COUNTABLE COMPACTIFICATIONS

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1. Introduction. It is assumed that all topological spaces discussed in this paper are Hausdorff. By a compactification αX of a space X we mean a compact space containing X as a dense subspace. If, for some positive integer n, $\alpha X - X$ consists of n points, we refer to αX as an n-point compactification of X, in which case we use the notation $\alpha_n X$. If $\alpha X - X$ is countable, we refer to αX as a countable compactification of X. In this paper, the statement that a set is countable means that its elements are in one-to-one correspondence with the natural numbers. In particular, finite sets are not regarded as being countable. Those spaces with n-point compactifications were characterized in **(3)**. From the results obtained there it followed that the only n-point compactifications of the real line are the well-known 1- and 2-point compactifications and the only n-point compactification of the Euclidean N-space, E^N $(N > 1)$, is the 1-point compactification. In this paper, we characterize those spaces that are locally compact and have countable compactifications. As a consequence, we obtain the fact that no Euclidean N-space has a countable compactification.

Let βX denote the Stone–Čech compactification of a completely regular space X, let card (Y) denote the cardinal number of a set Y, and finally, let c denote the cardinal number of the continuum. Since every compactification of a completely regular space X is a continuous image of βX, we conclude that card $(\alpha X) \leqslant$ card (βX) for every compactification αX of X. Now it is shown in **(1**; p. 131, 9.3**)** that card $(\beta R - R) = 2^c$ where R denotes the space of real numbers. The same technique yields the result that card $(\beta E^N - E^N) = 2^c$ for each Euclidean N-space E^N. Thus, if one wishes to assume the Generalized Continuum Hypothesis, one may conclude that if αR is any compactification of R, then card $(\alpha R - R)$ is 1, 2, c, or 2^c and if αE^N is any compactification of E^N $(N > 1)$, then card $(\alpha E^N - E^N)$ is 1, c, or 2^c.

We take this opportunity to acknowledge a considerable debt to the referee for his helpful suggestions, which resulted in the final form of Theorem (2.1) as well as a simpler proof of that theorem.

2. The Main Theorem and its corollaries.

THEOREM (2.1). *The following statements concerning a space X are equivalent:*

(2.1.1). *X is locally compact and $\beta X - X$ has an infinite number of components (maximal connected sets).*

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(2.1.2). *X is locally compact and there exists a compactification αX of X such that $\alpha X - X$ is infinite and totally disconnected.*

(2.1.3). *X is locally compact and has a countable compactification.*

(2.1.4). *X has an n-point compactification for each positive integer n.*

Proof. (2.1.1) $\Rightarrow$ (2.1.2). Let

$$\beta X - X = \bigcup\{H_a : a \in \Lambda\}$$

where $\{H_a : a \in \Lambda\}$ is the family of components of $\beta X - X$. Let $\alpha X = X \cup \Lambda$ and define a function h from βX onto αX by

$$h(p) = \begin{cases} p & \text{if } p \in X, \\ a & \quad p \in H_a. \end{cases}$$

Endow αX with the quotient topology induced by h. Then αX, being the continuous image of a compact space, is compact. In order to show that αX is Hausdorff, there are three cases to consider for distinct points p and q:

(1) p and q both belong to X,

(2) $p \in X$ and $q \in \alpha X - X$,

(3) p and q both belong to $\alpha X - X$.

The first case follows easily using the fact that X is locally compact and therefore an open subset of any compactification. This implies that any open subset of X is also an open subset of βX and hence also of αX. For the second case, we again use the local compactness condition of X to conclude that there exists an open subset G of X and a compact subset K of X such that $p \in G \subset K \subset X$. It follows that G and $\alpha X - K$ are disjoint, open subsets of αX containing p and q respectively. Now let us consider the third case. H_p and H_q are distinct components (and hence closed subsets of) $\beta X - X$, which is compact. Therefore, H_p and H_q are disjoint, closed subsets of βX and there are disjoint, open subsets G_p and G_q of βX containing H_p and H_q respectively. By (**1**, Theorem 16.15), the component of a point in a compact space is the intersection of all open-and-closed sets containing it. This implies that H_p is the intersection of all open-and-closed sets (relative to $\beta X - X$) containing it. Since $\beta X - X$ is compact and $G_p \cap [\beta X - X]$ is an open subset of $\beta X - X$ which contains H_p, it follows that the intersection of a finite number of the open-and-closed sets is contained in $G_p \cap [\beta X - X]$. Denote this intersection by V_p. Then V_p is an open-and-closed subset of $\beta X - X$ and

