BULL. AUSTRAL. MATH. SOC. 17B35, 17B70, 16S30, 16W55, 16N60
VOL. 56 (1997) [483–488]

PRIMENESS OF THE ENVELOPING ALGEBRA OF HAMILTONIAN SUPERALGEBRAS

MARK C. WILSON

In 1990 Allen Bell presented a sufficient condition for the primeness of the universal enveloping algebra of a Lie superalgebra. Let Q be a nonsingular bilinear form on a finite-dimensional vector space over a field of characteristic zero. In this paper we show that Bell's criterion applies to the Hamiltonian Cartan type superalgebras determined by Q, and hence that their enveloping algebras are semiprimitive.

1. INTRODUCTION

Let $L = L_+ \dotplus L_-$ be a finite-dimensional Lie superalgebra over a field of characteristic zero, and let $U(L)$ be its universal associative enveloping (super)algebra. In [**1**] Bell gave the following simple criterion for primeness of $U(L)$. Let $\{f_1, \ldots, f_n\}$ be a basis for the odd part L_- of L. Form the *product matrix* $M = ([f_i, f_j])$, considered as a matrix over the symmetric algebra $S(L_+)$. If $\det M \neq 0$ then $U(L)$ is prime.

Note that since $U(L)$ is a Jacobson ring (see for example [**5**]), if $U(L)$ is prime then it is also semiprimitive. As far as is known these last two properties may be equivalent for rings of the form $U(L)$.

The primeness question for enveloping algebras of the classical simple Lie superalgebras has been settled completely in [**1**] and [**3**]. An investigation into the applicability of Bell's criterion to the Cartan type Lie superalgebras was begun in [**8**], continued in [**10**] and is concluded in this paper and [**9**].

Here it is shown that the Hamiltonian algebras $H(Q)$ and $\widetilde{H}(Q)$ satisfy Bell's criterion. This immediately gives

THEOREM. *Let K be a field of characteristic zero, let $n \geqslant 4$ and let Q be a nonsingular bilinear form on a K-vector space of dimension n. Then $U(H(Q))$ and $U(\widetilde{H}(Q))$ are prime.*

As a consequence of the results of the above-mentioned papers we have the following theorem.

THEOREM. *Let L be a finite-dimensional simple Lie superalgebra over an algebraically closed field of characteristic zero. Then L satisfies Bell's criterion, and hence*

Received 24th February, 1997

https://doi.org/10.1017/S0004972700031282 Published online by Cambridge University Press

$U(L)$ is prime, unless L is of one of the types: $b(n)$ for $n \geqslant 3$; $W(n)$ for odd $n \geqslant 5$; $S(n)$ for odd $n \geqslant 3$.

2. The Hamiltonian superalgebras

Good references for basic facts about Lie superalgebras are [**2**] and [**7**].

Let K be a field of characteristic zero, n a positive integer and V an n-dimensional K-vector space. Let $\Lambda = \Lambda(V)$ be the Grassmann algebra of V. Recall that Λ is an associative $\mathbb{Z}$-graded superalgebra. Fix a basis $\{v_1, \dots, v_n\}$ for V. For each ordered subset $I = \{i_1, i_2, \dots, i_r\}$ of $N = \{1, 2, \dots, n\}$ with $i_1 < i_2 < \dots < i_r$, let v_I be the product $v_{i_1} v_{i_2} \cdots v_{i_r}$. The set of all such v_I forms a basis for Λ, where we interpret $1 = v_\emptyset$ as the empty product, and the homogeneous component Λ_r is spanned by the v_I with $|I| = r$. The anticommutativity of multiplication in Λ implies that

$$(1) \qquad v_I v_J = \begin{cases} \pm v_{I \cup J} & \text{if } I \cap J = \emptyset, \\ 0 & \text{if } I \cap J \neq \emptyset. \end{cases}$$

The algebra $W = W(V)$ is the $\mathbb{Z}$-graded Lie superalgebra consisting of all superderivations of Λ. Every element of W maps V into Λ and since it is a superderivation it is completely determined by its action on the generating subspace V. It follows that W can be identified with $\Lambda \otimes_K V^*$ and we shall henceforth do so.

Under this identification the map $\partial_i = \partial / \partial_{v_i}$ corresponds to the dual of v_i which we shall also denote ∂_i. The set of all $v_I \otimes \partial_i$ is then a homogeneous basis for W, the degree of such an element being equal to $|I| - 1$.

