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A NOTE ON p-CENTRAL GROUPS

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Abstract. A group G is *n-central* if $G^n \leq Z(G)$, that is the subgroup of G generated by n-powers of G lies in the centre of G. We investigate p^k-central groups for p a prime number. For G a finite group of exponent p^k, the covering group of G is p^k-central. Using this we show that the exponent of the Schur multiplier of G is bounded by $p^{\lceil \frac{c}{p-1} \rceil}$, where c is the nilpotency class of G. Next we give an explicit bound for the order of a finite p^k-central p-group of coclass r. Lastly, we establish that for G, a finite p-central p-group, and N, a proper non-maximal normal subgroup of G, the Tate cohomology $H^n(G/N, Z(N))$ is non-trivial for all n. This final statement answers a question of Schmid concerning groups with non-trivial Tate cohomology.

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1. Introduction. In 1970 Gupta and Rhemtulla introduced the notion of an n-central group which generalises both the notions of abelian and exponent n [**7**]. Let G be a group and n a natural number. Denote the centre of G by $Z(G)$ and the subgroup of G generated by nth-powers of elements of G by G^n.

Definition 1. A group G is *n-central* if $G^n \leq Z(G)$.

Clearly, a group G is n-central if and only if it satisfies the word $[x^n, y] = 1$ for all elements x and y in G. Thus, the n-central groups form a variety. (We note that some authors have used the term p-central to mean that all elements of order p in a finite p-group are central, this is a very different condition.)

Moravec [**15**, Theorem 2.5] has proved that for G a finitely generated soluble group of derived length d, then G is n-central if and only if G is isomorphic to a subgroup of the direct product of a finite soluble n-central group of derived length at most d and a free abelian group of finite rank. We are interested in p^k-central groups for p a prime number. Clearly, a finite p^k-central group is nilpotent and so, is a finite p-group modulo an abelian direct factor. Thus, we restrict our attention to finite p^k-central p-groups.

Several related concepts have been studied by authors, we recall a few of them. A group is said to be n-abelian if $(xy)^n = x^n y^n$ for all $x, y \in G$. It is easy to see that in an n-abelian group $[x^n, y] = [x, y]^n = [x^n, y^n] = [x, y]^{n^2}$ for all $x, y \in G$. Thus, a p^k-abelian p-group is p^k-central. Indeed, n-abelian groups have been classified by Alperin [**2**]: the variety of n-abelian groups is the join of the varieties of abelian groups, groups of exponent dividing n and groups of exponent dividing $n - 1$. More general than an

n-central group is an n-Bell group, that is one which satisfies the identity $[x^n, y] = [x, y^n]$ for all $x, y \in G$.

With the exception of recent papers of Moravec [**15–17**], Mann [**14**] and Thillaisundaram [**23**], it seems that little work has been done on n-central groups, with results often only occurring as a byproduct of results on other classes of groups. One such example is the result by Kappe and Morse [**12**, Theorem 13], which shows that a metabelian p-group G is p-central if and only if the exponent of the derived group of G divides p and G has nilpotency class at most p. In [**15**, Theorem 1.3] Moravec proves that the assumption that G is a p-group can be dropped. In the same paper Moravec classifies all finitely generated 2-central groups [**15**, Theorem 2.7] (finite 2-central groups had previously been classified [**6**]).

It is worth noting that for p odd all p-groups of order of at most p^4 are p-central. This is clear for groups of order $\leq p^3$. For groups of order p^4, by the result of Kappe and Morse mentioned above [**12**, Theorem 13], we just need to consider groups of nilpotency class 3, this case is covered in Proposition 2. For $p = 2$ the dihedral group of order 16 gives a group of order 2^4, which is not 2-central. The following presentation gives, for all primes p, a p-group of order p^5 that is not p-central

$$\langle x, y : x^{p^3} = 1 = y^{p^2}, y^{-1}xy = x^{(1+p)} \rangle.$$

Recall, the Nottingham group is a finitely generated pro-p group in which the p-powers in the group drop quickly down the lower central series, for details see [**3**]. Thus, it is not surprising that certain finite quotients of the Nottingham group give examples of p-central groups, for details see [**22**]. It is also interesting to note that p-groups with only one non-central conjugacy class size are p-central [**10**].

