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A supernilpotent non-special radical class

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Let F be the upper radical determined by all fields. The supernilpotent radical classes which are not special have thus far always contained F properly. The purpose of this note is to construct a countably infinite number of supernilpotent radical classes which are not special and each of which is properly contained in F. The construction involves a ring due to Leavitt which is interesting in its own right and is not generally known. All rings considered are associative.

1.

Let F be the upper radical determined by all fields. The supernilpotent radical classes which are not special (see [2] and [3]) have thus far always contained F properly. The purpose of this note is to construct a countably infinite number of supernilpotent radical classes which are not special and each of which is properly contained in F. The construction involves a ring due to Leavitt which is interesting in its own right and is not generally known. All rings considered will be associative and standard radical theory terminology can be found in [1].

If M is a hereditary class of rings we will let UM denote the upper radical class determined by M and SUM the corresponding semisimple class. Recall that a ring A is in SUM if and only if every nonzero ideal of A has a nonzero homomorphic image in M. In addition, β and G will denote the lower Baer and Brown-McCoy radicals respectively.

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Let UM be a supernilpotent radical class determined by the special class M. There are several conditions on UM which are known to be equivalent to the property that UM is a special supernilpotent radical class. We will need the following equivalence.

LEMMA 1. *If* UM *is a supernilpotent radical class then* UM *is special if and only if in every* UM*-semisimple ring* A *there exists a proper prime ideal* P *such that* A/P *is* UM*-semisimple.*

Proof. If UM is a special radical class then every UM-semisimple ring A is isomorphic to a subdirect sum of rings from the special class M [1, Lemma 80]. Each nonzero summand is of the form A/P where P is a proper prime ideal of A. Conversely, let N be the class of all prime UM-semisimple rings. Since UM is supernilpotent, N is a special class [1, Lemma 85] and by definition of an upper radical $UM \subseteq UN$ and hence $SUN \subseteq SUM$. Now let $A \in SUM$ and I be a nonzero ideal of A. Then $I \in SUM$ and by our assumption I has a nonzero prime UM-semisimple image. Thus, by definition $A \in SUN$. Hence $SUN = SUM$ and UM is a special radical class.

DEFINITION. A class W is a *weakly special class* of rings if it satisfies the following three conditions:

(1) every ring in the class W is semiprime;

(2) W is hereditary;

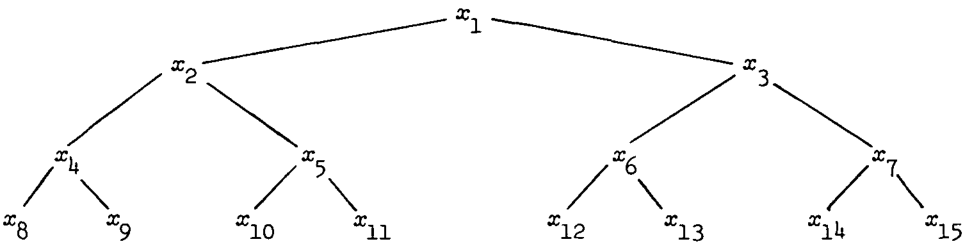
(3) if $A \in W$ and A is an ideal in a ring K where $A^* = 0$, then $K \in W$.

Here, A^* is the annihilator of A in K.

In [2], Rjabuhin shows that any upper radical determined by a weakly special class is supernilpotent. He then constructs a supernilpotent non-special radical class containing F properly. Snider [3] constructs another such class.

2.

With the aid of a ring due to Leavitt we are able to construct a supernilpotent radical class which is not special yet is contained properly in F. We first give the properties of this ring, call it A, which are best described by a diagram as follows:



The ring A is to be generated over Z_2, the field of order 2, by the countably infinite set of symbols $\{x_i\}$, with the following relations: any x_i is the sum of the two immediately below (thus $x_2 = x_4 + x_5$); any x_i is the identity for all those x_j equal to or below it but has zero product with incomparable x_j (thus $x_2x_9 = x_9x_2 = x_9$ but $x_2x_7 = x_7x_2 = 0$). By (x_i), below, we mean the set of x_j which are equal to x_i or below and comparable to x_i. Some of the properties of the ring A are:

(a) $A = (x_1)$ is isomorphic to the principal ideals, such as (x_5);

(b) every ideal of A is the direct sum of such principal ideals;

(c) the only prime image of A is Z_2 which you would get, say, by A/I where $I = (x_2, x_6, x_{14}, \ldots)$;

(d) the only prime ideals of A are maximal ideals;

(e) A has no nonzero nilpotent ideals and so is β-semisimple;

(f) the intersection of all maximal ideals is zero;

(g) A is a boolean ring with the identity x_1;

(h) A does not contain Z_2 as an ideal, for every ideal of A is countably infinite.

THEOREM 1. *Let S be a simple ring with unit which is not a field. Let T be a class of rings A satisfying the following* *conditions:*

(a) A *is a subdirect sum of copies of* S ;

(b) S *is not an ideal in* A .

