

## LETTERS TO THE EDITOR

### ON VARIATION IN BIRTH PROCESSES

M. J. FADDY,\* *University of Otago*

#### Abstract

Birth processes with piecewise linear birth rates are analysed, and numerical results suggest that, relative to the linear case, convex birth rates increase variability and concave birth rates decrease variability.

NON-LINEAR BIRTH RATE; RELATIVE VARIATION

#### 1. Introduction

A general Markov birth process  $\{X(t), t \geq 0\}$  may be defined in terms of infinitesimal transition probabilities:

$$(1.1) \quad P\{X(t + \delta t) = n + 1 \mid X(t) = n\} = \lambda(n) \delta t,$$

where  $\lambda(n)$  is a function of the population size. Non-linear forms for  $\lambda(n)$  can arise when the population size is bounded, as in the case of the simple stochastic epidemic (Bailey (1975), Chapter 5). Although explicit expressions for the probability distribution of the population size at varying times may be determined, they are analytically and computationally awkward, and do not reveal much about the qualitative behaviour of the process. Variation is an essential feature of any stochastic process: non-linear processes can be *linearised* into approximating processes which have the same *deterministic* solution (see, for example, Faddy (1977)), but their *stochastic* variation could be quite different (Morgan and Hinde (1976)). Here a class of piecewise linear forms for  $\lambda(n)$  is considered, where computationally convenient expressions for the moments can be established.

#### 2. Model formulation and results

Consider  $\lambda(n)$  given by:

$$(2.1) \quad \lambda(n) = \begin{cases} \lambda_1 n, & n \leq s \\ \alpha + \lambda_2 n, & n \geq s \end{cases}$$

with  $(\lambda_1 - \lambda_2)s = \alpha$ , for some  $s \geq n_0 + 1$  where  $n_0 (\geq 1)$  is the initial population size. It will be assumed that  $\lambda(n) \geq 0$ , so that if  $\lambda_2 < 0$  it will be necessary for  $\lambda(n) = 0$  for some  $n$ , making this value an upper bound on the population size. From (1.1) and (2.1) the process  $\{X(t), t \geq 0\}$  behaves initially as a simple linear birth process so that the state probabilities will be negative binomial (Cox and Miller (1965), Chapter 4); i.e., for  $n = n_0, n_0 + 1, \dots, s$ :

$$(2.2) \quad P\{X(t) = n\} = \frac{(n-1)!}{(n_0-1)! (n-n_0)!} (\exp(-\lambda_1 t))^{n_0} (1 - \exp(-\lambda_1 t))^{n-n_0}.$$

Received 5 December 1989; revision received 13 March 1990.

\* Postal address: Department of Mathematics and Statistics, University of Otago, PO Box 56, Dunedin, New Zealand.

Let the random time taken for the population size to reach the level  $s$  be  $T_s$ , then  $P(T_s > t) = \sum_{n=n_0}^{s-1} P\{X(t) = n\}$  and so the probability density function of  $T_s$  is:

$$\begin{aligned}
 f_s(t) &= - \sum_{n=n_0}^{s-1} \frac{d}{dt} P\{X(t) = n\} = \lambda_1(s-1)P\{X(t) = s-1\} \\
 (2.3) \quad &= \frac{\lambda_1(s-1)! (\exp(-\lambda_1 t))^{n_0} (1 - \exp(-\lambda_1 t))^{s-1-n_0}}{(n_0-1)! (s-1-n_0)!}.
 \end{aligned}$$

After reaching this size  $s$ , the population behaves as an *immigration–birth* process ((1.1) and (2.1)) with conditional probabilities  $p_n(t) = P\{X(t) = n \mid X(0) = s\}$ , for  $n = s, s+1, \dots$ , forming another negative binomial distribution (cf. exercise 8 on p. 200 of Cox and Miller (1965)):

$$(2.4) \quad p_n(t) = \frac{(\alpha/\lambda_2 + n - 1)!}{(\alpha/\lambda_2 + s - 1)! (n - s)!} (\exp(-\lambda_2 t))^{(\alpha/\lambda_2 + s)} (1 - \exp(-\lambda_2 t))^{n-s}.$$

Thus for  $n = s, s+1, \dots$ , the state probabilities  $P\{X(t) = n\}$  will be given by:

$$(2.5) \quad \int_0^t P\{X(t) = n \mid X(\tau) = s\} f_s(\tau) d\tau = \int_0^t p_n(t - \tau) f_s(\tau) d\tau,$$

since the population size must have reached  $s$  at some time.

