



COMPOSITIO MATHEMATICA

Actions of higher-rank lattices on free groups

Martin R. Bridson and Richard D. Wade

Compositio Math. **147** (2011), 1573–1580.

[doi:10.1112/S0010437X11005598](https://doi.org/10.1112/S0010437X11005598)



FOUNDATION
COMPOSITIO
MATHEMATICA

*The London
Mathematical
Society*





Actions of higher-rank lattices on free groups

Martin R. Bridson and Richard D. Wade

ABSTRACT

If G is a semisimple Lie group of real rank at least two and Γ is an irreducible lattice in G , then every homomorphism from Γ to the outer automorphism group of a finitely generated free group has finite image.

1. Introduction

In recent years, a powerful body of mathematics has emerged from efforts to extend rigidity phenomena from the context of irreducible lattices in semisimple Lie groups to a wider context that embraces mapping class groups of surfaces and automorphism groups of finitely generated free groups (see [BV06, Zim87] for references). An important focus of these efforts has been the conjecture that every map from an irreducible, higher-rank lattice Γ to a mapping class group or the automorphism group of a finitely generated free group must have finite image. This was proved in the case of mapping class groups by Farb, Kaimanovich and Masur [FM98, KM96]; subsequent proofs of their result have elucidated different aspects of the geometry of mapping class groups and their subgroups [BF02, DGO, DMS10].

When Γ is non-uniform, one obtains a short proof in the mapping class group case by combining the normal subgroup theorem [Zim84, Theorem 8.1.2] with the fact that all solvable subgroups of mapping class groups are virtually abelian [BLM83, Iva92]. A similar argument, using [Ali02, BFH04] in place of [BLM83, Iva92], shows that any homomorphism from Γ to the outer automorphism group of a finitely generated free group must also factor through a finite group; see [BF01]. Our main objective in this article is the corresponding result for uniform lattices.

THEOREM A. *Let Γ be a group. Suppose that no subgroup of finite index in Γ has a normal subgroup that maps surjectively to \mathbb{Z} . Then every homomorphism from Γ to the outer automorphism group of a finitely generated free group has finite image.*

In Theorem A and the variations on it in §3, we do not assume that Γ is finitely generated.

We say that a group satisfying the hypothesis of Theorem A is \mathbb{Z} -averse. The normal subgroup theorem of Kazhdan and Margulis [Zim84] tells us that irreducible lattices in connected, higher-rank, semisimple Lie groups with finite centre have no infinite normal subgroups of infinite index. Since such lattices are not virtually cyclic, it follows that they are \mathbb{Z} -averse.

COROLLARY B. *If G is a connected, semisimple Lie group of real rank at least two that has finite centre, and Γ is an irreducible lattice in G , then every homomorphism from Γ to the outer automorphism group of a finitely generated free group has finite image.*

An additional argument allows one to remove the hypothesis that G has finite centre (see Remark 3.3). Further examples of \mathbb{Z} -averse groups come from Bader and Shalom's recent work

Received 21 September 2010, accepted in final form 17 March 2011, published online 29 July 2011.

2010 Mathematics Subject Classification 20E36, 20F65 (primary), 22E40, 20F14 (secondary).

Keywords: automorphism groups of free groups, higher-rank lattices, superrigidity.

This work is funded by the EPSRC of Great Britain. Bridson is supported by a Senior Fellowship and Wade is supported by a DTA grant.

This journal is © Foundation Compositio Mathematica 2011.

on the normal subgroup theorem [BS06]. If a hereditarily just infinite group is not virtually cyclic, then it is \mathbb{Z} -averse; examples are described in [Wil00].

Corollary B has implications for the Zimmer programme [Zim87]. Specifically, it allows one to extend Farb and Shalen's theorem about actions of higher-rank lattices on 3-manifolds to the general case, removing the non-uniform hypothesis from [FS00, Theorem II] and improving their Theorem III as follows.