In this paper we consider three different aspects of p-central groups. The study of p^k-central groups is a natural setting to study the Schur Multiplier of a finite group of exponent p^k. The Schur Multiplier $M(G)$ of group G is given by the second cohomology group $H^2(G, \mathbb{C}^*)$. When G is finite $M(G)$ is also given by the second integral homology group $H_2(G, \mathbb{Z})$. In Schur's pioneering work at the beginning of the last century he proved that all groups have a covering group: H is a covering group of group G if H has a subgroup A isomorphic to $M(G)$ which satisfies $A \leq H' \cap Z(H)$ and $G \cong H/A$. So the covering group of a group of exponent p^k is a p^k-central group. In the next section we study the interplay between the p-power structure and the commutator structure of a p-central group. This leads to the following theorem about the exponent of the Schur multiplier of a finite group of exponent p.

THEOREM 1. *Let G be a finite group of exponent p and nilpotency class c. Then the exponent of $M(G)$ is bounded by $p^{\lceil \frac{c}{p-1} \rceil}$.*

This compares favourably with known results of Ellis [**5**] and Moravec [**16**] when p is large in comparison to the nilpotency class of the group. We note that a finite non-cyclic group of exponent p has non-trivial mutiplier [**13**, Corollary 3.4.11].

In the second section we consider p-central groups by coclass. A finite p-group of order p^n and nilpotency class c has coclass $n - c$, this invariant was introduced by Leedham-Green and Newman and suggests an interesting way to investigate p-groups. That a finite p^k-central p-group of coclass r has bounded order can be proven in a variety of ways. We use coclass theory to give an explicit bound, which, although not optimal, seems good.

THEOREM 2. *Let G be a finite p^k-central p-group of coclass r. Then the order of G is bounded by $p^{f(k,p,r)}$ where $f(k,p,r)$ is equal to $(k+1)(p-1)p^{r-1}+r$ when p is odd and $k \geq 2$, and $f(1,p,r)=2p^r+r-1$ when p odd. When p is even $f(k,2,r)=(2+k)2^{r+1}+r$ when $k\geq 2$ and $f(1,2,r)=2^{r+3}+r-1$.*

An interesting link between Schur Multipliers and coclass is given by Eick [**4**]. She proves that for an odd prime p there are at most finitely many p-groups G of coclass r with $|M(G)|\leq s$ for every r and s. She also shows that this does not hold for $p=2$ by constructing an infinite series of 2-groups with coclass r and trivial Schur Multiplier. These ideas are explored further by Moravec [**17**].

In the final section of this paper we look at the Tate cohomology of p-central groups. Recall that a finite p-group G is regular if given $x, y \in G$ there exists $s \in \gamma_2(\langle x, y\rangle)$ such that $(xy)^p = x^p y^p s^p$ [**18**, Lemma 1.2.10]. In [**20**] Schmid proved that for G a regular p-group, N a non-trivial normal subgroup of G and $Q = G/N$ non-cyclic then the Q-module $A = Z(N)$ has non-trivial cohomology. So in particular if G is a non-abelian regular p-group and Φ is the Frattini subgroup of G, then $H^n(G/\Phi, Z(\Phi)) \neq 0$ for all n; Schmid then asks whether this result holds more generally. Abdollahi [**1**] has given some cases where the result holds, and in the final section we prove the following result.

THEOREM 3. *Let G be a finite p-central p-group and N a proper, non-trivial normal subgroup of G that is not maximal. Let $Q = G/N$, then $H^n(Q, Z(N)) \neq 0$ for all n.*

Notation is standard. Given subsets X and Y of a group G, then $[X, Y]$ denotes the group generated by commutators $[x, y] = x^{-1}y^{-1}xy$, where $x \in X$ and $y \in Y$. For n, a natural number, $[X, {}_n Y]$ is defined inductively, $[X, {}_1 Y] = [X, Y]$ and $[X, {}_n Y] = [[X, {}_{n-1} Y], Y]$. The lower central series of group G is denoted by $\gamma_i(G)$ and defined inductively as $G = \gamma_1(G)$ and $\gamma_{i+1}(G) = [\gamma_i(G), G]$ for $i \geq 1$. We also use G' to denote the derived group of G. The centre of G is denoted by $Z(G)$. For $H \leq G$ we denote the subgroup generated by elements h^{p^i} with $h \in H$ by H^{p^i}.