Equations (2.5), (2.4) and (2.3) together with (2.2) thus make up the probability distribution of the population size, from which moments follow.

$$\begin{aligned}
 E\{X(t)\} &= n_0 \exp(\lambda_1 t) \sum_{k=0}^{s-n_0-1} \frac{(n_0+k)!}{n_0! k!} (\exp(-\lambda_1 t))^{n_0+1} (1 - \exp(-\lambda_1 t))^k \\
 (2.6) \quad &+ \frac{(\alpha/\lambda_2 + s) \exp(\lambda_2 t) (s-1)!}{(n_0-1)! (s-n_0-1)!} \int_0^{1-\exp(-\lambda_1 t)} y^{s-n_0-1} (1-y)^{n_0+\lambda_2/\lambda_1-1} dy \\
 &- \frac{\alpha/\lambda_2 (s-1)!}{(n_0-1)! (s-n_0-1)!} \int_0^{1-\exp(-\lambda_1 t)} y^{s-n_0-1} (1-y)^{n_0-1} dy.
 \end{aligned}$$

The sum and integrals on the right-hand side of (2.6) can all be expressed in terms of the incomplete beta function (Abramowitz and Stegun (1972), § 26.5),  $I_x(a, b)$ :

$$\int_0^x \frac{(a+b-1)!}{(a-1)! (b-1)!} y^{a-1} (1-y)^{b-1} dy = \sum_{k=0}^{b-1} \frac{(a+k-1)!}{(a-1)! k!} x^a (1-x)^k$$

so that (2.6) becomes:

$$\begin{aligned}
 E\{X(t)\} &= n_0 \exp(\lambda_1 t) I_{\exp(-\lambda_1 t)}(n_0 + 1, s - n_0) \\
 (2.7) \quad &+ (\alpha/\lambda_2 + s) \exp(\lambda_2 t) \frac{(s-1)! (n_0 + \lambda_2/\lambda_1 - 1)!}{(n_0-1)! (s + \lambda_2/\lambda_1 - 1)!} \\
 &\times I_{1-\exp(-\lambda_1 t)}(s - n_0, n_0 + \lambda_2/\lambda_1) \\
 &- \alpha/\lambda_2 I_{1-\exp(-\lambda_1 t)}(s - n_0, n_0).
 \end{aligned}$$

Also:

$$\begin{aligned}
 E\{X(t)(X(t) + 1)\} &= n_0(n_0 + 1) \exp(2\lambda_1 t) I_{\exp(-\lambda_1 t)}(n_0 + 2, s - n_0) \\
 (2.8) \quad &+ (\alpha/\lambda_2 + s)(\alpha/\lambda_2 + s + 1) \exp(2\lambda_2 t) \frac{(s-1)! (n_0 + 2\lambda_2/\lambda_1 - 1)!}{(n_0-1)! (s + 2\lambda_2/\lambda_1 - 1)!} \\
 &\times I_{1-\exp(-\lambda_1 t)}(s - n_0, n_0 + 2\lambda_2/\lambda_1) \\
 &- 2\alpha/\lambda_2 (\alpha/\lambda_2 + s) \exp(\lambda_2 t) \frac{(s-1)! (n_0 + \lambda_2/\lambda_1 - 1)!}{(n_0-1)! (s + \lambda_2/\lambda_1 - 1)!} \\
 &\times I_{1-\exp(-\lambda_1 t)}(s - n_0, n_0 + \lambda_2/\lambda_1) \\
 &+ \alpha/\lambda_2 (\alpha/\lambda_2 - 1) I_{1-\exp(-\lambda_1 t)}(s - n_0, n_0)
 \end{aligned}$$