THEOREM 1.1. *Let Γ be an irreducible uniform lattice in a semisimple Lie group of real rank at least two, and let M be any closed, orientable, connected 3-manifold. Then for every action $\Gamma \rightarrow \text{Homeo}(M)$ the image of Γ in $\text{Aut}(H_*(M, \mathbb{Q}))$ is finite.*

Our proof of Theorem A relies heavily on recent results of Bestvina and Feighn [BF10], Dahmani *et al.* [DGO] and Handel and Mosher [HM09]. The work of Bestvina and Feighn was inspired in part by the desire to prove Corollary B, following the lines of the proof given in the case of mapping class groups by Bestvina and Fujiwara [BF02], which invokes Burger and Monod's theorem that irreducible lattices in higher-rank Lie groups have trivial bounded cohomology [BM99]. One can replace this use of bounded cohomology with an argument of Dahmani, Guirardel and Osin that applies small cancellation theory to the study of purely pseudo-Anosov subgroups; this is used in [DGO] to prove an analogue of Theorem A for homomorphisms to mapping class groups.

Whether one uses bounded cohomology or the alternative end game from [DGO], the key step in the Bestvina–Feighn–Fujiwara approach is to get a finitely generated subgroup of $\text{Out}(F_n)$ to act in a suitable way on a hyperbolic metric space. Bestvina and Feighn [BF10] constructed such actions for subgroups of $\text{Out}(F_n)$ that contain a fully irreducible automorphism, and hence deduced that a higher-rank lattice cannot map onto such a subgroup. (Hamenstädt [Ham10] has a different way of constructing bounded cohomology classes for these subgroups.) If one could construct suitable actions for more general subgroups of $\text{Out}(F_n)$, then Theorem A would follow. A significant step in this direction was taken recently by Handel and Mosher [HM09], who proved that if a subgroup $H < \text{Out}(F_n)$ does not contain the class of a fully irreducible automorphism, then H has a subgroup of finite index that leaves the conjugacy class of a proper free factor of F invariant. Handel and Mosher also indicated that they hope to extend their work so as to prove Corollary B along the lines sketched above.

Arguments of a quite different sort allow one to see that a homomorphism from a uniform higher-rank lattice Γ to $\text{Aut}(F_n)$ cannot contain any polynomially growing automorphisms of infinite order: on the one hand, Piggott [Pig04] proved that the homomorphism $\text{Aut}(F_n) \rightarrow \text{GL}(m, \mathbb{Z})$ given by the action of $\text{Aut}(F_n)$ on the first homology of some characteristic subgroup of finite index in F_n will map a power of such an automorphism to a non-trivial unipotent; on the other hand, Margulis superrigidity implies that the image of any homomorphism $\Gamma \rightarrow \text{GL}(m, \mathbb{Z})$ can contain only semisimple elements.

Our proof of Theorem A proceeds as follows. In Proposition 2.1, we shall use the results of Handel–Mosher and Dahmani–Guirardel–Osin to see that if Γ is \mathbb{Z} -averse, then the image of every homomorphism $\Gamma \rightarrow \text{Out}(F_n)$ will have a subgroup of finite index that lies in the kernel $\overline{\text{IA}}_n$ of the map $\text{Out}(F_n) \rightarrow \text{GL}(n, \mathbb{Z})$ given by the action of $\text{Out}(F_n)$ on the first homology of F_n . In § 2.2, we use Lie methods, à la Magnus, to prove that every non-trivial subgroup of $\overline{\text{IA}}_n$ maps onto \mathbb{Z} (Corollary 2.9). (This result also appears in Bass and Lubotzky's work on central series [BL94].) As no finite index subgroup of Γ maps onto \mathbb{Z} , this completes the proof of Theorem A.

2. Proofs

We fix a \mathbb{Z} -averse group Γ . We do not assume that Γ is finitely generated.