For each symmetric bilinear form Q on V there are subalgebras of W denoted by $H(Q)$ and $\widetilde{H}(Q)$. Their (rather complicated) definition can be found in [**7**, p.194] or [**2**, section 3.3.2]. If we extend K to its algebraic closure then all such algebras become isomorphic to the algebra $\widetilde{H}(n)$ (respectively $H(n)$) defined below. Since Bell's criterion holds over a given field if and only if it holds over the algebraic closure of that field, it is sufficient to verify the criterion for the algebras $\widetilde{H}(n)$ and $H(n)$.

We now recall some basic facts about the Hamiltonian superalgebras. The subspace of W spanned by all superderivations of the form

$$D_\lambda = \sum_{i \in N} \partial_i(\lambda) \otimes \partial_i,$$

where $\lambda \in \Lambda$, is a Lie superalgebra called $\widetilde{H} = \widetilde{H}(n)$. $\widetilde{H}$ inherits a natural $\mathbb{Z}$-grading from W and we have

$$\widetilde{H} = \bigoplus_{r=-1}^{n-2} H_r.$$

The subalgebra $H = H(n) = \bigoplus_{r=-1}^{n-3} H_r = [\widetilde{H}, \widetilde{H}]$ is a simple Lie superalgebra of Cartan type.

The homogeneous component H_r is isomorphic as a vector space (in fact as an H_0-module) to Λ_{r+2} via $D_\lambda \mapsto \lambda$. Thus the superderivations $x_I = D_{v_I}$, where $\emptyset \neq I \subseteq N$, form a basis for $\widetilde{H}$, and $\dim H_r = \binom{n}{r+2}$.

3. Computation

It is known that the multiplication in H satisfies

$$[D_\lambda, D_\mu] = \pm D_{\{\lambda,\mu\}}$$

where $\{\lambda, \mu\} = \sum_i \partial_i(\lambda)\partial_i(\mu)$. Note that this differs slightly from the notation in [**2**], and that the exact multiplication formula is not needed for our purposes.

It follows from (1) that $\partial_i(v_I)\partial_i(v_J) = 0$ unless $I \cap J = \{i\}$, whence

$$[x_I, x_J] = \begin{cases} \pm x_{I \,\Delta\, J} & \text{if } |I \cap J| = 1, \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

where Δ denotes the symmetric difference (Boolean sum). Since Δ is the addition in the usual Boolean ring structure on the power set of N, this implies that for a given $A, I \subseteq N$, the equation $[x_I, x_J] = \pm x_A$ has at most one solution for J. This solution exists precisely when $A \,\Delta\, I \neq \emptyset$, that is when $I \not\subseteq A$ and $A \not\subseteq I$. Furthermore if $|I|$ is odd and $|A|$ even then $|J| = |A| + |I| - 2\,|I \cap A|$ is necessarily odd. Thus every even x_A appears (perhaps with a minus sign) in the product matrix, and each such x_A appears at most once in each row or column.

3.1. n even

Theorem 3.1. *Let $n \geqslant 4$ be an even integer. Then $H(n)$ and $\widetilde{H}(n)$ satisfy Bell's criterion.*

Proof: Write $n = 2m$. The highest odd degree occurring in $\widetilde{H}$ and H is $n-3$. It follows that if we group the basis elements x_I by increasing degree, then the product matrices for both H and $\widetilde{H}$ are the same and that this common matrix has the block reverse triangular structure

$$\begin{pmatrix} H_{-1,-1} & H_{-1,1} & \cdots & \cdots & H_{-1,n-3} \\ H_{1,-1} & H_{1,1} & \cdots & H_{1,n-5} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ H_{n-3,-1} & 0 & \cdots & 0 & 0 \end{pmatrix}$$

where $H_{r,s}$ is the block formed by the products of elements of degree r with those of degree s. Furthermore each block on the reverse diagonal is square, since if $r + s = n - 4$

then $\dim H_r = \binom{n}{r+2} = \binom{n}{s+2} = \dim H_s$. The product matrix is nonsingular if and only if each of these blocks is nonsingular.

