SOME EXAMPLES OF NONMEASURABLE SETS

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In a recent issue of this Journal, Pu [3] has given an interesting construction of a nonmeasurable subset A of R such that

$$(1) \qquad \lambda(A \cap I) = \lambda(I)$$

for all intervals I in R. [Throughout this note, the symbol λ denotes Lebesgue outer measure on R or Haar outer measure on a general locally compact group.] This solves a problem stated in [2], p. 295, Exercise (18.30).

We present in this note two examples, different from Pu's, that have properties like (1).

First consider any nonmeasurable [additive] subgroup B of R. There are many such subgroups: see for example [1], (16.13). A specific construction is the following. Let H be any Hamel basis for R over the rationals Q, let a be an element of H, and let B be the set of all linear combinations $\sum_{j=1}^{n} r_j h_j$ where $r_j \in Q$ and no h_j is equal to a. Steinhaus's theorem ([2], (10.43)) shows that B is nonmeasurable. Note that every measurable subset F of B has measure 0: if $F \subset B$ and $\lambda(F) > 0$, Steinhaus's theorem shows that $F - F$ contains an interval J about 0, and so $J \subset F - F \subset B - B = B$. This implies of course that $B = R$. Now if I is any open interval and U is an open set such that $I \backslash B \subset U \subset I$, then we have $\lambda(U) = \lambda(I)$, since otherwise we would have

$$0 < \lambda(I \backslash U) \text{ and } I \backslash U \subset B.$$

Hence the complement $R \backslash B$ has property (1).

Our second example is somewhat different. Following Sierpiński [4], we construct a set C of irrational numbers such that if x, y are irrational and $x + y$ is rational, then exactly one of x and y is in C. Well order the set S of irrational numbers in any fashion, by an ordering $\leqq$. Let the first element be in C. Now suppose that for all $x < y$, we have determined whether or not $x \in C$. If $y = -x + r$ for some $x < y$, $x \in C$, and rational r, then y is *not* in C. Otherwise, y is in C. If $u, v \in S$ and $u + v \in Q$, then plainly not both u and v are in C.

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Assume that neither u nor v is in C. Then there are a $u' < u$ and a $v' < v$ such that $u' + u$ and $v' + v$ are in Q, and u', v' are in C. We may suppose that $u' \leqq v'$. Then $(u' + v') + u + v$ is in Q, so that $u' + v'$ is in Q. an evident contradiction.

Next consider any rational number r and consider the mapping τ_r of R onto R defined by $\tau_r(x) = 2r - x$ [reflection in the point r]. It is clear that τ_r preserves Lebesgue outer measure: $\lambda(\tau_r(X)) = \lambda(X)$ for all $X \subset R$. Now consider any interval $I =]r - \alpha, r + \alpha[$, for $\alpha > 0$. It is simple to show that

$$\tau_r(I \cap C) = (I \backslash C) \cup (Q \cap I)$$

and so $\lambda(I \cap C) = \lambda(I \backslash C)$. Accordingly, we have

$$1 = \frac{\lambda(I)}{2\alpha} \leqq \frac{\lambda(I \cap C) + \lambda(I \backslash C)}{2\alpha},$$

$$= \frac{2\lambda(I \cap C)}{2\alpha},$$

that is,

$$\lambda(I \cap C) \geqq \tfrac{1}{2}\lambda(I).$$

Now given an arbitrary interval $]x, x + h[$, where $h > 0$, consider any h' such that $0 < h' < h$ and $x + \frac{1}{2}h'$ is rational. Then $]x, x + h'[$ is an interval of the type I, so that we have

$$\frac{\lambda(C \cap]x, x + h'[)}{h'} \geqq \frac{1}{2}.$$

We may also write

$$\frac{\lambda(C \cap]x, x + h[)}{h} \geqq \frac{\lambda(C \cap]x, x + h'[)}{h'} \cdot \frac{h'}{h}$$

$$\geqq \frac{1}{2} \frac{h'}{h}.$$

Since h'/h can be made arbitrarily close to 1, we see that

$$\frac{\lambda(C \cap]x, x + h[)}{h} \geqq \frac{1}{2}$$

for all $x \in R$ and $h > 0$. Exactly the same argument shows that

$$\frac{\lambda(]x, x + h[\backslash C)}{h} \geqq \frac{1}{2}$$

for all $x \in R$ and $h > 0$.

Plainly then C is nonmeasurable and

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$$\lambda(I \cap C) \geqq \tfrac{1}{2}\lambda(I) \tag{2}$$

for all intervals I.

Finally we note that our first exmaple admits a generalization to any locally compact Hausdorff group G with left Haar measure λ that contains a non λ-measurable subgroup B. [For example, every nondiscrete Abelian G contains such a subgroup: see [1], (16.13.c).] Then if E is a λ-measurable subset of G and $\lambda(E) < \infty$, the equality $\lambda(E \backslash B) = \lambda(E)$ holds.

References

[1] Edwin Hewitt, and Kenneth A. Ross, *Abstract Harmonic Analysis*, Vol. I (New York — Heidelberg — Berlin: Springer-Verlag 1963).

[2] Edwin Hewitt, and Karl R. Stromberg, *Real and Abstract Analysis*, 2nd printing. (New York — Heidelberg — Berlin: Springer-Verlag 1969).

[3] H. W. Pu, 'Concerning non-measurable subsets of a given measurable set', *J. Austral. Math. Soc.* 13 (1972), 267–270.

[4] W. Sierpiński, 'Sur un problème conduisant à un ensemble non mesurable'. *Fund. Math.* 10 (1927), 177–179.

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