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Isometric multipliers of Segal algebras

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We prove that for a large class of Segal algebras, the isometric multipliers consist of scalar multiples of translation operators.

1. Introduction

A bounded linear operator T on a commutative Banach algebra A is called a multiplier if for all $f, g \in A$,

$$T(f * g) = f * (Tg) = (Tf) * g .$$

If T is a multiplier of A then there exists a bounded continuous function $\hat{T}$ on Γ , the maximal ideal space of A , such that for all $f \in A$, $(Tf)\hat{}(\gamma) = \hat{T}(\gamma)\hat{f}(\gamma)$ and $\|\hat{T}\|_\infty \leq \|T\|$. The set of multipliers of A , denoted by $M(A)$, forms a commutative Banach algebra of operators under the operator norm. (For a detailed discussion of multipliers, see [*6*].)

Let S be a Segal algebra on a locally compact abelian group G . (For definition, examples, and properties of Segal algebras, see [*8*].) The maximal ideal space Γ of S is nothing but the dual of G . Let $M(G)$ denote the algebra of bounded regular Borel measures on G . For any Segal algebra S , $M(G)$ can be canonically imbedded in $M(S)$ by considering $\mu \in M(G)$ as a multiplier defined by $\mu(f) = \mu * f$, for all $f \in S$. This correspondence is norm decreasing; that is, $\|\mu\|_{M(S)} \leq \|\mu\|_{M(G)}$. For $S = L^1(G)$, this imbedding is an isometric isomorphism of $M(G)$ onto $M(S)$.

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For $a \in G$, let $\delta_a \in M(G)$ denote the unit point mass at 'a'. As an element of $M(S)$, δ_a is nothing but translation by 'a'; that is, $\delta_a(f) = f_a$, where $f_a(x) = f(x-a)$. By the definition of a Segal algebra, $\|f_a\| = \|f\|$. Hence $\lambda\delta_a$ for $|\lambda| = 1$ and $a \in G$ is an isometric multiplier of S. We shall prove that for a large class of Segal algebras, these are the only isometric multipliers. The crucial step in this attempt is provided by the following theorem which we prove in Section 2.

THEOREM 1. *Let A be a commutative, regular, semisimple Banach algebra which is tauberian. Then any isometric multiplier T of A is surjective and has a Gelfand transform of unit modulus.*

In Section 3, we prove the following theorems which give sufficient conditions for the set of isometric multipliers to coincide with the set of unimodular multiples of translation operators.

THEOREM 2. *Let the multiplier algebra $M(S)$ of a Segal algebra be isometrically isomorphic to $M(G)$. Then the isometric multipliers of S are just unimodular multiples of translation operators.*

THEOREM 3. *Let S be a Segal algebra on a locally compact abelian group G with connected dual Γ, and let $M(S)$ be isomorphic to $M(G)$. Then the isometric multipliers of S are just unimodular multiples of translation operators.*

In Section 4 we apply the results of Sections 2 and 3 to several specific Segal algebras and deduce that the isometric multipliers of all these Segal algebras consist of unimodular multiples of translation operators.

If G is a compact abelian group then any function of absolute value one on Γ defines an isometric multiplier of $L^2(G)$. Thus, in general, isometric multipliers need not be given even by measures. However, for noncompact groups, no simple example of a Segal algebra is known where isometric multipliers are not point masses. In Section 5, we construct such an example by using the notion of projective tensor products. (For a detailed discussion of results on projective tensor products, see [2].)

2.

We prove a few preliminary lemmas.

LEMMA 1. *Let A be a commutative, semisimple Banach algebra. If T is an isometric multiplier of A onto A, then $\hat{T}$ has absolute value one everywhere.*

Proof. We have $|\hat{T}(\gamma)| \leq \|T\| = 1$ for every γ in Γ, the maximal ideal space of A. The proof is completed by noting that T^{-1} is also an isometric multiplier of A [6]; hence the above inequality holds good with T replaced by T^{-1}.

The next lemma is a simple consequence of Wiener's Theorem ([3], 39.27). It has been observed by Burnham [1] also.

