is not based on limits, nor are numbers defined in terms of Cauchy sequences (or Dedekind cuts).

3 The sentence (2) follows (without using excluded middle) from the field axiom that every number other than zero has a multiplicative inverse. Suppose that a nilsquare e had a multiplicative inverse e^{-1}. Then $ee^{-1} = 1 = 1^2 = e^2e^{-2} = 0$. Recall that Robinson-style non-standard analysis (whose logic is classical) does have non-zero infinitesimals, and those do have inverses.

4 Recall that a formula $\varphi(x)$ is said to be *decidable* in a given theory (governed by intuitionistic logic) just in case $\forall x(\varphi(x) \vee \neg\varphi(x))$ is provable in that theory. Other versions of intuitionistic set theory have it that only decidable formulas can be used in separation. See, for example, [28, 29]. Because identity and membership are decidable in those theories, they will not be appropriate for SIA.

for addition; "minus" is the additive inverse; 1 is the identity for multiplication. And every member of R other than 0 has a multiplicative inverse:

$$\text{if } x \neq 0 \text{ then } \exists y(xy = 1).$$

Remember that we are using intuitionistic logic, and so we do not have

$$\forall x(x = 0 \vee x \neq 0).$$

So we do not have that every number either does or does not have a multiplicative inverse.

There is also an order relation "<" on R, with many (but not all) of the usual properties found in classical mathematics:

if $x < y$ and $y < z$ then $x < z$,
it is not the case that $x < x$,
if $x < y$ then $x + z < y + z$,
if $x < y$ and $0 < z$ then $xz < yz$,
either $0 < x$ or $x < 1$,
if $x \neq y$ then either $x < y$ or $y < x$.

We also assume that any positive member of R has a square root:

$$\text{if } x > 0 \text{ then } \exists y(y^2 = x).$$

So we can recapitulate all of the usual Euclidean constructions in R.

Let Δ be the set of nilsquares in R, the set of numbers n such that $n^2 = 0$. The key axiom driving SIA's differential calculus is that of *micro-affineness* (MA), also known as the Kock–Lawvere axiom, after its inventors, Anders Kock and William Lawvere. This axiom guarantees, in effect, that all functions on R are smooth: each has a derivative, the derivative has a derivative, etc.[5] The micro-affineness axiom states that the image of Δ under any function is a straight line:

AXIOM. *For any function $f : \Delta \to R$, there exists a unique* b *in R such that for all* e *in* Δ

$$f(e) = f(0) + be. \qquad \text{(MA)}$$

As Bell [3] describes it, the "micro-neighborhood" Δ of 0 is long enough to be translated and rotated, but it cannot be "bent." The set Δ is a kind of rigid rod, a micro-tangent vector to curves in R^2. And b measures its slope.

It follows from (MA) that for any function $f : R \to R$ and any $x \in R$ there is a unique $c \in R$ such that

$$\forall e \in \Delta f(x + e) = f(x) + ce.$$

The unique existence of this slope has three key consequences:

1. $\Delta \neq \{0\}$. To see this, assume that $\Delta = \{0\}$. Apply (MA) to the identity function, $f(e) = e$. If 0 were the only nilsquare, then f could only take the value 0, and the equation of the (MA) axiom would be satisfied for *any* value of b, in particular

[5] Classically smooth functions are described as members of C^∞.

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for both 0 and 1, which are distinct by the field axioms. This contradicts the uniqueness of b as required by (MA).

To emphasize: it is not just that in SIA one cannot prove that all nilsquares are zero; one can outright show that the set of nilsquares is not the singleton of zero (but, as noted, we can also show that no nilsquare is distinct from zero; thus the aforementioned "tension").

2. The principle of micro-cancellation: for any x and y in R, if $ex = ey$ for all e in Δ, then $x = y$. To see this, apply (MA) to the function $f(e) = ex$. Since $ex = ey$ for all e in Δ, both x and y serve as the unique b of the axiom.
3. The slope b of the axiom (MA) provides the means of defining the derivative of a function on R: given $f : R \to R$, fix x in R and define $g_x(e) = f(x + e)$. By (MA), $f(x + e) = g_x(e) = g_x(0) + b_x e$, where b_x is unique. Then introduce $f'(x)$ defined by

$$f(x + e) = f(x) + ef'(x).$$

It follows that all functions on R are continuous. We can also apply the same reasoning to f', getting the second-derivative of f. This clearly iterates, thus showing that all functions on R are smooth.[6]

For details, see [3].

One further axiom of SIA is needed for introducing integration.

$$\textbf{Integration Principle: } \text{Given any function } f : [0, 1] \to R, \text{ there exists a unique } g : [0, 1] \to R, \text{ such that } g' = f \text{ and } g(0) = 0. \quad \text{(IP)}$$

This suffices for the introduction of definite integrals, for the first and second fundamental theorems of the calculus, and, with definition of Cartesian products, the development of multivariate calculus and complex analysis, with standard applications to geometry and physics. Again, for details, see [3].[7]

Say that a subset $U \subseteq R$ is *detachable* if for all x, either $x \in U$ or $x \notin U$. One can show that the only detachable subsets of R are the empty set and R itself. Indeed, suppose that a subset U of R is detachable. Then there is a function f such that for all $x \in R$, $fx = 0$ if $x \in U$ and $fx = 1$ if $x \notin U$. Obviously, a function like this cannot be continuous if both U and its complement in R are non-empty.[8]

This theorem that no non-empty, proper subset of R can be broken into two non-empty pieces is consonant with a long-standing thesis that a continuous substance

6 It might be noted that the nilsquare infinitesimals of SIA are not subject to the trenchant criticism from George Berkeley's *The analyst* [5]. When (informally) calculating derivatives, one typically divides by an infinitesimal at one point, but later sets the infinitesimal to zero. Thus, as Berkeley famously put it, infinitesimals are "ghosts of departed quantities." Be this as it may, in SIA, one can never divide by a nilsquare e, since it is not the case that e is distinct from zero. Nor can one "set" e to zero, since zero is not the only nilsquare. Thanks to an anonymous referee for pointing this out. Recall, again, that in SIA, derivatives are not calculated as limits.

7 The Appendix to [3] provides a sketch of some topos models for SIA. See also [23] and, for more recent work, Chen (2022).

8 As in intuitionistic analysis, a set X is detachable if and only if membership in X is decidable. So in SIA, membership is decidable only for the entire domain R and in the empty set.

forms a unity—there is something that holds it together. In Book 5 of the *Physics*, Aristotle [1] writes:

> The continuous is just what is contiguous, but I say that a thing is continuous when the extremities of each at which they are in contact become one and the same and are (as the name implies) contained in each other. Continuity is impossible if these extremities are two. This definition makes it plain that continuity belongs to things that naturally, in virtue of their mutual contact form a unity. (227a6)

This (alleged) property of continuous substances is, of course, dropped in contemporary analysis.

1.2. The challenge of interpretation. As we have seen, the classical law of excluded middle (LEM) and attendant principles like double-negation elimination and some of the quantifier conversion rules cannot apply in SIA (and SDG) on pain of outright inconsistency. But that in itself provides no explanation or account of *why* those logical laws fail. Our goal is to provide an interpretation of SIA in a classical, modal framework, one which provides such an explanation, at least for those who adopt classical mathematics. In particular, we aim for an account of why the logic of SIA must be intuitionistic despite it's not being a constructive theory.

To be sure, we do not deny the legitimacy of pursuing the mathematics of SIA (and SDG) in the absence of such an account, taking the axioms at face value. We also do not claim that our interpretation provides an intended interpretation, nor that it is what the sentences in SIA really mean (whatever *that* may mean).

Providing a convincing interpretation here falls within the purview of foundations and philosophy of mathematics. For one thing, this will allow a classically minded philosopher or mathematician to make sense of SIA within her favored understanding of mathematics, perhaps an understanding that our classically minded philosopher or mathematician insists is the only one available for any and all mathematics. Even for philosophers and mathematicians who have a more pluralist, or eclectic orientation to mathematics,[9] a classical interpretation of SIA can lead to insights as to how this theory relates to other mathematical theories. And the classical interpretation can allow one to propose new axioms or principles which cannot even be formulated in the language of SIA—we do just that in Section 3.

In the case of intuitionist arithmetic (HA) and intuitionistic analysis, and the like, many (but not all) hold that conflicts with classical arithmetic (PA) and classical analysis (CA) are due to different interpretations of the logical connectives and quantifiers. According to some philosophers and mathematicians, the intuitionistic connectives and quantifiers are understood in terms of their proof conditions, codified in the so-called Brouwer, Heyting, Kolmogorov (BHK) reading.[10]

Hellman [14] and Shapiro [27] have (independently) argued that SIA does not conform to the constructivism underlying intuitionistic mathematics, and that the

9 See, for example, [2, 14, 27].
10 For a clear presentation of the BHK proof-conditions, see [8, 12ff].

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BHK interpretation does not apply to that theory. Let us take two examples. First, we have already seen that SIA proves

$$\neg\exists e[e^2 = 0 \wedge e \neq 0]. \quad (1)$$

Read via BHK, this says that it is (constructively) refutable that there is a (constructive) proof of existence of a non-zero nilsquare. Now in one sense, this is too weak: SIA should be seen as denying that there are *any* non-zero nilsquares, whether their existence is constructively provable or not. But in another way, the BHK reading is too strong: intuitively, SIA depends on "not ruling out" non-zero nilsquares, appealing to the fact that the micro-affineness axiom implies that

$$\neg\forall e[e^2 = 0 \rightarrow e = 0]. \quad (2)$$

As Bell puts it informally, non-zero nilsquares, though they can't be proved to exist, are taken as "possible," in some sense. But this "possible" cannot be read constructively, i.e., to mean that there somehow could be a constructive proof of existence of a non-zero nilsquare. The above displayed theorem (2) of SIA clearly implies that there could *not* be any such proof. So Bell's insight concerning the possibility of non-zero nilsquares must be understood in some other way. Providing such an understanding is one of the main aims of this paper.