Fix such a reverse diagonal block $H_{r,s}$ corresponding to products of elements of degree r by those of degree $s = n - 4 - r$. Using the identification of H_{n-4} with Λ_{n-2} we can index the basis elements of H_{n-4} by their (ordered) 2-element complements, for example $y_{13} = x_{N\setminus\{1,3\}}$. We now make the specialisation which sends y_{ij} to 0 unless $j - i = m$, and call the m remaining variables $z_1 = y_{11'}, \ldots, z_m = y_{mm'}$. For each i let $i' = i + m \pmod n$. Note that $(i')' = i$ and $z_i = z_{i'}$. The image B of the block $H_{r,s}$ under this specialisation is a matrix whose only possibilities for nonzero entries are $\pm z_i$ for some i.

We shall obtain a further block decomposition of B. By replacing all nonzero elements of B by 1's, we obtain a $(0,1)$ matrix which is the adjacency matrix of a unique graph $G = G(B)$. In other words, G has vertices the x_I and an edge joining x_I and x_J if and only if the product $[x_I, x_J]$ remains nonzero under our specialisation above. If for simplicity we label the vertex corresponding to x_I by I, there is an edge in G joining I to J if and only if $[x_I, x_J] = \pm z_i$ for some i. We shall say that in this case I and J are joined by an edge of *colour* i.

Finding a block decomposition of B is equivalent to decomposing G into disjoint subgraphs, which we now proceed to do. Fix $i \in N$. We determine exact conditions on I and J for there to exist an edge of colour i joining them. It follows from (2) that this occurs if and only if either $I \cap J = \{i\}$ and $I \cup J = N \setminus \{i'\}$, or $I \cap J = \{i'\}$ and $I \cup J = N \setminus \{i\}$. Thus there is an edge of colour i joining I and J if and only if $|I| + |J| = n$, one of i or i' belongs to both I and J and the other belongs to neither. Furthermore, for a given $I \neq N$, there is at most one edge of a given colour at the vertex I. Also there is at least one edge of some colour at the vertex I: since $I \neq N$, for some i we must have $i \in I$ and $i' \notin I$.

We now obtain a further block decomposition of B by showing that the set of colours occurring at a given vertex of $G(B)$ is constant on each component. To this end, we first show that vertices distance 2 apart have the same colours. Suppose that I and J are linked by an edge of colour i. Then without loss of generality $I \cap J = \{i\}$ and $I \cup J = N \setminus \{i'\}$. Let K be linked to J. If J and K are linked by an edge of colour j then either $\{i, i'\} = \{j, j'\}$, in which case $K = I$, or $\{i, i'\} \cap \{j, j'\} = \emptyset$. In the latter case we can assume $J \cap K = \{j\}$ and $J \cup K = N \setminus \{j'\}$. Thus $i' \in K$ since $i' \in J \cup K$ but $i' \notin J$. Let $X = J \cup \{i', j'\} \setminus \{i, j\}$. Then $|X| = |J|$, $K \cap X = \{i'\}$, $K \cup X = N \setminus \{i\}$ and so K and X are linked by an edge of colour i. Thus every colour occurring at I also occurs at K, and by symmetry I and K have the same colours.

It follows that if I and J are joined by an edge then they have the same colours, since if an edge of some colour i joins I and L, then J and L have the same colours by above and so the colour i occurs at J. By induction on the length of a path joining two vertices, the set of colours occurring at a vertex is constant on components. This decomposes $G(B)$ into a union of disjoint subgraphs, each corresponding to a given set of colours. Hence B decomposes as a direct sum of smaller blocks, each of which is parametrised by some nonempty subset of the set of colours.

Now fix such a block corresponding to a given set of colours. This matrix is such that in every row and column, each variable which is present occurs exactly once, perhaps with a minus sign. Then by specialising all but one of these variables to zero we obtain a nonsingular monomial matrix. This shows that the original product matrix for $H(n)$ and $\widetilde{H}(n)$ is nonsingular. $\square$

The fact that the Noetherian rings $R = U(H)$ and $S = U(\widetilde{H})$ are simultaneously prime is not a surprise. The component H_{n-2} is 1-dimensional, spanned by x say. Since $[x, H] \subseteq [\widetilde{H}, \widetilde{H}] = H$, $\operatorname{ad} x$ stabilises R. When n is even then $\operatorname{ad} x$ is an ordinary derivation and so S is the differential polynomial ring $R[x; \operatorname{ad} x]$. It is a well-known fact (see for example [**6**, Proposition 8.3.32]) that in this situation R is prime if and only if S is.