2.1 Controlling the action of Γ on homology

PROPOSITION 2.1. *For every subgroup of finite index $\Lambda \subset \Gamma$ and every homomorphism $\phi : \Lambda \rightarrow \text{Out}(F_n)$, the intersection $\phi(\Lambda) \cap \overline{\text{IA}}_n$ has finite index in $\phi(\Lambda)$.*

Proof. The proof is by induction on n . The case $n = 1$ is trivial. $\text{Out}(F_2)$ has a free subgroup of finite index and no subgroup of finite index in Γ can map onto a free group, so every homomorphism $\Lambda \rightarrow \text{Out}(F_2)$ has finite image.

Suppose $n \geq 3$. Recall that $\psi \in \text{Aut}(F_n)$ (and its image in $\text{Out}(F_n)$) is said to be *fully irreducible* if no power of ψ sends a proper free factor of F_n to a conjugate of itself. Let $[\psi]$ denote the image of ψ in $\text{Out}(F_n)$. Using the actions constructed in [BF10] and drawing on the approach to small cancellation theory developed in [DG08], Dahmani *et al.* [DGO] proved that if ψ is fully irreducible, then, for some positive integer N , the normal closure of $[\psi]^N$ is a free group. It follows that any subgroup of $\text{Out}(F_n)$ that contains a fully irreducible automorphism also contains an infinite normal subgroup that is free. In particular, $\phi(\Lambda)$ cannot contain a fully irreducible automorphism.

According to [HM09, Theorem 1.1], if $\phi(\Lambda)$ does not contain a fully irreducible automorphism, then a subgroup of finite index $H \subset \phi(\Lambda)$ leaves a free factor of F_n invariant up to conjugacy; say $F_n = L * L'$, where $\psi(L) = g_\psi^{-1} L g_\psi$ for all $[\psi] \in H$. Note that the image in $\text{Out}(L)$ of $x \mapsto g_\psi \psi(x) g_\psi^{-1}$, which we denote $[\psi]_L$, depends only on the image of ψ in $\text{Out}(F_n)$, and that $[\psi] \mapsto [\psi]_L$ defines a homomorphism from H to $\text{Out}(L)$. Likewise, the action on the quotient $F_n / \langle\langle L \rangle\rangle$ induces a homomorphism $H \rightarrow \text{Out}(L')$. By induction, we know that the induced action of H on the abelianization of both L and L' factors through a finite group. Thus, the action of H on the abelianization of $F_n = L * L'$ lies in a block triangular subgroup (with respect to a basis that is the union of bases for L and L')

$$\begin{pmatrix} G & 0 \\ * & G' \end{pmatrix} \leq \text{GL}(n, \mathbb{Z}),$$

where G and G' are finite. This matrix group is finitely generated and virtually abelian, whereas Γ , and therefore H , does not have a subgroup of finite index that maps onto \mathbb{Z} . Thus, the action of H on the homology of F_n factors through a finite group, and hence that of $\phi(\Lambda)$ does too, i.e. $\phi(\Lambda) \cap \overline{\text{IA}}_n$ has finite index in $\phi(\Lambda)$. This completes the induction. \square

2.2 Central filtrations of IA_n and $\overline{\text{IA}}_n$

Let γ_c be the c th term in the lower central series of F_n ; so $\gamma_1 = F_n$ and $\gamma_c = [\gamma_{c-1}, F_n]$. Let $\Gamma_c = F_n / \gamma_c$. As γ_c is characteristic, there is a natural map $\text{Aut}(F_n) \rightarrow \text{Aut}(\Gamma_c)$. Let G_{c-1} be the kernel of this map. Note that $G_0 = \text{Aut}(F_n)$ and $G_1 = \text{IA}_n$. Magnus [Mag35] showed that $\bigcap_{i=1}^\infty \gamma_i = 0$; it follows that $\bigcap_{i=0}^\infty G_i = 0$. In fact, Andreadakis proved that G_1, G_2, G_3, \dots forms a central series for G_1 with each quotient a finitely generated free abelian group [And65]. It now seems natural to regard his result in the context of *higher Johnson homomorphisms* [Day09, Mor93]. We include a proof for the convenience of the reader. Let $L_c = \gamma_c / \gamma_{c+1}$. Our commutator convention is $[x, y] := x^{-1} y^{-1} x y$.