Second, a BHK, or intuitionistic reading, of the logical apparatus of SIA cannot be applied to some of the key axioms of SIA without undermining them. Consider, for example, the key axiom of micro-affineness, MA. Read according to BHK it would say that, for any constructively given function f, a unique real b can be constructively found, along with a constructive proof that it satisfies,

$$\forall e[e \in \Delta \rightarrow f(e) = f(0) + be].$$

But what reason is there to suppose that there is a constructive method of finding a real number b with the desired property? After all, SIA is supposed to be providing an *alternative* to well-known limit methods of finding derivatives, and, as reviewed above, the micro-affineness axiom is the key to defining the derivative of a smooth function. Read constructively, the central MA axiom itself seems dubious. Again, a better interpretation of SIA seems called for that would explain and rationalize SIA's dependence on intuitionistic logic (for both constructively minded and classically minded philosophers and mathematicians).

1.3. Motivating the interpretation. Our interpretation of SIA uses a classical modal language. The key idea is a suggestion that was raised, but not developed in detail, in [14]. The proposal is that statements of identity between some members of R, such as between 0 and an arbitrary nilsquare, can sometimes be described as holding *indeterminately*, reflecting an inherent objectual or metaphysical indeterminacy concerning the identity of members of R. So, to invoke the framework of possible worlds, some worlds have nilsquares e that are distinct from zero ($e \neq 0$), but also satisfying a statement that e is not *determinately* distinct from 0—due to the inherent indefiniteness of e as an object. And, although this suggests, indeed entails, that

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this nilsquare e is *possibly* identical to 0, but depending on some choices that are available (see below), even that need not hold determinately, again due to the inherent indefiniteness of identity in this articulation of R.

As noted in Hellman [14], this idea runs up against a famous argument, due to Gareth Evans [10], that there cannot be inherently vague or indeterminate objects, on pain of contradicting a Leibniz law of identity, viz.

$$\Phi(x) \wedge \neg\Phi(y) \rightarrow x \neq y, \tag{ND}$$

the non-identity of discernibles. This is an immediate consequence of a standard axiom-scheme of identity in first-order logic with identity (whether classical or intuitionistic), namely

$$(x = y \wedge \Phi(x)) \rightarrow \Phi(y).$$

The Evans argument has been taken to refute the very idea of indeterminate identities, sometimes called “ontic” or “metaphysical” vagueness.[11] The argument is very short—[10] is one page long. We recapitulate it here in the context of SIA. Let $\nabla(\varphi)$ say that φ is indeterminate, i.e., that neither φ nor its negation is determinate. Let e be an arbitrary nilsquare, and assume that $\nabla(e = 0)$. Clearly, we have that it is determinate that $0 = 0$. So, by the axiom of identity noted above, we have that $e \neq 0$. The key assumption of the argument is that $e \neq 0$ is inconsistent with $\nabla(e = 0)$. If that is granted, and if the other inferences invoked in the argument are also correct, then an indeterminate identity leads to inconsistency and is thus incoherent (dialetheism aside).

Notice, first, that the Evans argument cannot even be formulated in SIA itself. That theory does have the usual identity axioms, including the Leibniz identity scheme, but it has no means of expressing statements of determinacy or indeterminacy; it has no modal vocabulary at all.

Nevertheless, the relevant vocabulary can be formulated in our classical modal theory that we use to interpret SIA. Like many principles and arguments in philosophy, the Evans argument is accepted by some and rejected on various grounds by others.[12] As noted, we are in the latter group (for more detail, see [14]).

Recall that a key assumption in the Evans argument is that $e \neq 0$ is inconsistent with $\nabla(e = 0)$. Notice that this fails unless we have that $e \neq 0$ entails that it is determinate

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11 It is generally agreed that the Evans argument does not refute the possibility of vague or indeterminate *statements* of identity (see, for example, [19]). A nice example, given by McGee and McLaughlin [22], is an identity statement about a mountain, e.g., Kilimanjaro, abbreviated K. Let p be a particular pebble at the base of the mountain and consider the identity statement,

$$K = K + p,$$

where the “+” is for mereological sum or fusion. Clearly there is no correct answer to many such statements and their negations. But such failures of the law of excluded middle can be explained entirely by appeal to the absence of any conventions of natural language usage of the name “Kilimanjaro” or the predicate “mountain” that would decide such cases.

12 See [24] for an independent, natural-language counterexample to the Evans argument.

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that $e \neq 0$. That entailment is not valid in our interpretation.[13] As shall be seen, this is a feature of our account, not a bug.

Just by definition, $\nabla(e = 0)$ entails that it is not determinate that $e \neq 0$ and it is not determinate that $e = 0$. In other words, it is *possible* that $e = 0$ and it is possible that $e \neq 0$. In terms of possible worlds, there is an accessible world in which $e = 0$ and there is another accessible world in which $e \neq 0$.

In our initial, and basic, interpretation, $\nabla(e = 0)$ is consistent with $e = 0$ and it is consistent with $e \neq 0$. We go on to propose two new (competing) axioms. One of them is inconsistent with the first conclusion the Evans argument, namely that $\nabla(e = 0)$ entails $e \neq 0$. The other proposal does accept that conclusion of the Evans argument, but this does not undermine the whole enterprise. As noted above, even in the basic interpretation, $e \neq 0$ does not entail that it is determinate that $e \neq 0$. Stay tuned for details, and some ramifications for how identity is treated in our interpretation.

§2. Formal developments of the core of a CM interpretation of SIA.

2.1. First pass. We now turn to the formal development of these motivating ideas. The first step is to introduce a classical modal-logical language, along with a model-theoretic semantics (based on possible worlds), and some axioms. We call this CM, for "classical modal."

We introduce a *determinateness* operator, D, behaving as a modal necessity operator on (closed and open) formulas. The key idea is to lay down axioms using this D operator such that it is not the case that non-identities are always determinate. Indeed, to take our key example, non-identities relating non-zero nilsquares to zero cannot be determinate, on pain of contradiction via the usual field axiom on multiplicative inverses. So we allow that there be cases (or worlds) where $x \neq y$, but $\neg D(x \neq y)$. In the crucial case of nilsquares of SIA, we recognize examples such as $\neg D(e = 0) \wedge \neg D(e \neq 0)$. This corresponds to the fact that, in SIA, there are failures of excluded middle in the form $e = 0 \vee e \neq 0$.

As noted, when it comes to identities, we interpret D, or rather $\neg$D, in terms of indeterminate identity, the notion rejected by Evans, Lewis and others, and defended by yet others. In general, if φ is a formula, then $D\varphi$ means that there is no indeterminacy concerning φ, either in its non-logical terms, the quantifiers, or the connectives. Using the metaphor (or heuristic) of possible worlds, we can say, informally, that $D\varphi$ implies that φ is permanent, irrevocable, irreversible, cannot be undone, ...

On this reading, it is very natural that D iterates: If φ is permanent, irrevocable, etc., then so is $D\varphi$. Taking the contrapositive, if $D\varphi$ could be "undone," then so could φ itself. So we adopt the following axiom for our D operator:

$$DA \to DDA,$$

which is the characteristic S4 axiom. The other axioms of S4 are also evident on interpreting necessity as determinateness. Since the background logic is not free, the Converse Barcan Formula follows:

$$\exists x \Diamond \phi(x) \to \Diamond \exists x \phi(x). \qquad \text{(CBF)}$$

[13] This is sometimes called the failure of negative stability of identity.

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Our basic modal logic is thus quantified S4 plus (CBF).[14] As in all, or at least most, classical modal logics, we adopt the following abbreviation for possibility:

$$\Diamond\Phi \leftrightarrow \neg \mathrm{D}\neg\Phi.$$

The next step is to apply the well-known translation, due to Gödel and others of a given intuitionistic theory into classical S4, i.e., our CM (see [13, 26]). Here are the clauses, where A^G is the Gödel-translate of the formula A:

- If A is atomic, A^G is DA.
- $(\neg A)^G$ is $\mathrm{D}(\neg A^G)$.
- $(A \wedge B)^G$ is $A^G \wedge B^G$.
- $(A \vee B)^G$ is $A^G \vee B^G$.
- $(A \rightarrow B)^G$ is $D(A^G \rightarrow B^G)$.
- $(\forall x A)^G$ is $\mathrm{D}(\forall x A^G)$.
- $(\exists x A)^G$ is $(\exists x A^G)$.

Our first pass at our interpretation is to take the Gödel-translates of the axioms of SIA (as in [3]) as axioms of our S4 classical-modal theory CM. If the background of SIA is Bell's intuitionistic set theory, then the translates of that are included in CM. This would make CM a first-order modal language. If, instead, the background is a higher-order (or plural) logic, then the translates of the axioms of that (including the instances of the comprehension scheme) are included in CM. This would make CM a classical higher-order (or plural) language. Recall that we are neutral on what the proper background for SIA should be. To repeat, the point here is that the axioms of the background theory should also be translated into the modal language (and, again, we are neutral on the choice of a background theory).

We then apply the key theorem that Gödel et al. proved governing the translation, namely it is proof-theoretically faithful. Let $\vdash_{\mathrm{int}}$ be deducibility in intuitionistic logic, and let $\vdash_{\mathrm{S4}}$ be deducibility in S4:

Theorem 1. *For any sentences, $A_1, \ldots A_n$,*

$$A_1, \ldots A_n, \vdash_{int} B \ \text{ if and only if } \ A_1^G, \ldots A_n^G \vdash_{S4} B^G. \qquad \text{(mirroring)}$$

On the formal side, there is (so far) not much new here. Theorem 1 is well known and works for (almost) any theory that invokes intuitionistic logic.[15] Our modest

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14 A referee asks whether we should adopt the following convergence principle:

$$\Diamond\Box p \rightarrow \Box\Diamond p, \qquad \text{(G)}$$

making the logic S4.2. This modal logic has been proposed as appropriate for certain kinds of potentialist accounts (see [20, 21]). The underlying idea is that the license to generate a mathematical object is not revoked if one does other constructions first (thus suggesting that the worlds be convergent). However, just as the present account is not intuitionistic, it is also not potentialist. The nilsquares are not thought of as "generated." We see no particular motivation for the principle (G) here, but we are open to persuasion that it should be included. We might add, however, that we do not know whether the strengthening to S4.2 here is even consistent—the mirroring Theorem 1 just below invokes S4. Linnebo and Shapiro [21] do establish a mirroring theorem for intuitionistic S4.2 (using a different translation), but that result requires atomic formulas to all be decidable in the intuitionistic theory. As above, neither identity nor membership is decidable in SIA.