2. Schur multipliers.

The Schur Multiplier of a group G, denoted $M(G)$ and introduced by Schur in 1904 [**21**], is given by the second cohomology group $H^2(G, \mathbb{C}^*)$. For a finite group $M(G)$ can be identified with the second integral homology group $H_2(G, \mathbb{Z})$. The study of Schur Multipliers is closely related to the study of central extensions of groups. A group H is a covering group of G if H has a subgroup $A \cong M(G)$ such that $G \cong H/A$ and $A \leq Z(H) \cap H'$. Schur proved that a covering group always exists, although it need not be unique. For more background on Schur Multipliers see [**13**]. So the covering group of a group of exponent p^k is a p^k-central group and information about the derived group of a p^k-central group yields information about the Schur Multiplier of a finite group of exponent p^k. This link has already been explored by Moravec [**16**].

We focus on p-central groups, and so Schur Multipliers of groups of exponent p. It is known that the derived group of a p-abelian group has exponent p, so identifying when a p-central group is p-abelian is useful.

LEMMA 1. *A finite p-group G is p-abelian if and only if it is p-central and regular.*

Proof. Clearly a p-abelian p-group is regular and it is p-central by [**8**] (or the comment in the Introduction). For the opposite direction, note that in a regular

p-group $[x^p, y] = 1$ yields $[x, y]^p = 1$ [**9**, Sec. III 10.6(b)] and furthermore $(G')^p = 1$ [**18**, Lemma 1.2.13(i)]. Weichsel [**25**] showed that G being p-abelian is equivalent to G being regular and satisfying $(G')^p = 1$. □

As a finite p-group of nilpotency class less than p is regular [**18**, Lemma 1.2.11(i)] this yields the following corollary.

COROLLARY 1. *A finite p-central group of nilpotency class less than p is p-abelian.*

Thus, the Schur Multiplier $M(G)$ of a finite group G of exponent p and nilpotency class $\leq p-2$ has exponent p. But by examining the interplay between the commutator and p-power structure of a p-central group we can do better than this. First we quote a technical lemma.

LEMMA 2 [**18**, Corollary 1.1.32]. *Let x and y be elements of G, and let p be a prime and r a positive integer. For $a, b \in \langle x, y\rangle$ define $K(a, b)$ to be the normal closure in $\langle x, y\rangle$ of the set of all basic commutators in $\{a, b\}$ of weight at least p^r and of weight at least two in b, together with the p^{r-k+1}th powers of all basic commutators in $\{a, b\}$ of weight less than p^k and of weight at least two in b for $1 \leq k \leq r$. Then,*

(i) $(xy)^{p^r} \equiv x^{p^r} y^{p^r} [y, x]^{\binom{p^r}{2}} [y, {}_2x]^{\binom{p^r}{3}} \ldots [y, {}_{p^r-1}x] \bmod K(x, y)$.

(ii) $[x^{p^r}, y] \equiv [x, y]^{p^r} [x, y, x]^{\binom{p^r}{2}} \ldots [[x, y], {}_{p^r-1}x] \bmod K(x, [x, y])$.

We isolate the next result to ease the proof of the following Proposition.

LEMMA 3. *Let G be a group, $S \subseteq G$ and p a prime. Suppose $L \leq G$ satisfies $(\gamma_2([S, G]))^p \leq L$ and $\gamma_p([S, G]) \leq L$. Further, suppose $[s, g]^p \in L$ for all $s \in S$ and $g \in G$. Then $[S, G]^p \leq L$.*

Proof. This follows inductively from Lemma 2(i). Note that an element of $[S, G]^p$ is of the form $([s_1, g_1] \ldots [s_n, g_n])^p$ for some $s_i \in S$ and $g_i \in G$ for $1 \leq i \leq n$. Write $x = ([s_1, g_1] \ldots [s_{n-1}, g_{n-1}])^p$ and by induction suppose $x \in L$. Then applying Lemma 2(i) to $(x[s_n, g_n])^p$ and noting the hypotheses of the lemma gives the required result. □

The next result shows how p-powers drop in a finite p-central group.