PROPOSITION 2.2. *For $c \geq 1$, there exists a homomorphism*

$$\tau_c : G_c \rightarrow \text{Hom}(H_1(F_n), L_{c+1})$$

such that $\ker(\tau_c) = G_{c+1}$.

Proof. For all $\psi \in G_c$ and $x \in F_n$, there exists $w_x \in \gamma_{c+1}$ with $\psi(x) = xw_x$. Define $\tau_c(\psi)([x]) = w_x\gamma_{c+2}$. Note that if $x \in \gamma_2 = [F_n, F_n]$, then

$$\psi(x)\gamma_{c+2} = x\gamma_{c+2}. \tag{1}$$

Indeed, the commutator relations $[x, yz] = [x, z][z, [y, x]][x, y]$ and $[xy, z] = [x, z][[x, z], y][y, z]$ imply

$$\begin{aligned} \psi([x, y]) &= [\psi(x), \psi(y)] \\ &= [xw_x, yw_y] \\ &= [xw_x, w_y][w_y, [y, xw_x]][xw_x, y] \\ &= [xw_x, w_y][w_y, [y, xw_x]][x, y][[x, y], w_x][w_x, y] \end{aligned}$$

and $[xw_x, w_y], [w_y, [y, xw_x]], [[x, y], w_x], [w_x, y]$ all lie in γ_{c+2} . It follows easily that τ_c is well defined and a homomorphism. The automorphism ψ belongs to G_{c+1} if and only if w_x is contained in γ_{c+2} for all x in F_n ; hence, $\ker \tau_c = G_{c+1}$. \square

Proposition 2.2 shows that G_c/G_{c+1} is isomorphic to a subgroup of the group $\text{Hom}(H_1(F_n), L_{c+1})$. The abelian group L_{c+1} is a subgroup of the finitely generated free nilpotent group Γ_{c+2} ; thus, it is finitely generated and free abelian.

COROLLARY 2.3. *If $c \geq 1$, then G_c/G_{c+1} is a finitely generated free abelian group.*

If S is a non-trivial subgroup of $\text{IA}_n = G_1$, then there exists $c \geq 1$ such that $S \leq G_c$ and $S \not\leq G_{c+1}$.

COROLLARY 2.4. *Every non-trivial subgroup of IA_n maps onto \mathbb{Z} .*

We would like to extend this analysis to subgroups of $\overline{\text{IA}}_n$. Let H_c be the image of G_c under the projection $\pi : \text{Aut}(F_n) \rightarrow \text{Out}(F_n)$. Our goal in the remainder of this section is to prove the analogue of Corollary 2.3 for H_1, H_2, \dots (cf. [BL94]). We make use of a theorem of Magnus that $L = \bigoplus_{c=1}^\infty L_c$, along with the bracket operation induced by commutation in F_n , is a *free Lie \mathbb{Z} -algebra* generated by the images in $H_1(F_n) = L_1$ of a basis for the free group. This theorem and the required background on free Lie algebras are explained in [MKS66, ch. 5]. Let p be a positive integer and let $(L)_p$ be the free Lie algebra obtained by taking the tensor product of $\mathbb{Z}/p\mathbb{Z}$ with L . Let L^c be the quotient algebra $L/\bigoplus_{i>c} L_i$. We will need the following fact, whose proof is sketched in [Bou89, Exercise 3.3, ch. 2]. Throughout, $Z(A)$ denotes the centre of a Lie algebra A .

PROPOSITION 2.5. *If $n \geq 2$, then $Z(L) = Z((L)_p) = 0$ and $Z(L^c)$ is the image of L_c under the quotient map $L \rightarrow L^c$.*

As $\overline{\text{IA}}_1$ is trivial, we restrict ourselves to $n \geq 2$ for the remainder of this section. For each non-trivial $y \in F_n$, there exists a unique c such that $y \in \gamma_c$ and $y \notin \gamma_{c+1}$. We identify y with its (non-trivial) image in the submodule L_c of the Lie algebra L . Let $\text{ad} : F_n \rightarrow \text{Aut}(F_n)$ be the map induced by the action of F_n on itself by conjugation.