15 See Section 2.3 for an explanation of the "almost."

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contribution (so far) is to provide an understanding of the modal operator that seems to fit in with SIA. In the case of intuitionistic theories, like Heyting arithmetic or intuitionistic analysis, the modal operator for the Gödel-translation is often taken to have an epistemic component. A formula in the form $\Box\Phi$ is supposed to be something like "Φ is knowable," or "Φ is (constructively) provable." As argued above, such a reading is not compatible with the main themes of SIA.[16]

Note that while we cannot prove the (actual) existence of non-zero nilsquares, we can prove that such are possible. Furthermore, it is straightforward to see that there are S4 models of our theory CM which have non-zero nilsquares in the base world.

2.2. Interlude on the Kripke semantics for intuitionistic logic. Saul Kripke [18] proposed a kind of model-theoretic semantics for intuitionistic logic. This is closely related to how the above Gödel-translation relates to the standard semantics for (most) modal logics (also due to Kripke [17]). It will prove instructive to briefly bring out this connection, as this plays a role in our development.

A *frame* is a pair $\langle W, R\rangle$, where W is a non-empty set of "worlds" and R is a reflexive and transitive relation on W. This, of course, is also a frame for the modal logic S4, and the connection is important here. If $F = \langle W, R\rangle$ is a frame, then an *interpretation* I on F is a function that assigns an interpretation to each $w \in W$, with the following conditions:

- For each $w \in W$, $O_I(w)$, the domain at w, is a non-empty set. And for $w, w' \in W$, we have that if Rww' then $O_I(w) \subseteq O_I(w')$. In words, the domains may grow but never shrink along the accessibility relation.[17]
- For each $w \in W$ and each constant c, $E_{Iw}(c)$ is a member of $O_I(w)$. It is the denotation of c at w. And we insist that if Rww' then $E_{Iw}(c) = E_{Iw'}(c)$. Constants do not change their denotations along the accessibility relation.
- For each $w \in W$ and each n-place predicate P, $E_{Iw}(P)$, the extension of P at w is a set of n-tuples of $O_I(w)$. And we insist that the interpretation of each atomic predicate is *monotonic*: if Rww' then $E_{Iw}(P) \subseteq E_{Iw'}(P)$. In the language of S4, this amounts to an insistence that for each atomic formula ϕ, we have the necessitation of the universal closure of

$$\phi \rightarrow \mathrm{D}\phi.$$

Note that this is usually not assumed in the semantics of S4 (or any other modal logic). So every Kripke interpretation is also an S4 interpretation, but not conversely. This difference will loom large below.

Let $F = \langle W, R\rangle$ be a frame and I an interpretation on F. Let $w \in W$. We then say what it is for a formula ϕ of the intuitionistic language to be *forced* at I, with respect

[16] It is also widely known that, despite Theorem 1, the usual possible-worlds model theory for S4 does not make much sense of intuitionistic theories. For example, any S4 model of the Gödel translation of intuitionistic arithmetic must have a copy of all of the natural numbers in each world (see, for example, [27]). And, to keep from also sanctioning excluded middle, such models must have worlds with infinite "numbers." We submit that, as we interpret the modal operator, the S4 models of SIA do make at least some philosophical sense of the enterprise of SIA.

[17] This last holds in S4 interpretations if the background logic is two-valued and not free.

to a variable assignment v, a function from the variables in the language to the domain $D_I(w)$ of w. This is written $I_v(w) \Vdash \phi$. The definition is given in Appendix A.

The Kripke semantics is sound and complete for intuitionistic logic: for any sentences $A_1, \dots A_n, B$,

> $A_1, \dots A_n, \vdash_{\text{int}} B$ if and only if for any Kripke interpretation I and any world W in the frame, if $I(w) \Vdash A_1$ and ... $I(w) \Vdash A_n$, then $I(w) \Vdash B$.

We can illustrate the connection between the Gödel translation from a language governed by intuitionistic logic and the usual semantics for S4. Recall that every Kripke interpretation is also an S4 interpretation (but not vice versa). It is straightforward to verify the following:

THEOREM 2. *Let A be any sentence in a theory governed by intuitionistic logic. Then*

$$I(w) \Vdash A \textit{ if and only if } A^G \textit{ is true in the corresponding } S4 \textit{ interpretation.}$$

2.3. The (annoying) matter of identity. The underlying logic of S4 here, of course, is classical, so we seem to have exactly what we've been seeking, a Classical-Modal (CM) interpretation of SIA. However, there is an issue with how the identity sign (=) of SIA is to be interpreted in CM—or translated into the language of CM. The most natural thing would be to just follow the general plan, and translate "=" from SIA as "=" in the classical modal theory CM. But the latter is the usual, fully determinate identity relation of classical mathematics. Its extension, in a given model or world is just the set of all pairs $\langle x, x\rangle$ of elements in the domain—we assume that the underlying logic and semantics of the meta-theory is classical.

The above results concerning Kripke semantics and the mirroring Theorem 1 (as well as Theorem 2) only work with respect to intuitionistic first-order logic *without identity*, or with intuitionistic theories (like arithmetic) in which identity is decidable. The "homophonic" interpretation of identity fails with respect theories like SIA and intuitionistic real analysis, for which identity is not decidable.

Let us examine this phenomenon. As noted, both SIA and intuitionistic analysis actually *refute* the decidability of identity. Each has a theorem that

$$\neg\forall x(x = 0 \vee x \neq 0). \tag{NDI}$$

The Gödel translation of this is

$$\mathrm{D}\neg\mathrm{D}\forall x(\mathrm{D}(x = 0) \vee \mathrm{D}\neg\mathrm{D}(x = 0)). \tag{NDI$^{\mathrm{G}}$}$$

This, in turn, is equivalent (given the underlying classical logic) to

$$\mathrm{D}\Diamond\exists x(\Diamond(x \neq 0) \wedge \Diamond(x = 0)). \tag{NDIc}$$

This entails that there is a world w with an object b such that there is an accessible world w' where b is distinct from zero and (another) accessible world w'' where b is identical to zero. But this is not possible on the usual classical understanding of identity. If b is identical to zero in one world, it must be identical to zero in every world in which b exists—not just in accessible worlds. Similarly, if b is distinct from zero in one world, then b must be distinct from zero in every world in which b exists. That's just what identity means (in classical contexts).

Here is another way to illustrate the issue here, one that will prove instructive later. In a modal language, a formula φ is said to be *positively stable* just in case the necessitation of the universal closure of φ is provable in the modal logic:

$$\varphi \rightarrow \mathrm{D}\varphi. \qquad \text{(Pos-stab)}$$

And φ is *negatively stable* just in case the necessitation of the universal closure of $\neg\varphi$ is provable in the modal logic:

$$\neg\varphi \rightarrow \mathrm{D}\neg\varphi. \qquad \text{(Neg-stab)}$$

Informally, a formula φ is positively stable if whenever it is decided that φ holds in a given world, concerning some objects that exist at that world, the interpretation never "changes its mind" about that. And, similarly, φ is negatively stable if whenever it is decided that φ does not hold in a given world, concerning some objects that exist at that world, it never "changes its mind" about that.

Genuine identity, symbolized "=," is both positively and negatively stable. If a world satisfies a sentence "$a = b$" then "a" and "b" denote the same object in all accessible worlds. Similarly if "a" and "b" denote different objects in a given world, then they denote different objects in all accessible worlds.

Identity, or a surrogate for identity, can be positively stable—it will be in any Kripke interpretation (see Section 3.1). But identity, or a surrogate, cannot be negatively stable in any S4 interpretation that satisfies the axioms and thus the theorems of SIA (or intuitionistic analysis). Recall our formula (NDIc):

$$\mathrm{D}\Diamond\exists x(\Diamond(x \neq 0) \wedge \Diamond(x = 0)).$$

Let w be a world in an S4 interpretation of SIA. Then the domain of w contains an object that is possibly distinct from zero and also possibly identical to zero. Classical logic (and bivalence) being what it is, either $e = 0$ at w or $e \neq 0$ at w. If the former, then by positive stability (if it holds), e can never be distinct from zero, and if e is distinct from zero at w then by negative stability, it can never be identical to zero.

Furthermore, as noted above, SIA forbids "$e \neq 0$," for any nilsquare e. Thus CM cannot allow the negative stability of identity, at least not in the classical context.

So, what should we do? One option would be to formulate the classical modal theory with an indeterminate identity relation. But that would undermine the present interpretative project. We are looking for a classical framework to capture (and perhaps explain) the subtle indeterminacies in SIA, using a modal language. It would not further that agenda if we have to just assume an indeterminate identity relation on the classical side (although we do make such an attempt in Section 4 below).

Our way around this obstacle is quite simple, and fairly common. We propose to translate SIA's "$x = y$" as a *non-logical* two-place relation, which we label "$E(x, y)$." It is a kind of surrogate identity relation.[18] Perhaps one way of reading this, in some contexts, is something like "x is indistinguishable from y." Since distinguishability is naturally understood as relative to available means of distinguishing, it is world-relative. So even though a nilsquare e is distinguishable from 0 in one world, w, in a world, w',

[18] We propose to keep the usual identity relation on the CM side. It will not appear in anything in the range of the translation.

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accessible from w, e could become indistinguishable from 0, while still counting as an object distinct from 0, using the usual identity sign to note this.[19]

This proposal requires that, in addition to translating SIA and the background set theory (or higher-order logic), we also have to translate the usual identity axioms. One of those is $\forall x(x = x)$. This translates as

$$\mathrm{D}\forall x \mathrm{D}E(x, x).$$

In words, it is determinate that everything bears E to itself. So E is reflexive, at least.

The other principle for identity is a scheme, the universal closure of:

$$\forall x \forall y((x = y \wedge \phi(x)) \rightarrow \phi(y)).$$

So, simplifying a bit, in CM we have

$$\mathrm{D}\forall x \forall y \mathrm{D}((\mathrm{D}(E(x, y) \wedge \phi^G(x)) \rightarrow \phi^G(y)).$$

So DE, the relation of being determinately E, is a congruence relation *on formulas in the range of the translation*. A fortiori, DE, is an equivalence relation on the CM side. But we do not verify that E itself is an equivalence.[20]

To get a better feel for how the Gödel-translation and mirroring theorem work, let us observe what happens when we apply these to the initially puzzling SIA result that, although we have

$$\vdash_{\mathrm{SIA}} \neg\forall e[e^2 = 0 \rightarrow e = 0], \tag{1}$$

we also have that

$$\vdash_{\mathrm{SIA}} \neg\exists x(x^2 = 0 \wedge x \neq 0). \tag{2}$$

Indeed, $x \neq 0$ entails that x has a multiplicative inverse, and so cannot be a nilsquare.