$$H_p \subset V_p \subset G_p \cap [\beta X - X].$$

Because V_p is both open and closed, it is the union of all H_a contained in it. Moreover,

$$V_p = V_p^* \cap [\beta X - X]$$

for some open subset V_p^* of βX where $V_p^* \subset G_p$. It follows that

$$V_p^* = V_p \cup [V_p^* \cap X].$$

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There exist sets V_q and V_q^* related to H_q in the same manner. Therefore V_p^* and V_q^* are disjoint. Now let

$$U_p = [V_p^* \cap X] \cup \{a : H_a \subset V_p\}, \qquad U_q = [V_q^* \cap X] \cup \{a : H_a \subset V_q\}.$$

Then $h^{-1}[U_p] = V_p^*$ and $h^{-1}[U_q] = V_q^*$. Since the latter are open subsets of βX and αX was given the quotient topology induced by h, it follows that U_p and U_q are disjoint, open subsets of αX containing p and q respectively. This proves that αX is a Hausdorff space. It follows easily that X is dense in αX; hence αX is indeed a compactification of X. In order to verify that $\alpha X - X$ is totally disconnected, we note that $U_p \cap [\alpha X - X]$ is open in $\alpha X - X$. Moreover,

$$h^{-1}[U_p \cap [\alpha X - X]] = V_p^* \cap [\beta X - X] = V_p$$

which is a closed subset of βX. This implies that $U_p \cap [\alpha X - X]$ is a closed subset of αX and therefore also of $\alpha X - X$. Hence p and q do not belong to the same component. Since p and q were any two distinct points of $\alpha X - X$, we conclude that the latter is totally disconnected.

(2.1.2) $\Rightarrow$ (2.1.3). Since X is locally compact, $\alpha X - X$ is an infinite, compact totally disconnected space and thus, by **(1**, Theorem 16.17**)**, has a basis of open-and-closed sets. Therefore, there exists a countable family $\{H_n\}_{n=1}^{\infty}$ of non-empty mutually disjoint subsets of $\alpha X - X$ which are both open and closed in $\alpha X - X$. Set

$$H_0 = [\alpha X - X] - \bigcup \{H_n\}_{n=1}^{\infty}.$$

Then $H_0 \neq \emptyset$ since $\alpha X - X$ is compact. Now define a function h from αX onto

$$X \cup \{n\}_{n=0}^{\infty} = \gamma X$$

by

$$h(p) = \begin{cases} n & \text{for } p \in H_n, \\ p & \phantom{\text{for }} p \in X. \end{cases}$$

Endow γX with the quotient topology induced by h. One can show as in the previous discussion that γX is Hausdorff and that X is dense in γX. Thus γX is a countable compactification of X.

(2.1.3) $\Rightarrow$ (2.1.4). Now suppose that X is locally compact and that γX is a countable compactification of X. Assume p and q are distinct points belonging to some connected subset H of $\gamma X - X$. Because $\gamma X - X$ is completely regular, there exists a continuous function f from $\gamma X - X$ into the closed unit interval I such that $f(p) = 0$ and $f(q) = 1$. Then $f[H]$ is connected and must be all of I since it contains both 0 and 1. This, of course, contradicts the cardinality of $\gamma X - X$; hence $\gamma X - X$ is totally disconnected. Again, since X is locally compact, $\gamma X - X$ is compact and we appeal once more to **(1**, Theorem 16.17**)** to conclude that $\gamma X - X$ has a basis of open-and-closed sets. Thus, for any positive integer n, there are n non-empty mutually disjoint subsets of

$\gamma X - X$ that are both open and closed and whose union is all of $\gamma X - X$. Denote these sets by $\{H_i\}_{i=1}^n$ and define a function h from γX onto

$$X \cup \{1, 2, 3, \ldots, n\} = \alpha_n X$$

by

$$h(p) = \begin{cases} i & \text{for } p \in H_i, \\ p & \phantom{\text{for }} p \in X. \end{cases}$$

Let $\alpha_n X$ have the quotient topology induced by h. One shows as in previous cases that $\alpha_n X$ is indeed a (Hausdorff) compactification of X.