COROLLARY 2.6. *The kernel of the map $F_n \rightarrow \text{Aut}(\Gamma_{c+1})$ induced by ad is γ_c . Hence, $\text{ad}(y)$ belongs to G_c if and only if y is in γ_c .*

Proof. If $y \in \gamma_c$, then $x^{-1}y^{-1}xy$ lies in γ_{c+1} for each $x \in F_n$; therefore, $\text{ad}(y)$ lies in G_c . Conversely, if y is not in γ_c , then, by Proposition 2.5, its image in L^c is not central. As L^c is generated by the images of a basis x_1, \dots, x_n for F_n , this means that some $[x_i, y] \neq 0$ in L^c ; hence, $x_i^{-1}y^{-1}x_iy \notin \gamma_{c+1}$. \square

THEOREM 2.7. For $c \geq 1$, the sequence

$$0 \rightarrow \gamma_c/\gamma_{c+1} \rightarrow G_c/G_{c+1} \rightarrow H_c/H_{c+1} \rightarrow 0$$

is an exact sequence of free abelian groups, where the second and third maps are induced by ad and π , respectively.

Proof. Surjectivity of the map $G_c/G_{c+1} \rightarrow H_c/H_{c+1}$ is trivial, and exactness of the remaining maps follows from Corollary 2.6. It remains to show that $H_c/H_{c+1} \cong (G_c/G_{c+1})/\text{ad}(\gamma_c)$ is torsion free. Suppose that $\psi \in G_c$ and there exist $y \in \gamma_c$ and $p \geq 1$ such that $\text{ad}(y)G_{c+1} = \psi^p G_{c+1}$. For each x_i in a free generating set of F_n , we have

$$\psi(x_i) = x_i w_i$$

for some w_i in γ_{c+1} . Equation (1) in Proposition 2.2 tells us that $\psi(w_i)\gamma_{c+2} = w_i\gamma_{c+2}$; therefore,

$$\psi(x_i)^p \gamma_{c+2} = x_i w_i^p \gamma_{c+2}.$$

As $\text{ad}(y)G_{c+1} = \psi^p G_{c+1}$, this rearranges to

$$x_i^{-1} y^{-1} x_i y \gamma_{c+2} = w_i^p \gamma_{c+2};$$

therefore, $[x_i, y] = 0$ in the associated Lie algebra $(L)_p$. However, $(L)_p$ is generated by the images of x_1, \dots, x_n , so the image of y lies in $Z((L)_p) = 0$. It follows that y lies in the kernel pL of the map $L \rightarrow (L)_p$. Hence, there exists y_0 in γ_c such that $y\gamma_{c+1} = y_0^p \gamma_{c+1}$ and $\text{ad}(y_0)^p G_{c+1} = \psi^p G_{c+1}$. But, G_c/G_{c+1} is free abelian, so $\text{ad}(y_0)G_{c+1} = \psi G_{c+1}$. \square

The final fact we need concerning $\text{Out}(F_n)$ is well known.

LEMMA 2.8. The intersection $\bigcap_{i=1}^{\infty} H_i$ is trivial in $\text{Out}(F_n)$.

Proof. By [LS77, Proposition I.4.9], if $\psi(u)$ is conjugate to u in Γ_c for all c , then $\psi(u)$ is conjugate to u in F_n . And, if ψ takes every element of F_n to a conjugate of itself, then ψ is inner [Gro74, Lemma 1]. \square

It follows that if S is a non-trivial subgroup of $\overline{\Gamma}_n = H_1$, then there exists $c \geq 1$ such that $S \leq H_c$ and $S \not\leq H_{c+1}$. In Theorem 2.7, we saw that for $c \geq 1$ the quotient H_c/H_{c+1} is a finitely generated free abelian group.

COROLLARY 2.9. Every non-trivial subgroup of $\overline{\Gamma}_n$ maps onto \mathbb{Z} .

As we explained at the end of the introduction, this completes the proof of Theorem A.