Applying the Mirroring Theorem to the first SIA theorem, we have

$$\mathrm{D}\neg\mathrm{D}\forall x \mathrm{D}[\mathrm{D}E(x^2, 0) \rightarrow \mathrm{D}E(x, 0)].$$

In the classical context, this becomes

$$\mathrm{D}\Diamond\exists x \Diamond[\mathrm{D}(E(x^2, 0)) \wedge \neg\mathrm{D}(E(x, 0))].$$

Thus, in CM we have the *possible existence* of nilsquares that possibly do not determinately bear E to 0. Of course, this cannot be derived in SIA itself, as SIA lacks the modal operators needed to express it. It is only equivalent in CM to the CM translate of the SIA theorem displayed above.[21]

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19 As we shall see in Section 3.2 below, on one of our proposed new axioms, a nilsquare can "change its mind" about its E-relation to 0, back and forth repeatedly. This axiom requires that both positive and negative stability fail.

20 On one of our two proposed new axioms in Section 3 E itself is an equivalence, but, in all cases, it is not identity, since E cannot be negatively stable in any interpretation.

21 It is perhaps of interest to observe that, in any CM model-structure, there cannot be an endpoint world, i.e., a world such that it itself is the only world accessible to it. In an endpoint world, a formula $\Box\varphi$ is equivalent to φ. So the Gödel translations of (1) and (2) would be equivalent to (1) and (2), and in the classical context, those are contradictory.

This result gives (classical modal) substance to Bell's [3] remark that "non-zero" nilsquares exist only in a "potential sense,"as their outright existence cannot be proved or even consistently added to SIA, as already seen above.[22]

Another SIA theorem is the following:

$$\forall x(x^2 = 0 \rightarrow \neg\neg(x = 0)). \tag{DN=}$$

The translation of this is

$$\mathrm{D}\forall x\mathrm{D}(\mathrm{D}E(x^2, 0) \rightarrow \mathrm{D}\neg\mathrm{D}\neg\mathrm{D}E(x, 0)). \tag{DNE}$$

And this is equivalent to

$$\mathrm{D}\forall x\mathrm{D}(\mathrm{D}E(x^2, 0) \rightarrow \mathrm{D}\Diamond\mathrm{D}E(x, 0)).$$

In words, every determinate nilsquare e is possibly determinately "identical" to zero. If e is a determinate nilsquare in a world w in a CM interpretation of SIA, then there is an accessible world w' for which e is determinately zero.

There is some fallout from the above considerations for how functions are to be treated in CM. The mantra of SIA is that all *functions* are smooth. So we need to think about which functions exist (in SIA) and how this translates to CM. Note that the very notion of a function involves identity—a function must be *single-valued*. A key item in every relevant version of SIA is a comprehension scheme for functions.[23] One instance is the universal closure of

$$\forall x\exists! y\Phi(x, y) \rightarrow \exists f\forall x\Phi(x, fx),$$

where Φ is any formula not containing f free. How should statements like this be translated into the language of CM? We cannot think of f there as a variable ranging over CM functions, since the latter would invoke the identity relation of CM, not the surrogate E that we use to render SIA's identity symbol.

We propose to slightly reformulate SIA using only relation variables. And the various axioms of SIA, such as the Kock–Lawvere axiom, would be restricted to relations that happen to represent functions. We forgo tedious details, as it is fairly straightforward. To keep things manageable, we will stick to one-place functions. Those correspond to binary relations R where we have (in SIA):

$$\forall x\exists y\forall v(Rxv \leftrightarrow v = y).$$

As above, our translation of this is (equivalent) to

$$\mathrm{D}\forall x\exists y\mathrm{D}\forall v\mathrm{D}(\mathrm{D}Rxv \leftrightarrow \mathrm{D}E(v, y)).$$

These are the relations that fall under the translation of the Kock–Lawvere axiom in CM. All of these relations enjoy (the translation of) being smooth.

§3. Proposals for additional axioms. In this section, we present and discuss two additional axioms for CM. They are competitors in the sense each of them entails the

[22] The word "potential" is perhaps misleading here, thus the scare quotes. As noted, SIA is not a potentialist theory in the sense, for example, of Linnebo [20] and Linnebo and Shapiro [21].

[23] In Bell's own version, this comes from the background (intuitionistic) set theory. As noted, it is possible to use higher-order logic or plural logic instead.

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negation of the other. In both cases, there is the burden of showing that the axiom in question is consistent with CM. They both involve the surrogate identity relation E.

3.1. The positive stability of "identity". Recall that in a classical, modal setting like CM, a predicate ϕ is *positively stable* just in case the necessitation of the universal closure of the following holds:

$$\phi \to \mathrm{D}\phi.$$

A formula ϕ is *negatively stable* just in case the necessitation of the universal closure of the following holds:

$$\neg\phi \to \mathrm{D}\neg\phi.$$

And ϕ is *stable* if it is both positively and negatively stable.

We saw in the previous section that the translation of the identity predicate in SIA cannot be negatively stable—that conflicts with the translation of the Kock–Lawvere axiom. Indeed, negative stability leads directly to contradiction by implying the existence of multiplicative inverses of some nilsquares. For that reason (among others), we translated the identity symbol "=" as a non-logical predicate "E."

Notice that, so far, it is left open whether the translation of the identity predicate is *positively* stable. Our first proposed new axiom for CM is the positively stability of E. In CM, this can be stated as a sentence:

$$\mathrm{D}\forall x \forall y (E(x, y) \to \mathrm{D}E(x, y)). \tag{PSE}$$

The consistency of (PSE) with CM is almost immediate. Recall that in any Kripke-interpretation of a theory whose background logic is intuitionistic, it is stipulated that all atomic predicates are monotonic: if ϕ is an atomic predicate and a world w' is accessible from a world w, then the extension of ϕ in w is a subset of the extension of ϕ in w'. It follows that if the interpretation in question is construed as an S4 interpretation, all atomic predicates are positively stable. Since the surrogate identity predicate E is atomic, (PSE) holds in any Kripke interpretation, construed as an S4 interpretation. And given that SIA is consistent, there are Kripke interpretations for it.

The main feature of (PSE) is that it allows the predicate E to function more like an identity relation than it otherwise would (*ceteris paribus*). As we have seen, E is (determinately) reflexive, but there is no guarantee that E is symmetric or that it's transitive. We do have that E is a congruence on formulas in the range of the Gödel translation, but only on that.

Under the translation, "$t = u$" becomes "$\mathrm{D}(E(t, u))$." Under (PSE), this is equivalent to just $E(t, u)$. In SIA, of course we have the usual (intuitionistic) identity axioms. In particular, SIA proves that identity is symmetric:

$$\forall x \forall y (x = y \to y = x),$$

and that identity is transitive:

$$\forall x \forall y \forall z ((x = y \land y = z) \to (x = z)).$$

The translations of these into CM are (equivalent to)

$$\mathrm{D}\forall x \forall y \mathrm{D}(\mathrm{D}E(x, y) \to \mathrm{D}E(y, x))$$

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and

$$\mathrm{D}\forall x \forall y \forall z \mathrm{D}((\mathrm{D}E(x,y) \wedge \mathrm{D}E(y,z)) \rightarrow \mathrm{D}E(x,z)).$$

Now, under (PSE), we have (in CM) that $E(t,u)$ is equivalent to $\mathrm{D}E(t,u)$. So the translations of the SIA theorems that identity is symmetric and transitive are equivalent to

$$\mathrm{D}\forall x \forall y \mathrm{D}(E(x,y) \rightarrow E(y,x))$$

and

$$\mathrm{D}\forall x \forall y \forall z \mathrm{D}((E(x,y) \wedge E(y,z)) \rightarrow E(x,z)).$$

So CM plus (PSE) implies

$$\mathrm{D}\forall x \forall y (E(x,y) \rightarrow E(y,x))$$

and

$$\mathrm{D}\forall x \forall y \forall z ((E(x,y) \wedge E(y,z)) \rightarrow E(x,z)).$$

That is, it is determinate that E is symmetric and transitive (and thus an equivalence).

As noted, we have that in CM, DE is a congruence on formulas in the range of the translation. So in CM plus (PSE) E itself is also a congruence on formulas in the range of the translation. Moreover, it is straightforward that E is also a congruence on all formulas that have no instances of modal operator D.

The contrapositive of (PSE) is

$$\forall x \forall y (\neg \mathrm{D}E(x,y) \rightarrow \neg E(x,y)).$$

In a sense, this is the first conclusion of the Evans argument (using E as the surrogate identity relation): if x is not determinately identical to y then x is distinct from y. But, as should be familiar by now, this conclusion is harmless. The Leibniz principle, formulated with E, does not hold in CM. And, again, E is not negatively stable.

3.2. An axiom that sustains indeterminate "identity". We now set (PSE) aside and move to another (competing) axiom. Recall that the axioms of CM are just the Gödel-translations of the axioms of SIA (and the background intuitionistic set theory, higher-order logic, or plural logic)—replacing the identity symbol "=" with a non-logical predicate "E."

The proposed new axiom is

$$\mathrm{D}\forall e[(E(e^2,0) \wedge \Diamond(\neg E(e,0)) \rightarrow \Diamond(E(e,0) \ \wedge \neg \mathrm{D}E(e,0))]. \qquad \text{(New Ax)}$$

In words: "Determinately (i.e., in all worlds) any nilsquare that is possibly non-zero is possibly 'identical' to 0 but not determinately so."

Notice that (New Ax) is incompatible with (PSE), the positive stability of the surrogate identity relation E. Under (PSE), $E(e,0)$ entails $\mathrm{D}E(e,0)$ and so is not consistent with $\neg\mathrm{D}(E(e,0))$. This means that (New Ax) is not satisfied in *any* Kripke interpretation of SIA (construed as an S4 interpretation). So we have to establish the consistency of (New Ax) with CM.