LEMMA 2. *Suppose that A is a commutative, semisimple, regular, tauberian Banach algebra. Then a multiplier T of A has dense range if and only if it has nonvanishing Gelfand transform.*

Proof. In view of the Wiener Tauberian Theorem, it is enough to observe the following:

(1) $T(A)$ is an ideal in A;

(2) the hull of $T(A)$ is just the set of those points of Γ where $\hat{T}$ vanishes.

Proof of Theorem 1. By Lemmas 1 and 2, it suffices to show that the zero set E of $\hat{T}$ is void. If E is nonempty, let I denote the closure of the ideal consisting of those elements of A whose Gelfand transforms vanish in a neighbourhood of E.

First we show that $T(I) = I$. The inclusion of $T(I)$ in I being obvious, it is enough to prove that the hull of the closed ideal $T(I)$ is E because I is the smallest closed ideal with E as the hull.

If γ does not belong to E, then there is an x in A such that $\hat{x}(\gamma) = 1$ and $\hat{x}$ has compact support disjoint from E. Then x belongs to I, but $(Tx)\hat{}(\gamma) \neq 0$. Hence γ is not in hull $T(I)$.

Thus T is an isometric multiplier of I onto I and so $\hat{T}$ has unit modulus outside E. This implies that E is open and closed. If E is not null, then a non-zero x in A can be found with $\hat{x}$ supported in

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E . For such an x , $Tx = 0$ as its Gelfand transform vanishes identically. This contradiction shows that E is void and completes the proof.

COROLLARY 1. *Any isometric multiplier of a Segal algebra* S *on a locally compact abelian group is surjective and has unimodular Fourier transform.*

Corollary 1 is very easy to prove in the case of a compact abelian group G . For, in this case, $\gamma \in S$ and

$$T(\gamma) = T(\gamma * \gamma) = T(\gamma) * \gamma = \hat{T}(\gamma)\gamma ,$$

for every $\gamma \in \Gamma$.

It is not, of course, true that every multiplier with unimodular Fourier transform is an isometry in general. For example, let μ be a measure with real valued Fourier-Stieltjes transform and let $\nu = \exp(i\mu)$. Then $\hat{\nu}$ has unit modulus, but is not an isometry on any Segal algebra whose isometric multipliers are given by unimodular multiples of Dirac measures. (For examples of such Segal algebras, see Section 4.)

3.

Proofs of Theorems 2 and 3 will become trivial after we prove the following lemma.

LEMMA 3. *Let* $M(S)$ *be isomorphic to* $M(G)$. *If* $\mu \in M(G)$ *defines an isometric multiplier of* S , *then* $\hat{\mu}$ *is a piecewise affine map from* Γ *into the circle group* T .

Proof. Since $M(S)$ is isomorphic to $M(G)$ and $\|\mu\|_{M(S)} \le \|\mu\|_{M(G)}$, it follows that both the norms are equivalent. If μ defines an isometric multiplier, μ is invertible by Corollary 1 of Theorem 1 and $\|\mu^n\|_{M(S)} = 1$ for $n = 0, \pm 1, \pm 2, \ldots$. Hence $\|\mu^n\|_{M(G)} \le K$, a fixed constant for $n = 0, \pm 1, \pm 2, \ldots$. By Theorem 4.7.3 of [9], $\hat{\mu}$ is then a piecewise affine map from Γ into T . This completes the proof.

If $M(S)$ is isometrically isomorphic to $M(G)$ then the constant K appearing in the above proof can be taken to be 1 , and then the homomorphism ψ of $l^1(z)$ into $M(G)$ defined by

$$\psi(f) = \sum_{n=-\infty}^{\infty} f(n)\mu^{-n}$$

(see the proof of Theorem 4.7.3 of [9]) is of norm less than or equal to 1 , and by 4.6.3 (b) of [9], $\hat{\mu}$ is actually an affine map of Γ into T . Since $\hat{\mu}$ is defined on Γ , it follows that μ is a translate of a continuous homomorphism of Γ into T ; that is, $\mu = \lambda\delta_a$ for $|\lambda| = 1$ and $a \in G$. This proves Theorem 2.