PROPOSITION 1. *Let G be a finite p-central group and H a subset of G. Define $H_1 = H$ and $H_{i+1} = [H, {}_iG] \leq G$ for $i \geq 1$. Then $(H_i)^p \leq H_{i+p-1}$ for all $i \geq 2$.*

Proof. Let $i \geq 2$, $x \in H_{i-1}$ and $y \in G$. We begin by showing that $[x, y]^p \in (H_{i+1})^p H_{i+p-1}$. Applying Lemma 2(ii) to $[x^p, y]$ yields

$$1 \equiv [x, y]^p [x, y, x]^{\binom{p}{2}} \ldots [x, y, {}_{p-1}x] \bmod K(x, [x, y]).$$

Note that

$$[x, y, x]^{\binom{p}{2}} \ldots [x, y, {}_{p-2}x]^p \in [H_{i-1}, G, G]^p \leq H_{i+1}^p,$$

and $[x, y, {}_{p-1}x] \in [H_{i-1}, {}_pG] \leq H_{i+p-1}$. Now consider the normal subgroup $K(x, [x, y])$. First note that $H_i \leq \gamma_i(G)$ and $[H_i, \gamma_j(G)] \leq H_{i+j}$. Thus, commutators of weight at least p and of weight at least two in $[x, y]$ lie in H_{2i+p-2}. Similarly, p^{th}-powers of commutators of weight less than p and weight of at least two in $[x, y]$ lie in $(H_{2i+1})^p$. Thus, $K(x, [x, y]) \leq (H_{2i+1})^p H_{2i+p-1} \leq (H_{i+1})^p H_{i+p-1}$ and consequently $[x, y]^p \in (H_{i+1})^p H_{i+p-1}$.

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Applying the previous lemma with $H_{i-1} = S$ and $L = (H_{i+1})^p H_{i+p-1}$, we have

$$(H_i)^p \le (H_{i+1})^p H_{i+p-1}$$

for $i \ge 2$. Substituting the above result for H_{i+1} yields

$$(H_i)^p \le ((H_{i+2})^p H_{i+p})H_{i+p-1} \le (H_{i+2})^p H_{i+p-1}.$$

Continuing in this manner, and noting G is nilpotent so $(H_{i+k})^p$ is a strictly descending series of subgroups, yields

$$(H_i)^p \le H_{i+p-1}.$$

□

COROLLARY 2. *Let G be a finite p-central group then $(\gamma_i(G))^p \le \gamma_{i+p-1}(G)$ for all $i \ge 2$.*

Using the above proposition we can gain information about the Schur Multiplier of a finite group of exponent p.

THEOREM 1. *Let G be a finite group of exponent p and nilpotency class c. Then the exponent $M(G)$ is bounded by $p^{\lceil \frac{c}{p-1} \rceil}$.*

Proof. Suppose H is the covering group of G, then it is sufficient to prove that the exponent of H' is bounded by $p^{\lceil \frac{c}{p-1} \rceil}$. As G has exponent p it follows that H is a p-central group, so we can apply the previous proposition and thus $(H')^p \le \gamma_{p+1}(H)$. Now proceed inductively. Since $(H')^{p^k} \le ((H')^{p^{k-1}})^p$, it follows that $(H')^{p^k} \le \gamma_{2+k(p-1)}(H)$. As $\gamma_{c+2}(H) = 1$, it follows that $(H')^{p^k} = 1$ when $2 + k(p-1) \ge c + 2$, the result follows. □

This improves known results when p is large compared to c. For example, Ellis has shown that for G a finite p-group of nilpotency class $c \ge 2$, the exponent of $M(G)$ divides $(\exp G)^{\lceil c/2 \rceil}$ [**5**]. More recently Moravec has bounded the exponent of $M(G)$ by $p^{k\lceil \log_2 c \rceil}$ where k is a function dependent on p and the exponent of G [**16**].

In a previous version of this paper we commented that we did not know of a finite p-central group which had derived group not of exponent p. By results of Kappe and Morse [**12**] such an example would need to have derived length ≥ 3 and $p \ne 2$ or 3. The referee kindly supplied us with the following example. Take the class 10 quotient of the free group on two generators subject to the laws $x^{25} = 1$ and $[x^5, y] = 1$, call this group G. Using GAP one can readily check that G is a 5-central group of order 5^{55} and exponent 25 satisfying $\exp(G') = 25$ [**19**]. In particular, the two generators g_1 and g_2 of G satisfy $[g_1, g_2]^5 \ne 1$. This example demonstrates that the class of p-central groups is indeed different from the class of p-Levi groups, that is groups which satisfy $[x, y^p] = [x, y]^p$ for all $x, y \in G$ [**11**].

However, our follow-up question, whether the Schur Multiplier of a finite p-group of exponent p necessarily has exponent p (see the related question of Moravec [**17**, Question 1.5]) remains unanswered, since for G in the example above the exponent of $G/(G' \cap Z(G))$ is 25.