We do, however, insist that both (PSE) and (New Ax) are each worth exploring. We begin with a comment on the advantages of (New Ax) in relation to the present project. We saw that (PSE) makes the surrogate identity relation E an equivalence relation, and thus more like genuine identity. As far as we can determine, (New Ax) does not. However, it further highlights the phenomenon of "indeterminate identity" as an objective feature of SIA's smooth worlds, a phenomenon whose possibility critics of the Evans argument have been at pains to support.

Perhaps surprisingly to all those from Russell to Evans to Lewis, who have doubted the mere coherence of objectually indeterminate identity, SIA, through the lense of CM, is a remarkable example of a consistent theory with indeterminate identity from within mathematics itself, one that thrives on the coherence of objectually indeterminate identity of its infinitesimals. The necessity of the positive instability of the E relation in (New Ax) can be seen as an expression of this indeterminateness, thus a *feature* of the system, rather than a bug.

To invoke another metaphor, all versions of CM allow a kind of fusion: things that are "distinct" (such as a given nilsquare and zero) can merge and become just one thing. According to (PSE), fusion is permanent. Once merged, they stay merged. In effect, zero is a fateful strong attractor. Against this, with (New Ax) we have fission as well as fusion.

We suggested above that one possible gloss on the relation E is as a kind of indistinguishability. A skilled fiction writer could perhaps tell an entertaining tale of "life as a nilsquare," in the regime of (New Ax), as either "forever" (i.e., determinately) merging with 0 or repeatedly first merging with 0, then splitting off from 0, then merging again with 0, then splitting off, *ad infinitum*, while retaining its identity throughout. This is a never-ending identity crisis for some of our poor nilsquares.

In this scenario, the necessity of translating SIA's "$=$" as non-logical "E" becomes a virtue. Again, an advantage of the rival axiom (PSE) is that E is an equivalence relation. "Indistinguishability" is presumably symmetric, but it is plausible that it is not transitive. As in cases involving what cognitive psychologists call "just noticeable differences," one expects that in sufficiently long chains of samples (e.g., of colors or textures or sounds, etc.), items along such a chain will eventually exceed the threshold of just noticeable difference.[24]

In sum, in the regime of (New Ax), a nilsquare, e, can keep changing along a sequence of worlds from $\neg E(e,0)$ to $E(e,0)$ but then back to $\neg E(e,0)$, and so *ad infinitum, without e's ever losing its individual identity!* That is, not only is *merging or fusion* (with 0) possible, but also *splitting or fission* (from 0) is possible *while e remains the same object throughout*![25]

We repeat that all of this makes sense in the context of our classical CM, but not in SIA itself. Recall that SIA cannot countenance any nilsquare e that is distinct from zero, on pain of contradiction (given the field axiom that any member of R that is distinct from zero has a multiplicative inverse). And SIA lacks modal operators and so cannot even entertain possibilities (metaphysical, epistemic, linguistic, or otherwise). As explained in [14], one of the problems with the Evans argument in the context of SIA is its use of lambda abstraction to yield a property of "being the object e,"

[24] This makes for an at least superficial connection with vagueness and vague statements of identity.

[25] Stating this point requires that the meta-language (or the meta-meta-language) have a genuine identity relation, one we use to track objects in different worlds. Incidentally, note that examples of fission-fusion-fission in, say, cell biology and in particle physics do *not* exhibit this "preservation of identity" across the transitions. For the latter, see [12].

where e is a variable ranging over Δ, the set of nilsquares. The only instance where this "property" makes sense is 0, but of course SIA thrives on its theorem,

$$\neg\forall e[e \in \Delta \to e = 0].$$

In the context of CM, however, such lambda abstraction does make sense. As noted, in the presence of (PSE), the first part of the Evans argument goes through, although it is harmless (since E is not negatively stable). As explained above, in the regime of (New Ax), even the first step of the Evans argument fails.

We do have to establish the consistency of (New Ax). Let (New Ax$^-$) be the result of dropping the initial D from (New Ax). Our construction of a model of CM + (New Ax) proceeds in three stages. The first concentrates on getting the initial world of a model of CM to satisfy an instance of (New Ax$^-$) for a single non-zero nilsquare e. In the second stage, the construction is seen to generalize to all nilsquares of all the worlds of the assumed model of CM, yielding a CM model in which all worlds of the assumed CM model satisfy (New Ax$^-$). The third stage invokes an inductive construction that insures that all worlds *ever added* to M via iterations of the first construction also satisfy (New Ax$^-$), with the result that our extended CM model satisfies the full (New Ax), as desired. Details of the proof are in Appendix B.

It is worth commenting on the cardinalities of nilsquares that may arise in our models of New Ax- and New Ax. Very little about this can be inferred from SIA and the Gödel-translations of its axioms. Essentially all we know on this basis is that, in a classical model of the translates of the SIA axioms, there are at least as many nilsquares as there are members of the smooth line R, as follows from the fact that for any $r \in R$ and $e \in \Delta$, $r \cdot e \in \Delta$ as well, and if $r \neq r'$, then also $r \cdot e \neq r' \cdot e$, for any non-zero nilsquare, e. But beyond that, no information is available based on SIA itself (as opposed to particular topos models) and the Gödel translation. We can say, however, that the infinite cardinality of the set of nilsquares of the model $M+$ of New Ax– just is the cardinality of the set of nilsquares of the final model, p_ω, of CM + New Ax. This follows from the fact that the operations of adding worlds and their nilsquares to the given initial model M of the CM axioms (translates of the SIA axioms) are countable, i.e., only countable sequences of worlds with copies of descendants from worlds of M get added (as in stages 1 and 2 above); and the pre-models p_n leading to the final model, p_ω, form an ω-sequence. Thus, for example, the cardinality of the totality of nilsquares of the given CM model, M (hence of p_ω), may be either regular (i.e., not the union of a lesser cardinality of sets or pluralities of nilsquares all of lesser cardinality) or singular.

§4. Improving on CM: transition to "semi-classical modal framework". The fact that our modal framework CM is bivalent caused some awkwardness in the treatment of identity. Given the classical logic of CM, we have that every object a in a given world w in an S4 interpretation either satisfies $a = 0$ or $a \neq 0$—even if w also satisfies $\neg\mathrm{D}(a = 0)$ and $\neg\mathrm{D}(a \neq 0)$ (i.e., $\Diamond(a \neq 0)$ and $\Diamond(a = 0)$). So we translated the identity symbol "=" of SIA as a non-logical relation "E." Admittedly, this at least partially thwarts our aim of satisfying the possible existence of nilsquares e such that $\neg\mathrm{D}(e = 0)$ where it is not the case that $e \neq 0$ *simpliciter.* So long as we remain within the confines of CM, this aim cannot be realized as fully as we might like.

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Our goal here is to present a (partial) solution. Instead of a fully bivalent, classical modal logic, we introduce a three-valued modal logic. The three truth-values are T, for truth, F, for falsity, and I, for indefinite.[26] We adopt the so-called strong Kleene interpretation of the logical terminology. The usual clause for atomic formulas allows for worlds in which the sentences or formulas $a = 0$ and $a \neq 0$, for example, are both assigned I. The clause for D is:

- A formula in the form $D\phi$ is T in a world w of an interpretation, under a variable assignment, if ϕ is T in every world accessible from w under the same variable assignment; $D\phi$ is F otherwise. In other words, $D\phi$ is F at w under the variable assignment just in case there is at least one accessible world in which ϕ is either F or I.

Notice that any sentence of our language beginning with a D will be evaluated as either T or F at any world, never I.

Let us call the language and theory here SCM. Notice also that what we may call the meta-language of SCM is itself fully bivalent and, of course, classical. We insist, for example, that every sentence is either evaluated as T or it is not evaluated as T in any world in any interpretation. The latter happens if the sentence is evaluated as either F or I. Similarly, we have that for any two objects a, b, either $a = b$ is evaluated as T in a given world or it is not so evaluated. Those distinctions cannot be made in the object language. This will loom large.

Now consider the Gödel-translation φ^G of an arbitrary sentence φ in the language of SIA. Notice that in SCM, φ^G is equivalent to a sentence that begins with a D. So φ^G will be evaluated as either T or F at every world in every interpretation. In effect, φ^G is bivalent, having its usual bivalent truth-conditions in any S4 interpretation.

It is thus straightforward to verify that the mirroring Theorem from Section 2 above is good for SCM:

Theorem 1. *for any sentences,* $A_1, \dots A_n$,

$$A_1, \dots A_n, \vdash_{int} B \text{ if and only if } A_1^G, \dots A_n^G \vdash_{S4} B^G.$$

4.1. The annoying matter of identity, in this context. As noted, in SCM we have the freedom to satisfy, for example, $\neg D(a = 0)$ and $\neg D(a \neq 0)$ while still satisfying neither $a \neq 0$ nor $a = 0$, since those can be assigned I. Notice that, so far, we have not said anything about the formal properties of identity in SCM. If we end up translating the identity of SIA as the identity of SCM, then we will have the translations of the usual axioms of identity, namely

- $D\forall x(D(x = x))$.
- The determinate universal closure of

  $$D\forall x \forall y D((D(x = y) \wedge \phi^G(x)) \rightarrow \phi^G(y)),$$

  one instance for every formula ϕ in the language of SIA.

But this says nothing about how identity is to be treated for formulas that are not in the range of the translation. There are several options for how to proceed here. One

[26] We must distinguish indefiniteness, the gloss on the third truth-value (in the meta-language) from indeterminacy which is a modal notion in the object language: $\neg D$.

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is to not add anything special concerning identity on the SCM side. Our theory just is the translation of the axioms of SIA. This has the advantage of formally working: it is a consistent rendering of SIA into a classical, three-valued modal language, using the strong Kleene interpretations of the connectives and quantifiers.