3. Alternative hypotheses

It emerges from the proofs in the previous section that one can weaken the hypotheses of Theorem A as follows. Note that since we have not assumed Γ to be finitely generated, condition (1) is not equivalent to assuming that every finite index subgroup of Γ has finite abelianization.

THEOREM 3.1. Let Γ be a group. Suppose that each finite-index subgroup $\Lambda \subset \Gamma$ satisfies the following conditions:

- (1) Λ does not map surjectively to \mathbb{Z} ;
- (2) Λ does not have a quotient containing a non-abelian, normal, free subgroup.

Then every homomorphism $\Gamma \rightarrow \text{Out}(F_n)$ has finite image.

Proof. The only additional argument that is needed concerns the normal closure I of $[\psi]^N$ in $\text{Out}(F_n)$, as considered in the second paragraph of the proof of Proposition 2.1. We must exclude

the possibility that the intersection of I with the image of $\phi : \Lambda \rightarrow \text{Out}(F_n)$ is cyclic, generated by $[\psi]^m$ say. But, if this were the case, $\langle [\psi]^m \rangle$ would be normal in $\phi(\Lambda)$. Since the normalizer in $\text{Out}(F_n)$ of the subgroup generated by any fully irreducible element is virtually cyclic [BFH97], it would follow that $\phi(\Lambda)$ itself was virtually cyclic, contradicting the fact that no finite-index subgroup of Λ maps onto \mathbb{Z} . \square

Remark 3.2. The class of groups that satisfy the hypotheses of Theorem 3.1 is closed under certain extension operations. For example, if $1 \rightarrow A \rightarrow \hat{\Gamma} \rightarrow \Gamma \rightarrow 1$ is a short exact sequence and, in the notation of Theorem 3.1, we suppose that:

- every finite-index subgroup of A satisfies (2);
- every finite-index subgroup of $\hat{\Gamma}$ satisfies (1); and
- every finite-index subgroup of Γ satisfies both (1) and (2),

then an elementary argument shows that every finite-index subgroup of $\hat{\Gamma}$ satisfies (2). (Hence, every homomorphism $\hat{\Gamma} \rightarrow \text{Out}(F_n)$ has finite image.)

Remark 3.3. Let \hat{G} be an arbitrary semisimple Lie group of real rank at least two, with centre $Z(\hat{G})$. Let $\hat{\Gamma} < \hat{G}$ be an irreducible lattice, let $A = \hat{\Gamma} \cap Z(\hat{G})$ and let $\Gamma = \hat{\Gamma}/A$. The abelianization of any subgroup of finite index in $\hat{\Gamma}$ is finite [Mar91, p. 333] and Γ is an irreducible lattice in the centreless semisimple Lie group $G = \hat{G}/Z(\hat{G})$. Thus, the above remark allows us to remove from Corollary B the hypothesis that the centre of G is finite.

In Theorem 3.1, condition (2) is used only to exclude the possibility that a homomorphic image of Γ in $\text{Out}(F_n)$ might contain a fully irreducible element. An alternative way of ruling out such images is to use bounded cohomology, as in [BF10]. We briefly review some notation. A map $f : \Gamma \rightarrow \mathbb{R}$ is a *quasi-homomorphism* if the function

$$(g, h) \mapsto |f(g) + f(h) - f(gh)|$$

is bounded on $\Gamma \times \Gamma$. Let $V(\Gamma)$ be the vector space of all quasi-homomorphisms from Γ to \mathbb{R} . Two natural subspaces of $V(\Gamma)$ are $B(\Gamma)$, the vector space of bounded maps from Γ to \mathbb{R} , and $\text{Hom}(\Gamma; \mathbb{R})$, the vector space of genuine homomorphisms. Define $\widetilde{\text{QH}}(\Gamma) = V(\Gamma)/(B(\Gamma) + \text{Hom}(\Gamma; \mathbb{R}))$.