(2.1.4) $\Rightarrow$ (2.1.1). Let any positive integer n be given and let $\alpha_n X$ be an n-point compactification of X. Then $\beta X - X$ must have at least n components since there exists a continuous function mapping it onto $\alpha_n X - X$. Since this is true for every positive integer, $\beta X - X$ must have infinitely many components. Finally, any space with a finite compactification is locally compact and the proof is complete.

(3, Theorem (2.6)) states that if every compact subset of X is contained in a compact subset whose complement has at most N components, then X has no n-point compactification for $n > N$. This fact and Theorem (2.1) of this paper result in

COROLLARY (2.2). *Suppose X is locally compact and there exists a positive integer N such that every compact subset of X is contained in a compact subset whose complement has at most N components. Then X has no countable compactification.*

(3, Theorem (2.9)) states that if (X, d_1) and (Y, d_2) are two unbounded, connected metric spaces such that for all points $x_0 \in X$ and $y_0 \in Y$ and every positive number r, the sets

$$\{x \in X : d_1(x, x_0) \leqslant r\} \quad \text{and} \quad \{y \in Y : d_2(y, y_0) \leqslant r\}$$

are compact, then $X \times Y$ has no n-point compactification for $n > 1$. This and Theorem (2.1) of this paper yield

COROLLARY (2.3). *Let (X, d_1) and (Y, d_2) be two unbounded, connected, locally compact metric spaces and suppose that for all points $x_0 \in X$ and $y_0 \in Y$ and every positive number r, the sets*

$$\{x \in X : d_1(x, x_0) \leqslant r\} \quad \textit{and} \quad \{y \in Y : d_2(y, y_0) \leqslant r\}$$

are compact. Then $X \times Y$ has no countable compactification.

It follows from these corollaries that no Euclidean N-space has a countable compactification. The space

$$X = I - [\{0\} \cup \{1/n\}_{n=1}^{\infty}]$$

(as before, I denotes the closed unit interval) is an example of a locally compact subspace of R that has a countable compactification; namely, I itself. X, of

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course, is not connected. Indeed, Corollary (2.2) implies that no connected subspace of R will have a countable compactification. There are, however, locally compact, connected subspaces of the Euclidean plane E^2 that have countable compactifications. For example, let

$$Y_n = \{(x, y) \in E^2 : y = x/n,\ x \geqslant 0, \text{ and } x^2 + y^2 < 1\}.$$

Let

$$Y = [\bigcup\{Y_n\}_{n=1}^{\infty}] \cup \{(x, 0) : 0 \leqslant x < 1\},$$

and finally, let

$$K = \{(1, 0)\} \cup \{(x, y) : y = x/n \text{ and } x^2 + y^2 = 1\}_{n=1}^{\infty}.$$

Then Y is a locally compact, connected subspace of E^2 and $K \cup Y$ is a countable compactification of Y.

By (**3**, Theorem (2.1)), a space X has an n-point compactification if and only if it is locally compact and contains a compact subset K whose complement consists of n mutually disjoint open subsets $\{G_i\}_{i=1}^{n}$ such that $K \cup G_i$ is not compact for each i. This, in conjunction with Theorem (2.1) of this paper, results in

COROLLARY (2.4). *A locally compact space X has a countable compactification if and only if for each positive integer n it contains a compact subset K_n whose complement is the union of n mutually disjoint open subsets $\{G_{n,i}\}_{i=1}^{n}$ with the property that $K_n \cup G_{n,i}$ is not compact for each i.*

We conclude with

COROLLARY (2.5). *Suppose X is locally compact and is the union of an infinite number of mutually disjoint open subsets. Then X has a countable compactification. In particular, every infinite discrete space has a countable compactification.*

Proof. We make use of the previous corollary. For any positive integer n, let $K_n = \emptyset$. Since X can be regarded as the union of n mutually disjoint open noncompact subsets, the proof is complete.

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