But that leaves a philosophical issue concerning whether the identity symbol of SCM really captures an *identity* relation. Recall that SIA has the following as a theorem:

$$\neg\forall x(x = 0 \vee x \neq 0). \qquad \text{(NDI)}$$

The Gödel translation NDI^{Q} of this is

$$\text{D}\neg\text{D}\forall x(\text{D}(x = 0) \vee \text{D}\neg\text{D}(x = 0)).$$

This, in turn, is equivalent (given the underlying classical logic) to

$$\text{D}\Diamond\exists x(\Diamond(x \neq 0) \wedge \Diamond\text{D}(x = 0)). \qquad \text{(NDIc)}$$

This entails that there is a world w with an object b such that there is an accessible world w' where b is distinct from zero and (another) accessible world w'' where b is determinately identical to zero. One may question this, even in the three-valued context. If b is (determinately) identical to zero in one world, it may (in the three-valued context) end up failing to be zero in a different world, but (perhaps) it cannot be *distinct* from zero anywhere. Perhaps this stretches the notion of identity too far. Perhaps. We submit that it is not clear how solid intuitions are concerning identity in the three-value context.

If one insists the notion of identity has been stretched too far, a second option is to follow the lead of our classical, bivalent CM, and translate the identity symbol of SIA as a non-logical predicate E. The properties of that relation indicate what features (and bugs) the surrogate "identity" relation "=" of SIA will have in the classical context. As in the two-valued CM, there is an issue concerning functions and the like, but everything goes pretty much as in CM.

In the rest of this section, we go with the first option and translate "=" homophonically as "=." That fits better with the motivation to go with a three-valued language in the first-place. Our primary question will be what properties and features are enjoyed (or not) by the identity relation of SCM.

Consider the usual axioms of identity. One of those, $\forall x(x = x)$ is not problematic. That sentence follows from the translation of the corresponding axiom in SIA. However, the so-called Leibniz axiom scheme—the indiscernibility of identicals—formulated (as usual) in terms of formulas, is problematic in the three-valued context. Here is an instance of the scheme:

$$\forall x\forall y((x = y \wedge \phi(x)) \rightarrow \phi(y)).$$

This is indefinite, value I, in every SCM interpretation. A variant on the first Evans argument shows this: let x be 0, and let y be a number that is not determinately identical to zero (in a given world) and also not determinately distinct from 0 in that world; *a*

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might be an arbitrary nilsquare. Then consider the following instance of the identity scheme:

$$(0 = a \wedge \mathrm{D}(0 = 0)) \rightarrow \mathrm{D}(0 = a).$$

In that world in question, the antecedent is I and the consequent is F, thus making the conditional I. So the corresponding instance of the identity scheme is also I.

Following the theme of Section 3.1 above, concerning (PSE), one might think that it should at least be consistent with the framework to have identity be positively stable, which is formalized as

$$\forall x \forall y (x = y \rightarrow \mathrm{D}(x = y)).$$

This also comes out as indefinite in any SCM interpretation. Suppose that $a = 0$ is indefinite in a given world. Then $\mathrm{D}(a = 0)$ is false, and so the conditional is indefinite.

One might think that it is safe to restrict the identity scheme to formulas that contain no modal operators, the universal closure of

$$\forall x \forall y ((x = y \wedge \phi(x)) \rightarrow \phi(y)),$$

where ϕ does not contain any modal operators. Something like this is to be expected. As Frege [11], Quine [25], and a host of others point out, one can be led astray by substituting "equals for equals" in modal and other intensional contexts.

Unfortunately, even this restriction to non-modal formulas fails in SCM. The following non-modal instance of the scheme

$$(a = b \wedge b = 0) \rightarrow a = 0$$

is indefinite if the atomic identities are indefinite.

As noted above, the original Evans [10] note generated a number of responses in defense of indefinite (or indeterminate) identity. Some of those responses employ a three-valued framework, and some of those invoke the strong-Kleene interpretation (although, as far as we know, none employs a modal framework).[27] Parsons and Woodruff [24] give the strong Kleene readings of negation, conjunction, and disjunction, but decline to include a conditional, since that would be "too complicated."Here we are trying to give plausible identity axioms for SCM, and the identity scheme uses a conditional. So does SIA. For better or worse, we use the strong-Kleene conditional in SCM.

Bruce Johnsen [15] employs a three-valued (non-modal) language, with the strong Kleene interpretation. He makes the above point that this framework makes the identity scheme indefinite (in some models), and thus invalidates the first conclusion of the Evans argument. Of course, if that were our only goal, we would be done. However, like us, Johnsen wants to say something about how identity does behave in the three-valued framework (although Johnsen is not dealing with SIA, and his formal language is not modal).

[27] Broome [6] employs a three-valued (non-modal) framework, but the connectives and quantifiers are not interpreted via strong Kleene. For Broom, a conditional is true in a given interpretation if either its antecedent is either indefinite or false (i.e., if its antecedent fails to be true) or the conclusion is true.

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We have not said anything yet about how *validity* is to be defined in our three-valued (modal) context, SCM. Inspired by Johnsen, it is natural to think of an argument as valid if it necessarily preserves truth:

> If Γ is a set of sentences and ϕ is a sentence, then $\Gamma \vDash \phi$ if and only if for every world w in any interpretation, if every member of Γ is T then ϕ is T.

Notice that the usual introduction rule for the conditional is not valid on this conception. Also note that if every member of Γ and ϕ are all either T or F in every world of a given interpretation, then this is the usual, classical definition of validity (in which case the usual introduction rule for the conditional is valid).

So one option is to formulate the identity scheme as a rule of inference (or, equivalently, as a restriction on interpretations). There are two cases:

1. If ϕ contains no modal operators, then $\{a = b, \phi(a)\} \vDash \phi(b)$.
2. $\{\mathrm{D}(a = b), \phi(a)\} \vDash \phi(b)$.

Notice that if ϕ is in the range of the translation, then it is equivalent to a formula that begins with D, and so is true or false in every world, and that if $\mathrm{D}(a = b)$ is true in a world w, then it is also true in every accessible world. So, by clause (2), this version is valid, and so the mirroring theorem still holds.

There is at least one other option, or perhaps better, a program for developing another option. This involves introducing operators for truth, falsity, and indefiniteness in the object language. For details, see Appendix C.

To sum up, here are the options concerning how identity is to be treated in SCM:

- Follow the lead of CM and translate the identity relation "=" of SIA as a non-logical predicate E. Then it does not matter what we say about the identity relation of SCM, since the symbol for that does not occur in any formula in the range of the translation.
- Translate the identity symbol of SIA homophonically of SCM, but don't add principles for this symbol, except the translations of the identity axioms in SIA.
  Admittedly, on the first two options, there is no guarantee that the identity symbol of SCM is a genuine identity relation.
- Translate the identity symbol of SIA homophonically, and thus accept the translations of the identity axioms of SIA; and adopt the instances of (LL_{nv}) (see Appendix C) in which the formula ϕ contains no modal operators, and $\mathrm{LLD}_{v'}$.

4.2. The new axioms. We begin with (PSE), the positive stability of the surrogate identity relation. In the context of SCM, this could be the positive stability of identity (PS=). We saw above that the sheer statement of this

$$\forall x \forall y (x = y \rightarrow \mathrm{D}x = y)$$

is indefinite in all models of SCM. Following the earlier lead, (PS=) should be formulated as a rule of inference or a restriction on interpretations:

$$t = u \vDash \mathrm{D}(t = u). \tag{PS=}$$

We turn to (New Ax) in SCM. We can be brief here, as the general situation is essentially the same. As with several other axioms (the identity axioms and (PS=)),

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we cannot just assert (New Ax), since some of the instances of it (i.e., (New Ax⁻)) are indefinite in some worlds, and so (New Ax) is false in typical SCM models.

Once again, the solution is to formulate (New Ax) as a rule of inference or as a restriction on models. The result would be

$$(e^2 = 0 \wedge \Diamond e \neq 0) \vDash \Diamond(e = 0 \wedge \Diamond e \neq 0).$$

§5. Concluding remarks. Our primary goal in this paper was to provide a classical (modal) interpretation of Smooth Infinitesimal Analysis (SIA). We have actually provided several: CM, SCM, both with or without one or the other of our proposed new axioms (PSE/=) and (New Ax). We do *not* claim that one of these classical modal theories just is SIA, nor that it gives the meaning (or real meaning) of the language of SIA, nor that this is the proper ground for SIA, nor that this is what advocates of SIA have in mind. So what is the value of this interpretative enterprise?

In Section 1.2 above, we noted that the interpretation allows a classically minded philosopher or mathematician to make sense of SIA, within her favored practice. From that perspective, this would locate SIA (or a surrogate) within the established foundation for mathematics, ZFC. To be sure, some philosophers and mathematicians are more eclectic, willing to understand, say, SIA, intuitionistic analysis, and classical analysis on their own terms, putting aside any apparent conflicts (on the model of Euclidean and non-Euclidean geometries).

Our modal theories also help fulfill one of our main desiderata, viz. explaining *why* the logic of SIA should be intuitionist, going beyond the mere observation that classical logic results directly in inconsistency and that SIA based on intuitionist logic has models and so is consistent.[28] In their distinctive ways, our classical or semi-classical modal theories accommodate indeterminate identity of nilsquares (other than 0). This is expressed by (the negation of) our modal operator, D, and also, in the SCM theories, by the intermediate "Indefinite" truth value, I. Of course, SIA cannot express ¬D or I, but it can simply not accept sentences of the form, "$e = 0 \vee e \neq 0$," or "$e = f \vee e \neq f$," for nilsquares e, f. Our account of SIA's reliance on intuitionist logic is implemented by the Gödel et al. translation scheme, in effect, by reading the biconditionals of the mirroring theorem in the reverse direction, i.e., going from the classical-modal deducibility relations, governed by classical S4, back to the SIA deducibility relations, governed by intuitionist logic. Thus, our classical-modal theories with their treatment of indeterminate identity, along with the Gödel et al. translation scheme, do help explain why the logic of SIA indeed has to be intuitionistic.

[28] Michael Dummett [[9], p. 216] once made the following quip (in a different, but analogous context):

> We can gain some grasp on the idea of a totality too big to be counted ...but once we have accepted that totalities too big to be counted may yet have numbers, the idea of one too big even to have a number conveys nothing at all. And merely to say, "If you persist in talking about the number of all cardinal numbers, you will run into contradiction," is to wield the big stick, but not to offer an explanation.

We claim here to be offering an explanation (at least from some perspectives).