THEOREM 3.4. *Let Γ be a group. Suppose that for every finite index subgroup $\Lambda \subset \Gamma$ the second bounded cohomology of Λ is finite dimensional and $\text{Hom}(\Lambda, \mathbb{R}) = 0$. Then every homomorphism $\phi : \Gamma \rightarrow \text{Out}(F_n)$ has finite image.*

Proof. Bestvina and Feighn [BF10] showed that if $H < \text{Out}(F_n)$ contains a fully irreducible automorphism, then either H is virtually cyclic or $\widetilde{\text{QH}}(H)$ is infinite dimensional. If $\text{Hom}(\Lambda; \mathbb{R}) = 0$, then a surjective map $\Lambda \rightarrow H$ induces an injection $\widetilde{\text{QH}}(H) \rightarrow \widetilde{\text{QH}}(\Lambda)$. The vector space $\widetilde{\text{QH}}(\Lambda)$ injects into the second bounded cohomology of Λ (see [Mon06]). Therefore, for all finite-index subgroups $\Lambda \subset \Gamma$ and integers m the image of a homomorphism $\Lambda \rightarrow \text{Out}(F_m)$ cannot contain a fully irreducible automorphism. It follows from Corollary 2.9 and the arguments in Proposition 2.1 that $\phi(\Gamma)$ is finite. \square

In the light of [BM99], this alternative to Theorem A also implies Corollary B.

ACKNOWLEDGEMENT

We thank Laurent Bartholdi for a helpful conversation concerning central filtrations.

REFERENCES

- Ali02 E. Alibegović, *Translation lengths in $\text{Out}(F_n)$* , *Geom. Dedicata* **92** (2002), 87–93.
- And65 S. Andreadakis, *On the automorphisms of free groups and free nilpotent groups*, *Proc. Lond. Math. Soc.* (3) **15** (1965), 239–268.
- BS06 U. Bader and Y. Shalom, *Factor and normal subgroup theorems for lattices in products of groups*, *Invent. Math.* **163** (2006), 415–454.
- BL94 H. Bass and A. Lubotzky, *Linear-central filtrations on groups*, in *The mathematical legacy of Wilhelm Magnus: groups, geometry and special functions (Brooklyn, NY, 1992)*, *Contemporary Mathematics*, vol. 169 (American Mathematical Society, Providence, RI, 1994), 45–98.
- BF10 M. Bestvina and M. Feighn, *A hyperbolic $\text{Out}(F_n)$ -complex*, *Groups Geom. Dyn.* **4** (2010), 31–58.
- BFH97 M. Bestvina, M. Feighn and M. Handel, *Laminations, trees, and irreducible automorphisms of free groups*, *Geom. Funct. Anal. (GAFA)* **7** (1997), 215–244, 1143.
- BFH04 M. Bestvina, M. Feighn and M. Handel, *Solvable subgroups of $\text{Out}(F_n)$ are virtually abelian*, *Geom. Dedicata* **104** (2004), 71–96.
- BF02 M. Bestvina and K. Fujiwara, *Bounded cohomology of subgroups of mapping class groups*, *Geom. Topol.* **6** (2002), 69–89.
- BLM83 J. S. Birman, A. Lubotzky and J. McCarthy, *Abelian and solvable subgroups of the mapping class groups*, *Duke Math. J.* **50** (1983), 1107–1120.
- Bou89 N. Bourbaki, *Lie groups and Lie algebras*, in *Elements of mathematics (Berlin)* (Springer, Berlin, 1989), Chapters 1–3.
- BF01 M. R. Bridson and B. Farb, *A remark about actions of lattices on free groups*, *Topology Appl.* **110** (2001), 21–24.
- BV06 M. R. Bridson and K. Vogtmann, *Automorphism groups of free groups, surface groups and free abelian groups*, in *Problems on mapping class groups and related topics*, *Proceedings of Symposia in Pure Mathematics*, vol. 74, ed. B. Farb (American Mathematical Society, Providence RI, 2006), 301–316.
- BM99 M. Burger and N. Monod, *Bounded cohomology of lattices in higher rank Lie groups*, *J. Eur. Math. Soc. (JEMS)* **1** (1999), 199–235.
- DGO F. Dahmani, V. Guirardel and D. Osin, *Hyperbolic embeddings and rotating families in groups acting on hyperbolic spaces*, Preliminary version, www-fourier.ujf-grenoble.fr/~dahmani/Files/VRF.pdf.
- Day09 M. B. Day, *Extensions of Johnson’s and Morita’s homomorphisms that map to finitely generated abelian groups*, Preprint (2009), arXiv:0910.4777.
- DG08 T. Delzant and M. Gromov, *Courbure mésoscopique et théorie de la toute petite simplification*, *J. Topol.* **1** (2008), 804–836.
- DMS10 C. Drutu, S. Mozes and M. Sapir, *Divergence in lattices in semisimple Lie groups and graphs of groups*, *Trans. Amer. Math. Soc.* **362** (2010), 2451–2505.
- FM98 B. Farb and H. Masur, *Superrigidity and mapping class groups*, *Topology* **37** (1998), 1169–1176.
- FS00 B. Farb and P. Shalen, *Lattice actions, 3-manifolds and homology*, *Topology* **39** (2000), 573–587.
- Gro74 E. K. Grossman, *On the residual finiteness of certain mapping class groups*, *J. Lond. Math. Soc.* (2) **9** (1974), 160–164.
- Ham10 U. Hamenstädt, *Lines of minima in outer space*, Preprint (2010), arXiv:0911.3620.
- HM09 M. Handel and L. Mosher, *Subgroup classification in $\text{Out}(F_n)$* , Preprint (2009), arXiv:0908.1255.
- Iva92 N. V. Ivanov, *Subgroups of Teichmüller modular groups*, *Translations of Mathematical Monographs*, vol. 115 (American Mathematical Society, Providence, RI, 1992).
- KM96 V. A. Kaimanovich and H. Masur, *The Poisson boundary of the mapping class group*, *Invent. Math.* **125** (1996), 221–264.