3.2. n ODD This case reduces rather easily to the previous one.

THEOREM 3.2. *Let $n \geqslant 5$ be odd. Then $H(n)$ and $\widetilde{H}(n)$ satisfy Bell's criterion.*

PROOF: Let $M, \widetilde{M}$ be the product matrices for $H(n), \widetilde{H}(n)$ respectively. The top degree $n-2$ occurring in $\widetilde{H}(n)$ is odd, and $\dim H_{n-2} = 1$. Thus $\widetilde{M}$ is obtained from M by adding another row and column. Since this procedure either leaves the rank unchanged or increases the rank by 1, it suffices to show that $\widetilde{M}$ is nonsingular.

We decompose $\widetilde{M}$ into 4 blocks as follows. Group the rows indexed by those I for which $n \in I$ together and follow them by the rows for which $n \notin I$. Do the same for the columns. This gives an obvious 2×2 block structure. Make the specialisation which sets all even x_I with $n \in I$ to zero. Then $\widetilde{M}$ specialises to a matrix of the form $\begin{pmatrix} x & 0 \\ 0 & Y \end{pmatrix}$. It suffices to show that X and Y are nonsingular.

The matrix Y has entries which are the pairwise products of the x_I with $I \subseteq \{1, \ldots, n-1\}$ and hence is just a product matrix for $H(n-1)$. Thus Y is nonsingular by Theorem 3.1.

Now choose I with $n \in I$. Since $I \neq N$, both $I \not\subseteq N \setminus \{n\}$ and $N \setminus \{n\} \not\subseteq I$ hold and so there is precisely one J with $n \in J$ for which $[x_I, x_J] = \pm x_{N \setminus \{n\}}$. Thus in X every row and column has precisely one occurrence of $\pm x_{N\setminus\{n\}}$, so specialising to zero all variables except this one yields a nonsingular monomial matrix. $\square$

It is not as obvious *a priori* that the rings $R = U(H)$ and $S = U(\widetilde{H})$ should be simultaneously prime. Let x span H_{n-2}. Then $[x, x] = 2x^2 = 0$ and so $S = R[x; \delta]/I$, where δ is the skew derivation $\operatorname{ad} x$ and I is the ideal generated by x^2. Obviously S prime implies R prime but the converse for extensions of this type (see [**4**]) requires extra hypotheses regarding the action of δ on the symmetric Martindale quotient ring of R which seem difficult to verify in our situation.

References

[1] A.D. Bell, 'A criterion for primeness of enveloping algebras of Lie superalgebras', *J. Pure Appl. Algebra* **69** (1990), 111–120.

[2] V.G. Kac, 'Lie superalgebras', *Adv. in Math.* **26** (1977), 8–96.

[3] E. Kirkman and J. Kuzmanovich, 'Minimal prime ideals in enveloping algebras of Lie superalgebras', *Proc. Amer. Math. Soc.* **124** (1996), 1693–1702.

[4] A. Milinski, 'Actions of pointed Hopf algebras on prime algebras', *Comm. Algebra* **23** (1995), 313–333.

[5] I.M. Musson, 'Enveloping algebras of Lie superalgebras: a survey', *Contemp. Math.* **124** (1992), 141–149.

[6] L. Rowen, *Ring theory* (Academic Press, Boston, 1988).

[7] M. Scheunert, *The theory of Lie superalgebras*, Lecture Notes in Mathematics **716**, (Springer-Verlag, Berlin, Heidelberg, New York, 1979).

[8] M.C. Wilson, 'Primeness of the enveloping algebras of a Cartan type Lie algebra', *Proc. Amer. Math. Soc.* **124** (1996), 383–387.

[9] M.C. Wilson and G. Pritchard, 'Primeness of the enveloping algebra of the special Lie superalgebras', *Arch. Math. (Basel)* (to appear).

[10] M.C. Wilson, G. Pritchard and D.H. Wood, 'Bell's primeness criterion for $W(2n+1)$', *Experiment. Math.* (to appear).

Department of Mathematics
University of Auckland
Private Bag 92019 Auckland
New Zealand
e-mail: wilson@math.auckland.ac.nz