Furthermore, a classical modal interpretation of SIA can lead to insights as to how the theory relates to other mathematical theories (or how it might relate under certain assumptions). This is an instance of a general theme in mathematics. When a theory comes along, or is developed in a certain direction, one of the first things to be explored is how that theory relates to other theories. We need an arena to explore that. In the case of classical theories, that arena can be ZFC.[29]

As we have seen, the richer classical modal framework can allow one to propose new axioms or principles which cannot even be formulated in the language of SIA. Such principles might concern formulas that are not in the range of the translation. More important, perhaps, the richer theory can express and adjudicate modal claims that cannot even be stated in the non-modal SIA. To use a metaphor of Linnebo [20], the classical modal language represents a finer resolution than that of the original SIA. Given the various mirroring theorems, this resolution can be turned on or off at will.

We have illustrated this with the consideration of possible axioms for CM and SCM, beyond the Gödel translations of the SIA axioms. In Section 2.4 there was the principle stating that there are non-zero nilsquares, and in Sections 3 and 4.2 there are treatments of the (competing) axioms (PSE/=) and (New Ax). In every case, it is required to show that the proposed new axiom is consistent with CM or SCM. This is sometimes relatively easy, but more often at least *prima facie* difficult.

Another possible area of inquiry concerns comparisons between the smooth reals R of SIA and the ordinary, classical real numbers $\mathbb{R}$. Let's use SCM here, since there we translate the identity symbol "=" of SIA homophonically as "=" Within SIA, the smooth real numbers R are an ordered field. So there are constants 0,1. As usual, we can define the natural numbers, within SIA, as the closure of the set $\{0,1\}$ under addition, the integers as the closure of $\{0,1\}$ under addition and subtraction, and the rational numbers as the closure of the integers under division.

So we can expect to find isomorphic copies of the natural numbers, integers, and rational numbers in each world of each model of SCM and, we suspect, CM. The translations of the SIA theorems about these numbers will constitute a theory equivalent to intuitionistic arithmetic, integral, and rational analysis. In the classical context, we can then define the real numbers $\mathbb{R}$ as the closure of the rational numbers under Dedekind cuts (or equivalence classes of Cauchy sequences) as usual.

So each world in SCM (and CM) will contain an extension for the smooth real numbers R and also an isomorphic copy of the classical real numbers $\mathbb{R}$. These two sets (or properties, or pluralities) overlap (or at least they can overlap): the rational numbers are members of both. There are members of R that are not in $\mathbb{R}$: The latter cannot contain any nilsquares that are not (determinately) zero. And there are members of $\mathbb{R}$ that are not in R. Bell [3] shows that the intermediate value theorem fails in SIA, and so the least upper bound principle has to fail.

In sum, the classical modal contexts, here (S)CM with or without the various proposed new axioms, might provide fertile ground for exploring various connections between the smooth real numbers of SIA and their classical cousins.

[29] Some maintain that the category of all categories, or other related theories is a better foundation, but we need not pursue that issue here.

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§A. Forcing in the Kripke semantics. Let $F = \langle W, R\rangle$ be a frame for the Kripke semantics for intuitionistic logic and I an interpretation on F (see Section 2.2). Let $w \in W$. We define what it is for a formula ϕ of the intuitionistic language to be *forced* at I, with respect to a variable assignment v, a function from the variables in the language to the domain $D_I(w)$ of w. This is written $I_v(w) \Vdash \phi$.

- If ϕ is an atomic formula $Pt_1 \ldots t_n$, then $I_v(w) \Vdash \phi$ if and only if the n-tuple consisting of the denotations of $t_1 \ldots t_n$ is in the extension $E_{Iw}(P)$ of P. This, of course, is the usual clause for atomic formulas in classical modal logics.
- $I_v(w) \Vdash \phi \wedge \psi$ if and only if $I_v(w) \Vdash \phi$ and $I_v(w) \Vdash \psi$. This is also the usual clause for conjunctions in classical modal logics.
- $I_v(w) \Vdash \phi \vee \psi$ if and only if $I_v(w) \Vdash \phi$ or $I_v(w) \Vdash \psi$. This is also the usual clause for disjunctions in classical modal logics.
- $I_v(w) \Vdash \phi \rightarrow \psi$ if and only if *for every* w' such that Rww' if $I_v(w') \Vdash \phi$ then $I_v(w') \Vdash \psi$. This, of course, is not the usual clause for conditionals in classical modal logics. In words $I_v(w)$ forces $\phi \rightarrow \psi$ just in case for every accessible world w', if $I_v(w')$ forces ϕ then $I_v(w')$ forces ψ.
- $I_v(w) \Vdash \neg\phi$ if and only if for every w' such that Rww' it is not the case that $I_v(w') \Vdash \phi$. This, too, is not the usual clause for negation in classical modal logics. In words $I_v(w)$ forces $\neg\phi$ just in case there is no world w' accessible from w such that $I_v(w')$ forces ϕ.
- $I_v(w) \Vdash \exists x\phi$ if and only if there is an object a in the domain $O_I(w)$ of w such that $I_{v'}(w) \Vdash \phi$ for the variable assignment v' that assigns x to a and agrees with v on every other variable. This, too, is the usual clause for existential formulas in classical modal logics. In words, we have that $I_v(w) \Vdash \exists x\phi$ just in case there is an object that exists in w that witnesses the existential.
- $I_v(w) \Vdash \forall x\phi$ if and only if for every world w such that Rww' and every variable assignment v' on w' that agrees with v on every variable except possibly x, $I_v(w) \Vdash \phi$. This is *not* the usual clause for universal quantifiers in classical modal logics. In words $I_v(w) \Vdash \forall x\phi$ just in case all accessible worlds force ϕ. In effect, the existential quantifier just ranges over the objects in w, while the universal quantifier ranges over the objects in all accessible worlds.

It is straightforward (if a bit tedious) to verify that the semantics is *monotonic* in the same sense as atomic formulas are:

$$\text{for all worlds } w, w', \text{ if } I_v(w) \Vdash \phi \text{ and } Rww' \text{ then } I_v(w') \Vdash \phi.$$

§B. The consistency of CM plus (New Ax). The meta-theory (or maybe the meta-meta-theory) here is either Zermelo–Fraenkel set theory with choice (ZFC) or an axiomatic second-order logic or plural logic, with unrestricted comprehension and the axiom of choice applied to sets, properties, or pluralities of worlds of model-structures of S4.

Recall that (New Ax) is

$$\mathrm{D}\forall e[(E(e^2, 0) \wedge \Diamond(\neg E(e, 0)) \rightarrow \Diamond(E(e, 0) \ \wedge \neg \mathrm{D}E(e, 0))].$$

In words: "Determinately (i.e., in all worlds) any nilsquare that is possibly non-zero is possibly 'identical' to 0 but not determinately so."

And, as above, let (New Ax⁻) be the result of dropping the initial D from (New Ax). We assume that SIA is consistent (see the Appendix to [3] or [23], or [7]), and so CM is, by the mirroring theorem. Start with a fixed model M of CM. Since the logic of CM is classical, every world of M satisfies either $E(a,0)$ or $\neg E(a,0)$ for any object a in the domain of the world. Recall that it is a theorem of SIA that not all nilsquares are identical to 0. The Gödel translation of that theorem entails that every world of M satisfies

$$\Diamond\exists e(E(e^2,0) \wedge \Diamond\neg E(e,0)).$$

In words, we have that in every world of M, it is possibly that there is a nilsquare which is possibly non-zero (as formulated with the predicate E).

The construction below is to be carried out for *every* non-zero nilsquare occurring in every world of M. In SIA, it is easy to show that for any nilsquare e and any member r of the smooth line, re is also a nilsquare and, if $r \neq 0$ and $r \neq r'$, then $re \neq r'e$. It follows that there are infinitely many possibly non-zero nilsquares in each world of M. This at least indicates that the Axiom of Choice will be needed.

Let w_0 be the initial world of M. Suppose that w_0 does not have any non-zero nilsquares. Since all worlds must satisfy the above displayed formula, move to an accessible world, call it w^* (appealing to the axiom of choice if necessary), such that $w^* \models \neg E(e,0)$, for some nilsquare e in w^*. Now consider w^* along with all its "descendants," that is all worlds accessible from w^*. Since all those worlds are part of the starting CM model, M, it follows that w^* together with all its descendants is also a model of CM, so we can label that as our CM model M.

Next, add to the model M a denumerable sequence, $\langle w_j \rangle$, $j = 1, 2, 3, \ldots$ of worlds accessible from w^*, such that, for j odd, we stipulate that $w_j \models E(e,0) \wedge \neg \mathrm{D}E(e,0)$, and for j even, $wj \models$ exactly the sentences that w^* does (including of course $\neg E(e,0)$, fulfilling the condition of the previous world that $\neg \mathrm{D}E(e,0)$). Further, for j even, on a separate branch from the main one containing w_{j+1}, add, as descendants of w_j, *copies of all the worlds accessible to* w^*, *preserving their accessibility relations*. This will insure, for instance, that, for *all* j, $w_j \models \Diamond \mathrm{D}E(e,0)$, as required by the translation of the SIA theorem,

$$\forall e[e^2 = 0 \rightarrow \neg\neg e = 0],$$

which states that, determinately, for any nilsquare, determinately it is possible that determinately $E(e,0)$. Thus, for j even, w_j together with all its descendants along the branch just described meets all the requirements for a CM model that the original w^* does. And for j odd, w_j also meets all such requirements. Indeed, since every axiom of CM begins with a D, and by the characteristic S4 axiom, $\mathrm{D}\Phi \rightarrow \mathrm{DD}\Phi$, and since, by transitivity of the S4 accessibility relation, every added world (along with all others) is accessible from w^*, the original world of the sequence for nilsquare e—which, as a world of the original CM model, M, by hypothesis satisfies all the CM axioms—it follows that every added world $w_j \models \mathrm{D}Ax$, where Ax is an axiom of CM. (This must be so also by our construction, because each w_j, for even j, is an exact copy of w^*, and for odd j, w_j is also a copy of w^*, except that the formula $\neg E(e,0)$ is replaced by $E(e,0) \wedge \neg \mathrm{D}E(e,0)$.) Thus far, then, we have a world, w^*, of M which satisfies the instance of (New Ax⁻) with respect to the nilsquare e. This completes stage 1 of our construction.