3. Coclass. Recall that the coclass of a finite p-group G of order p^n and nilpotency class c is given by $n - c$. As all finite p-groups have finite coclass, the coclass gives a useful invariant for investigating finite p-groups. To study p-groups of coclass 1, also known as p-groups of maximal class, a chain of normal subgroups is introduced:

$$G = P_0 > P_1 > P_2 > \cdots > P_n = \langle 1 \rangle.$$

For $i \geq 2$ the P_i are just the terms of the lower central series and P_1 is a 2-step centralizer, for more details see [**18**, Chap. 3]. In a p-group of coclass 1 the p-powers drop in a uniform way, this gives us the following dichotomy.

PROPOSITION 2. *Let p be an odd prime and G a finite p-group of order p^n and coclass 1. Then G is p^k-central if and only if $n \leq k(p-1)+2$.*

Proof. That G is p^k-central if $n \leq p+1$ follows from [**18**, Proposition 3.3.2]. For $n > p+1$ we have that G has positive degree of commutativity by [**18**, Theorem 3.3.5]. So, by [**18**, Lemma 3.3.1] if $t \notin P_1$ then $t^p \in P_{n-1}$. Now to consider $P_1^{p^k}$. From [**18**, Corollary 3.3.6(i)] it follows that $P_1^{p^k} = P_{1+k(p-1)}$ when $1+k(p-1) \leq n$ and $P_1^{p^k} = 1$ otherwise. Thus, G is p^k-central if and only if $1+k(p-1) \geq n-1$ which gives the result. □

More generally we can give a bound on the order of a finite p^k-central group of coclass r. Although the bound is not best possible (compare with the previous proposition), it seems better than bounds provided by alternative methods.

THEOREM 2. *Let G be a finite p^k-central p-group of coclass r. Then the order of G is bounded by $p^{f(k,p,r)}$ where for odd p*

$$f(k,p,r) = \begin{cases} (k+1)(p-1)p^{r-1} + r & \text{if } k \geq 2 \\ 2p^r + r - 1 & \text{if } k = 1 \end{cases}$$

and

$$f(k,2,r) = \begin{cases} (2+k)2^{r+1} + r & \text{if } k \geq 2 \\ 2^{r+3} + r - 1 & \text{if } k = 1. \end{cases}$$

Proof. Let p be odd and c the nilpotency class of G. When $k \geq 2$, suppose $c > (k+1)(p-1)p^{r-1}$ and when $k = 1$, suppose $c \geq 2p^r$. Equivalently, for p^n the order of G, we have $n > (k+1)(p-1)p^{r-1} + r$ when $k \geq 2$ and $n \geq 2p^r + r$ when $k = 1$. By [**18**, Theorem 6.3.9], there exists $m = m(p,r) = (p-1)p^{r-1}$ such that G acts uniserially on $\gamma_m(G)$ and $(\gamma_i(G))^p = \gamma_{i+d}$ for all $i \geq m$ and for some $d = (p-1)p^s$ with $0 \leq s \leq r-1$. Since G acts uniserially on $\gamma_m(G)$, it follows that $|\gamma_i(G) : \gamma_{i+1}(G)| = p$ for all $i \geq m$ and thus $(\gamma_m(G))^{p^k} = \gamma_{m+kd}$. But $m + kd \leq (k+1)(p-1)p^{r-1} < c$ and thus $(\gamma_m(G))^{p^k}$ does not lie in the centre of G. Hence, G is not p^k-central.

For $p = 2$ we refer to [**18**, Theorem 6.3.8], in this case $m(2,r) = 2^{r+2}$ and $d = 2^s$ with $0 \leq s \leq r+1$. We suppose $c > (2+k)2^{r+1}$ when $k \geq 2$ and $c \geq 2^{r+3}$ when $k = 1$. Equivalently $n > (2+k)2^{r+1} + r$ when $k \geq 2$ and $n \geq 2^{r+3} + r$ when $k = 1$. Then $m + kd \leq (2+k)2^{r+1} < c$, and G is not p^k-central.