If Γ is connected, any piecewise affine map of Γ into T is automatically affine and Theorem 3 follows as above.

REMARK. The proof of Lemma 3, and hence that of Theorems 2 and 3, depends on some deep results on homomorphisms between measure algebras of groups. However, we also have a very elementary proof of Theorem 2.

Suppose $\mu \in M(G)$ defines an isometric multiplier of S . Then μ is invertible, $\|\mu\| = \|\mu^{-1}\| = 1$ and $|\hat{\mu}| = 1$. Let $\mu = \mu_c + \mu_d$ be the decomposition of μ into its continuous and discrete parts. Since μ is invertible, μ_d cannot be zero. We claim that $\mu_c = 0$. If $\mu_c \neq 0$ then $\|\mu_d\| < 1$ and since $\delta_0 = \mu^{-1} * \mu_c + \mu^{-1} * u_d$, we get that $\left\|\delta_0 - \mu^{-1} * \mu_c\right\| = \left\|\mu^{-1} * \mu_d\right\| < 1$. But this implies that $\mu^{-1} * \mu_c$, a continuous measure, is invertible and this is absurd.

Thus $\mu = \mu_d$ is of the form

$$\mu = \sum a_n \delta_{x_n} ,$$

where the x_n's are distinct points of G and $\sum |a_n| = 1$. To complete the proof of the theorem, we shall show that μ is supported in a point. Suppose this is not the case. Then we can assume that $a_1 \neq 0$ and $a_2 \neq 0$ in the representation of μ . Now, for every $\gamma \in \Gamma$,

$$1 = |\hat{\mu}(\gamma)| = \left|\sum a_n(-x_n, \gamma)\right|$$

$$\leq |a_1(-x_1, \gamma)+a_2(-x_2, \gamma)| + \sum_{n>2} |a_n|$$

$$\leq \sum |a_n| = 1 .$$

Therefore,

$$|a_1(-x_1, \gamma)+a_2(-x_2, \gamma)| = |a_1| + |a_2| .$$

This implies that $a_1\bar{a}_2(x_2-x_1, \gamma) \geq 0$ for every $\gamma \in \Gamma$. But this will imply that $x_1 = x_2$, contrary to our hypothesis.

4.

Wendel [14] proved that the isometric multipliers of $L^1(G)$ are unimodular multiples of Dirac measures for any arbitrary locally compact group G. For abelian groups, Theorem 2 gives an alternative proof of this fact. We now list a few Segal algebras for which we can conclude that the isometric multipliers are unimodular multiples of Dirac measures:

(i) $C(G)$ for compact G with sup norm;

(ii) $L^1 \cap C_0(G)$ for noncompact G with norm $\|f\| = \|f\|_{L^1} + \|f\|_\infty$;

(iii) $L^1 \cap L^p(G)$ for noncompact G with norm $\|f\| = \|f\|_{L^1} + \|f\|_{L^p}$ for $1 \leq p < \infty$;

(iv) $A_p(G) = \{f \in L^1(G) : \hat{f} \in L^p(\Gamma)\}$ with norm $\|f\| = \|f\|_{L^1} + \|\hat{f}\|_{L^p}$ for $1 \leq p < \infty$;

(v) $A_p(\mu)$, defined as in (iv) with Haar measure on Γ replaced by any positive unbounded Radon measure μ;

(vi) $E_p(\mu)$, defined as in (v) with $L^1(G)$ replaced by $C(G)$ for compact G;

(vii) $S(\alpha)$ for a locally bounded unbounded function α on Γ, defined by

$$S(\alpha) = \{f \in L^1(G) : \alpha\hat{f} \text{ vanishes at infinity}\} ,$$

with norm $\|f\| = \|f\|_{L^1} + \|\alpha\hat{f}\|_\infty$;

(viii) $S_p(G)$ (for definition, see Unni [13]).