- LS77 R. C. Lyndon and P. E. Schupp, *Combinatorial group theory* (Springer, Berlin, 1977).
- Mag35 W. Magnus, *Beziehungen zwischen Gruppen und Idealen in einem speziellen Ring*, Math. Ann. **111** (1935), 259–280.
- MKS66 W. Magnus, A. Karrass and D. Solitar, *Combinatorial group theory: presentations of groups in terms of generators and relations* (Interscience Publishers (John Wiley), New York–London–Sydney, 1966).
- Mar91 G. A. Margulis, *Discrete subgroups of semisimple Lie groups* (Springer, Berlin, 1991).
- Mon06 N. Monod, *An invitation to bounded cohomology*, in *International congress of mathematicians, Vol. II* (European Mathematical Society, Zürich, 2006), 1183–1211.
- Mor93 S. Morita, *The extension of Johnson’s homomorphism from the Torelli group to the mapping class group*, Invent. Math. **111** (1993), 197–224.
- Pig04 A. Piggott, *The topology of finite graphs, recognition and the growth of free-group automorphisms* DPhil thesis, University of Oxford (2004).
- Wil00 J. S. Wilson, *On just infinite abstract and profinite groups*, in *New horizons in pro-p groups*, Progress in Mathematics, vol. 184 (Birkhäuser, Boston, MA, 2000), 181–203.
- Zim84 R. J. Zimmer, *Ergodic theory and semisimple groups*, Monographs in Mathematics, vol. 81 (Birkhäuser, Basel, 1984).
- Zim87 R. J. Zimmer, *Actions of semisimple groups and discrete subgroups*, in *Proceedings of the international congress of mathematicians, Vols. 1, 2 (Berkeley, CA, 1986)* (American Mathematical Society, Providence, RI, 1987), 1247–1258.

Martin R. Bridson bridson@maths.ox.ac.uk
 Mathematical Institute, 24–29 St Giles’, Oxford OX1 3LB, UK

Richard D. Wade wade@maths.ox.ac.uk
 Mathematical Institute, 24–29 St Giles’, Oxford OX1 3LB, UK