The result follows. □

4. Tate cohomology. Let G be a finite p-group, N a normal subgroup of G and $A = Z(N)$, the centre of N. Then A is a $Q = G/N$-module and one can investigate the

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Tate cohomology groups $H^n(Q, A)$. The Q-module A is called cohomologically trivial if $H^n(K, A) = 0$ for all integers n and all subgroups K of Q. By the result of Uchida [**24**] we know that A is cohomologically trivial if $H^r(Q, A) = 0$ for just one integer r. In [**20**] Schmid investigates when the cohomology is non-trivial, he proves that if G is a regular p-group and $Q = G/N$ is not cyclic then $H^n(Q, Z(N)) \neq 0$ for all n. So, in particular, if G is a non-abelian regular p-group and Φ is the Frattini subgroup of G then $H^n(G/\Phi, Z(\Phi)) \neq 0$ for all n, Schmid then asks whether this holds more generally. Abdollahi addresses this question in [**1**] (and uses the alternative definition of p-central as mentioned in our Introduction) and poses the following more general question:

Question 1 [**1**, Question 1.2]**.** For which finite p-groups G and which normal subgroups N of G do we have $H^n(\frac{G}{N}, Z(N)) \neq 0$ for all integers n?

In this section, using the methods of Schmid and Abdollahi, we prove the following.

THEOREM 3. *Let G be a finite p-central p-group and N a proper, non-trivial normal subgroup of G that is not maximal. Let $Q = G/N$, then $H^n(Q, Z(N)) \neq 0$ for all n.*

By Uchida's result we will be able to restrict our attention to $H^0(Q, Z(N))$. Recall, $H^0(Q, A) = A_Q/A^\tau$, where A_Q denotes the fixed points of A under the action of Q, and A^τ denotes the image of A under the trace map $\tau = \tau_Q$. The trace map is given by $\tau_Q : a \mapsto a\sum_{x\in Q} x$.

We analyse the trace map for a finite p-central group G. Let A be an abelian normal subgroup of G, let $a \in A$ and $x \in G$. Then $a^{1+x+\cdots+x^{p-1}} = a^p z$ for some central element z of G. This is clear since $a^{1+x+\cdots+x^{p-1}} = x^{-p}(xa)^p \in Z(G)$ and $a^p \in Z(G)$. The following lemma says slightly more, proving that the central element z in the above statement is the commutator $[a,{}_{p-1}x]$ and consequently that a is a p-Engel element.

LEMMA 4. *Let G be a finite p-central p-group and suppose A is a normal abelian subgroup of G. Let $a \in A$ and $x \in G$ then $a^{1+x+\cdots+x^{p-1}} = a^p[a,{}_{p-1}x]$ and $[a,{}_{p-1}x] \in Z(G)$.*

Proof. Apply Lemma 2(i) to $(xa)^p$ and note that $K(x, a) = 1$. Next we show that most of the terms in this expression for $(xa)^p$ vanish. Let $H = \langle A, x\rangle$. Then $H' = [A, x] = \{[a, x] : a \in A\}$ since A abelian. Now by applying Lemma 2(ii) to $[a^p, x]$ and noting that all terms vanish except $[a, x]^p$, we see that $[a, x]^p = 1$ and thus H' has exponent p. So returning to our expression for $(xa)^p$ yields $(xa)^p = x^p a^p z$ where $z = [a,{}_{p-1}x] \in Z(G)$. □

To prove the theorem we need the following proposition due to Schmid.

PROPOSITION 3. [**20**, Proposition 1] *Suppose $A \neq 0$ is a cohomologically trivial Q-module where A and Q are finite p-groups. Then for every subgroup H of Q, the centralizer $C_Q(A_H) = H$.*

The ideas behind the proof of the theorem follow very closely the ideas of Schmid [**20**] and Abdollahi [**1**] but are included for completeness.

Proof of Theorem 3. Suppose for a contradiction $H^n(Q, Z(N)) = 0$ for some integer n. Then by [**24**, Theorem 4], it follows that $A = Z(N)$ is a cohomologically trivial Q-module. Thus, $H^0(H/N, A) = 0$, where H is a subgroup of G containing N such that $|H : N| = p$. So $A_{H/N} = A^{\tau_{H/N}}$. By Lemma 4, for each $a \in A$, there exists a central element z_a such that $\tau_{H/N}(a) = a^p z_a$. Thus, $C_{G/N}(A^{\tau_{H/N}}) = C_{G/N}(A^p) = G/N$ since G is p-central. However, Proposition 3 gives $C_{G/N}(A^p) = C_{G/N}(A_{H/N}) = H/N$. The result follows. □

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