This, however, is just an instance of a general pattern: the same construction applies to any other nilsquare e' evaluated at any world of M as $\neg E(e', 0)$.

PROPOSITION. The result of applying the above construction to every nilsquare of every world of M is a model, $M+$, of *CM* and (*New Ax*$^-$) in which every world of M satisfies *CM* and (*New Ax*$^-$).[30]

Proof. By hypothesis, every world w of the original M satisfies all the axioms of CM. But (New Ax$^-$) is also satisfied by those worlds: so long as a nilsquare e satisfies $\neg E(e, 0)$ at some world, w, of M, our construction guarantees that, at an accessible further world, w_1, we have $E(e, 0)$, but this is indeterminately the case, since at a later world w_2, we again have $\neg E(e, 0)$. But this too will hold only indeterminately, as required for consistency, since at an accessible w_3, we have $E(e, 0)$, and so *ad infinitum.* But e here is arbitrary, as the same construction applies to any nilsquare e', where we have $\neg E(e', 0)$. Thus, the demands of (New Ax$^-$) are met by our constructions with respect to any nilsquare of any world of M. □

This completes stage 2 of our modeling, a CM model, $M+$, in which each of the worlds of M satisfies (New Ax$^-$). We refer to this as a "*pre-model*" of (New Ax), the necessitation of (New Ax$^-$).

But we aim for more, namely a genuine CM model of (New Ax), i.e., a model in which *every world ever added* by the construction of stage 1 satisfies (New Ax$^-$), not merely the worlds of the original model M.

Clearly, if all worlds ever added to M as well as those of $M+$ satisfy (New Ax$^-$), then they all also satisfy (New Ax), by the semantics of S4 modal logic.

To this end, we deploy a further construction via (ordinary arithmetical) recursion on pre-models:

- p_0 is just the model $M+$ of CM + (New Ax$^-$).
- For each natural number n, p_{n+1} is the result of applying the construction of stage 1 above to all nilsquares of all worlds added at p_n.
- p_ω is $\bigcup\{p_n | n = 1, 2, \dots\}$.

THEOREM. p_ω is a model of CM + (*New Ax*).

Proof. Clearly every world of p_ω satisfies all the CM axioms. But by the construction of stage 1 and the definition of the p_n, every world of p_ω also satisfies (New Ax$^-$): every such world occurs in some pre-model, p_n, and all its nilsquares are subject to the construction of stage 1, which insures that, at pre-model, p_{n+1}, all those worlds satisfy (New Ax$^-$). Thus all worlds of p_ω satisfy (New Ax$^-$), so they also satisfy (New Ax), the necessitation of (New Ax$^-$).

The theorem is thus proved. □

COROLLARY. *As nothing was assumed about the CM model M beyond that it is indeed a model of the CM axioms, the above constructions apply to every non-zero nilsquare of any world of* any *CM model, including those with increasing domains of nilsquares along accessibility-paths of worlds emanating from the base world.*[31]

30 This Proposition only applies to all nilsquares of the worlds of the original model M. It does not (yet) apply to all nilsquares of all the worlds added by the construction of the linear alternating sequences and their additional descendants. Thus a further construction is needed, and will be carried out in the next stage.

31 Of course, M itself may be such a model.

§C. Identity in SCM. As noted above, several of the authors that resist the Evans argument have adopted a three-valued logic, just as we do here. Some of them introduce operators in the object language for truth, falsity, and indefiniteness. All of those are "bivalent" in that they only take the values of T and F (like our determinacy operator D). We can formulate the definitions in our modal framework:[32]

- A formula $\mathrm{T}(A)$ is T at a given world, under a variable assignment, if A is evaluated as T in that world, under that assignment; $\mathrm{T}(A)$ is F otherwise (i.e., if A is evaluated either F or I).
- A formula $\mathrm{F}(A)$ is T at a given world, under a variable assignment, if A is evaluated as F in that world, under that assignment; $\mathrm{F}(A)$ is F otherwise (i.e., if A is evaluated either T or I).
- A formula $\mathrm{I}(A)$ is T at a given world, under a variable assignment, if A is evaluated as I in that world, under that assignment; $\mathrm{I}(A)$ is F otherwise (i.e., if A is evaluated either T or F).

Notice that $\mathrm{I}(A)$ is not equivalent to the statement $\neg\mathrm{D}(A)$, nor to $(\neg\mathrm{D}(A) \wedge \neg\mathrm{D}(\neg A))$. The latter pair are modal statements, and their truth values at a given world w depend on the truth values of A at accessible worlds. The new operators just concern the truth-value of A at the world w in question.[33]

Johnsen [15, p. 109] formulates three different versions of the Leibniz Law (concerning the indiscernibility of identicals). We translate each of them as a scheme concerning identity, and using the above operators:

$$\forall y \forall z[(\mathrm{T}(\phi(y) \vee \mathrm{F}(\phi(y)) \vee (\mathrm{T}(\phi(z) \vee \mathrm{F}(\phi(z))] \rightarrow [y = z \rightarrow (\phi(y) \leftrightarrow \phi(z))]. \quad \text{(LL)}$$

In words, if at least one of $\phi(y)$ and $\phi(z)$ is either true or false, then if y is identical to z then $\phi(y)$ if and only if $\phi(z)$.

$$\forall y \forall z[(\mathrm{T}(\phi(y) \vee \mathrm{F}(\phi(y)) \wedge (\mathrm{T}(\phi(z) \vee \mathrm{F}(\phi(z))] \rightarrow [y = z \rightarrow (\phi(y) \leftrightarrow \phi(z))]. \quad (\mathrm{LL}_{nv})$$

In words, if both $\phi(y)$ and $\phi(z)$ are either true or false, then if y is identical to z then $\phi(y)$ if and only if $\phi(z)$.

$$\forall y \forall z[(\mathrm{T}(y = z) \vee \mathrm{F}(y = z)) \wedge (\mathrm{T}(\phi(y) \vee \mathrm{F}(\phi(y)) \wedge (\mathrm{T}(\phi(z) \vee \mathrm{F}(\phi(z))] \rightarrow [y = z \rightarrow (\phi(y) \leftrightarrow \phi(z))]. \quad (\mathrm{LL}_v)$$

Notice that if $(y = z)$ is false in a given world (under a given variable assignment), then both the antecedent and the consequent of (LL_v) are true. So (LL_v) can be

[32] We use the same letters, T, F, I for the operators and for the truth values in the three valued framework. Context will indicate which is meant (in particular, the operators are always followed by a formula in parentheses).

[33] Any one of the three operators could be defined in terms of the two others. For example, if we just had $\mathrm{T}(A)$ and $\mathrm{F}(A)$, then $\mathrm{I}(A)$ could be defined as $(\neg\mathrm{T}(A) \wedge \neg\mathrm{F}(A))$. Similarly, Johnsen [15] defines two other operators, both of which are interdefinable with the present operators: $\Delta(A)$ is our $(\mathrm{T}(A) \vee \mathrm{F}(A))$; and $\nabla(A)$ is its negation, our $(\neg\mathrm{T}(A) \wedge \neg\mathrm{F}(A))$. The first of these, of course, uses our symbol for the set of nilsquares.

simplified to

$$\forall y \forall z[(\mathrm{T}(y=z)) \wedge (\mathrm{T}(\phi(y) \vee \mathrm{F}(\phi(y))) \qquad \wedge(\mathrm{T}(\phi(z) \vee \mathrm{F}(\phi(z))] \rightarrow [\phi(y) \leftrightarrow \phi(z)] \qquad (\mathrm{LL}_{v'})$$

In words, if $y = z$ is true, and both $\phi(y)$ and $\phi(z)$ are either true or false, then $\phi(y)$ if and only if $\phi(z)$.[34]

For present purposes, (LL) is a non-starter. Suppose that, $\phi(y)$ is $y = y$ and $\phi(z)$ is $y = z$, and suppose that $y = z$ is I. For example, z might be 0 and y an arbitrary nilsquare. Then the antecedent of this instance of (LL) is T, but the consequent is I, So this instance of (LL) is I.

Notice that (LL_{nv}), (LL_v), and ($\mathrm{LL}_{v'}$) each imply that identity is positively stable. Let $\phi(y)$ be $\mathrm{D}(y = y)$ and let $\phi(z)$ be $\mathrm{D}((y = z))$. Suppose that $y = z$ is T at a world w. Recalling that any formula beginning with a D is bivalent (always either true of false), the antecedents of all three principles are true, and, again, so is $(y = z)$. We also have $\mathrm{D}(y = y)$. So, at w, $\mathrm{D}(y = z)$.

The contrapositive of this is also valid: $\neg\mathrm{D}(y = z)$ implies $(y \neq z)$. This is one way to state the first conclusion of the Evans argument in the three-valued context. As with CM, this is harmless here (since identity is not negatively stable), but one of the purposes of going to the three-valued framework was to block the first Evans argument.

Our proposal (for this option), then, is to add two principles. One of those is the instances of (LL_{nv}) in which the formula ϕ contains no modal operators. So it just concerns the truth values that occur in each world. And the second is to adopt a modalized version of ($\mathrm{LL}_{v'}$):

$$\forall y \forall z[(\mathrm{D}(y=z)) \wedge (\mathrm{T}(\phi(y) \vee \mathrm{F}(\phi(y))) \wedge (\mathrm{T}(\phi(z) \vee \mathrm{F}(\phi(z))] \rightarrow [\phi(y) \leftrightarrow \phi(z)]. \qquad (\mathrm{LLD}_{v'})$$

In words, if y is *determinately* identical to z and both $\phi(y)$ and $\phi(z)$ are either true or false, then $\phi(y)$ if and only if $\phi(z)$. It is straightforward to see that the mirroring theorem still holds.

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[34] Johnsen is keen to examine the Evans argument in the three-valued (non-modal) context:

> Proponents of Evans can and should take this opportunity to advance [the argument] by invoking ... (LL_{nv}), instead of LL as reformulation of the argument ... [The result is a] ... valid deduction of the incoherence identity. Nonetheless, whether [this revised argument] is sound must depend upon whether or not LL_{nv}, is in fact a law. After all, it is not at all clear that Evans's supporters are entitled to invoke anything stronger than (LL_v) ... Yet it is evident that no argument analogous to Evans's can succeed by invoking merely LL_v. (p. 109)

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