Let F *be the class of all fields. Then if* $M = F \cup T$, M *is a weakly special class.*

Proof. Every ring in M is G-semisimple and hence β-semisimple and thus semiprime. To show that M is hereditary let $a \in M$ and $I \neq 0$ be an ideal of A . Since A is a subdirect sum of copies of S , A contains a class of maximal ideals $\{M_i\}$ such that $\cap M_i = 0$ and $A/M_i \cong S$ for each i . Hence $I \cap M_i$ is a prime ideal in I for each i and $\cap(I \cap M_i) = 0$. Thus I is a subdirect sum of rings $I/(I \cap M_i) \cong (M_i+I)/M_i$. It is clear that $I \not\subseteq M_i$ for at least one i and in this event $(M_i+I)/M_i \cong A/M_i \cong S$. On the other hand, if $I \subseteq M_i$ then $(M_i+I)/M_i = 0$. Hence I is a subdirect sum of copies of S . If S was an ideal in I then $0 \neq (S')^3 \subset S \subset S' \subset I \subset A$ where S' denotes the ideal of A generated by S . But S is simple so $(S')^3 = S$ and S would be an ide˙ of A . Hence M is hereditary.

Now let $B \in M$ where B is an ideal in the ring A with $B^* = 0$. If $B \in F$ then $A = B + C$ and $C \subset B^* = 0$. Hence $A = B \in M$. Now suppose $B \in T$. Then B contains a class of maximal prime ideals $\{P_i\}$ with $\cap P_i = 0$ and $B/P_i \cong S$ for each i . Define for each P_i , $P_i^* = \{x \in A \mid Bx \subseteq P_i\}$. Then P_i^* is an ideal in A and we claim that $P_i = P_i^* \cap B$ for each i . It is clear that $P_i \subseteq P_i^* \cap B$ so let $b \in P_i^* \cap B$. Then $Bb \subseteq P_i$, and hence $B(b)_r \subseteq P_i$, where $(b)_r$ denotes the right ideal of B generated by b . Therefore $(b)_r^2 \subseteq P_i$, which implies $(b)_r \subseteq P_i$, for P_i is prime in B , and hence $b \in P_i$. Now we show $A = P_i^* + B$ for each i . Let $e + P_i$ be the identity in B/P_i , $e \in B$. Then $ye = y$ (P_i) for any $y \in B$, so $yex = yx$ (P_i) for any

$y \in B$ and any $x \in A$. But then $y(x-ex) = yx - yex \equiv yx - yx \ (P_i)$, so $y(x-ex) \in P_i$ for any $y \in B$, $x \in A$. This implies that $x - ex \in P_i^*$ for any $x \in A$. From $x = (x-ex) + ex$ with $x - ex \in P_i^*$ and $ex \in B$ it follows that $A = P_i^* + B$. Hence $A/P_i^* = (B+P_i^*)/P_i^* \cong B/(B \cap P_i^*) = B/P_i \cong S$. Also $\cap P_i = B \cap (\cap P_i^*) = 0$ which says $\cap P_i^* \subseteq B^* = 0$, so $\cap P_i^* = 0$. Hence A is a subdirect sum of copies of S.

Finally, suppose that S is an ideal in A. Since $B \in M$, S is not an ideal of B so $S \cap B = 0$. Thus $SB = BS = 0$ or $S \subseteq B^* = 0$, a contradiction. Hence S is not an ideal of A and $A \in T \subset M$ completing the proof.

By choosing an appropriate simple ring with unit we show in the following lemma that the class T of the preceding theorem is non-empty. It is here that the ring of Leavitt is used. The matrix theory used in the proof is an elementary consequence of using a ring with unit.

LEMMA 2. *Let $S = (Z_2)_n$, the ring of $n \times n$ matrices over the field Z_2. The class T of rings which are subdirect sums of copies of S and which do not contain S as an ideal is non-empty.*

Proof. The ring S is a simple ring with unit but not a field. Where A denotes Leavitt's ring, we show that $A_n \in T$. The ring A has a unit, so every ideal in A_n has the form I_n where I is an ideal in A. First we show that if I_n is a prime ideal in A_n, then $A_n/I_n \cong S$. Since $A_n/I_n \cong (A/I)_n$ it follows that $(A/I)_n$ is a prime ring and hence A/I is a prime ring. The only prime image of A is Z_2 so $A/I \cong Z_2$ and $A_n/I_n \cong (A/I)_n \cong (Z_2)_n = S$. This also shows that every prime ideal in A_n is maximal.

Since $\beta(A) = 0$, $\beta(A_n) = (\beta(A))_n = 0$. Hence A_n is a subdirect sum of prime rings, that is, there exists a class $\{(P_i)_n\}$ of prime ideals in A_n with $\cap(P_i)_n = 0$. From the above we get $A_n/(P_i)_n \cong S$ for any $(P_i)_n$. Hence A_n is a subdirect sum of copies of S. As Z_2 is not an

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ideal in A, $(Z_2)_n = S$ is not an ideal in A_n. Hence $A_n \in T$ and T is non-empty.

The class $M = T \cup F$ is a weakly special class and hence UM is supernilpotent [2]. We show that UM is not special by contradicting the equivalence of Lemma 1.

THEOREM 2. *Let T be the class of all subdirect sums of copies of $(Z_2)_n$ not containing $(Z_2)_n$ as an ideal and let F be the class of all fields. Then the upper radical class UM determined by $M = T \cup F$ is supernilpotent, not special, and $UM \subset F$.*

Proof. There only remains to show that UM is not special. $A_n \in T$ so A_n is UM-semisimple. The only prime image of A_n is $(Z_2)_n$. Since $(Z_2)_n$ is simple and not in M, $(Z_2)_n$ must be UM-radical. Hence for A_n there does not exist a proper prime ideal P_n such that A_n/P_n is UM-semisimple and hence UM is not special by Lemma 1.

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