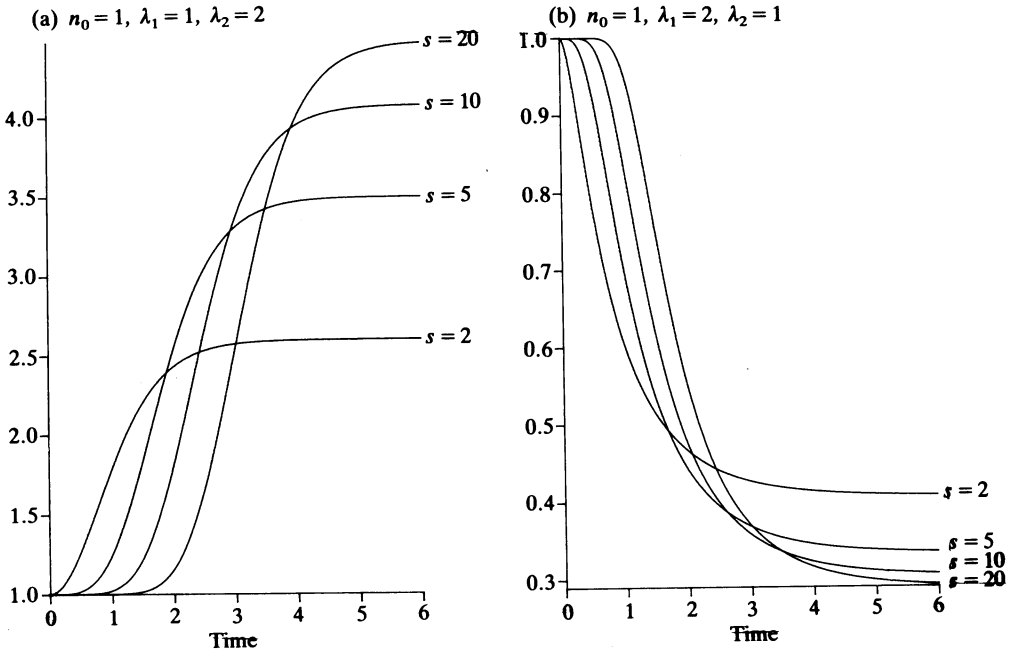


Figure 1. Relative variation

from which the variance of the population size may be determined.

Now had the birth process  $\{X(t), t \geq 0\}$  been linear, the negative binomial distribution (2.2) would have held over all population sizes  $\geq n_0 = X(0)$ , in which case (Cox and Miller (1965), Chapter 4)  $\text{Var}\{X(t)\} = E\{X(t)\}[E\{X(t)\}/n_0 - 1]$ ; so

$$(2.9) \quad \frac{\text{Var}\{X(t)\}}{E\{X(t)\}[E\{X(t)\}/X(0) - 1]}$$

seems a natural measure of relative variation for birth processes. Shown in Figure 1 are some numerical calculations of this quantity (2.9) from (2.7) and (2.8) for some typical birth rates  $\lambda(n)$  from (2.1): (a) convex  $\lambda(n)$  ( $\lambda_1 < \lambda_2$ ) and (b) concave  $\lambda(n)$  ( $\lambda_1 > \lambda_2$ ). There is more relative (to the linear case) variation for convex  $\lambda(n)$  and less for concave  $\lambda(n)$ , with a similar pattern being observed for other numerical choices of  $\lambda_1$  and  $\lambda_2$ : it is tempting to conjecture that this is generally true, and not just for the specific form (2.1) of  $\lambda(n)$ .

The opposite result for non-linear death processes was conjectured by Faddy (1985) and subsequently proved by Ball and Donnelly (1987). Their proof rested not on general expressions for the mean and variance of the population size, but on expressing a similar relative variation in terms of correlations between pairs of individuals initially present in the population. For birth processes, (2.9) must exemplify some interaction between individuals in the population; just how is not at all clear, and a proof remains elusive!

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