For cases (i) to (iv) our assertion follows from Theorem 2. For case (vii), let T be an isometric multiplier of $S(\alpha)$; then for any $f \in S(\alpha)$,

$$\begin{aligned}\|f\|_{L^1(G)} + \|\alpha\hat{f}\|_\infty &= \|Tf\|_{L^1(G)} + \|\alpha\hat{T}\hat{f}\|_\infty \\ &= \|Tf\|_{L^1(G)} + \|\alpha\hat{f}\|_\infty .\end{aligned}$$

The last equality holds since $|\hat{T}| = 1$. Thus $\|Tf\|_{L^1(G)} = \|f\|_{L^1(G)}$, and T gives an isometric multiplier of $L^1(G)$. Hence the result follows from the corresponding result for $L^1(G)$. The proofs in cases (v), (vi), and (viii) are similar, the general philosophy being the following. In all these cases the fact that $|\hat{T}| = 1$ implies that T defines an isometric multiplier of $L^1(G)$ or $C(G)$, as the case may be, and the result follows from Theorem 2.

The isometric multipliers of $A_p(G)$ were determined by Tewari [11]. The result for $A_p(\mu)$ for noncompact G is stated in Krogstad [5], but in his proof some restriction on μ appears to be necessary. The merit of our approach lies in the fact that it disposes of several classes in one stroke.

Not all Segal algebras on abelian groups whose isometric multipliers are known to be point measures are subsumed by the results given here. Strichartz [10] and Parrott [7] proved that for an arbitrary G , the isometric multipliers of $L^p(G)$, $p \neq 2, \infty$, are multiples of translations.

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5.

Finally, we give an example of a Segal algebra on a noncompact group whose isometric multipliers are not the 'usual ones'. Let G, H be locally compact abelian groups and let $S(G)$, $S(H)$ be Segal algebras. The projective tensor product of $S(G)$ and $S(H)$ can be identified with a Segal algebra $S(G \times H)$ on $G \times H$ (see Kapoor [4] for details). If T_1, T_2 are multipliers of $S(G)$, $S(H)$ respectively, then $T_1 \otimes T_2$ is a multiplier of $S(G \times H)$. (This is not difficult to prove - see Tewari [12].) Let us show that if T_1, T_2 are isometries, then so is $T_1 \otimes T_2$:

$$\begin{aligned}\|(T_1 \otimes T_2)(f)\| &= \left\|(T_1 \otimes T_2)\left\{\sum f_n \otimes g_n\right\}\right\| \\ &= \left\|\sum T_1 f_n \otimes T_2 g_n\right\| \\ &= \inf\left\{\sum \|f_n'\|\|g_n'\| : \sum T_1 f_n \otimes T_2 g_n = \sum f_n' \otimes g_n'\right\} .\end{aligned}$$

Now T_1, T_2 are isometric multipliers and are therefore surjective. Hence $f_n' = T_1 h_n$, $g_n' = T_2 k_n$ for some h_n in $S(G)$ and k_n in $S(H)$. Thus

$$(T_1 \otimes T_2)(f) = \sum T_1 h_n \otimes T_2 k_n = (T_1 \otimes T_2)\left\{\sum h_n \otimes k_n\right\}$$

with $\|f_n'\| = \|h_n\|$ and $\|g_n'\| = \|k_n\|$. But $T_1 \otimes T_2$ is bijective since T_1 and T_2 are, and so $f = \sum h_n \otimes k_n$. It follows that

$$\begin{aligned}\|(T_1 \otimes T_2)(f)\| &= \inf\left\{\sum \|h_n\|\|k_n\| : f = \sum h_n \otimes k_n\right\} \\ &= \|f\| .\end{aligned}$$

Now let G be noncompact and H be compact. Let $S(G) = L^1(G)$, $S(H) = L^2(G)$, and let T_1, T_2 be isometric multipliers of $S(G)$, $S(H)$ where T_2 is not given by a measure. Then

(i) $G \times H$ is noncompact,

(ii) $T_1 \otimes T_2$ is an isometric multiplier of $S(G \otimes H)$, and

(iii) $T_1 \otimes T_2$ is not given by a measure [12].

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