CHARACTERIZATIONS OF DEVELOPABLE TOPOLOGICAL SPACES

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1. Introduction. The class of developable topological spaces, which includes the metrizable spaces, has been fundamentally involved in investigations in point set topology. One example is the remarkable edifice of theorems relating to these spaces constructed by R. L. Moore **(13)**. Another is the role played by the developable property in several metrization theorems, including Alexandroff and Urysohn's original solution of the general metrization problem **(1)**.

This paper presents an anslysis of the concept of developable space in terms of certain more extensive classes of spaces satisfying the first axiom of countability: spaces with a base of countable order and those having what is here called a θ-base. The analysis is given in the characterizations of Theorems 3 and 4 below.

Arhangel'skiĭ introduced bases of countable order in a significant metrization theorem **(4)**. He proved that a T_2 paracompact space is metrizable if and only if it has a base of countable order. This generalizes the theorem that a T_2 paracompact developable space is metrizable, which, as Ponomarev **(15)** among others observed, is a restatement of the original Alexandroff–Urysohn theorem. Theorem 3 connects the concepts of developability and base of countable order. It may be viewed as an analogue, for developable spaces, of the metrization results mentioned above, in which paracompactness is replaced by a generalization called θ-refinability. Theorem 4 gives a characterization in terms of a θ-base. This concept generalizes the notions of σ-discrete base and σ locally finite base, which were used by Bing **(6)** and Nagata and Smirnov **(14, 16)**, respectively, to obtain metrization theorems different in character from those mentioned above. Theorem 4 is analogous to these theorems and another theorem of Arhangel'skiĭ (given as Theorem D in Section 5 below).

The concepts involved here have forerunners in some that are used in metrization theorems; certain of the techniques of proof have not. A technical difficulty not met in analysing paracompact T_2-spaces—namely, that it is not true even for all metacompact regular T_1-spaces that for every collection H of open sets covering space there exists a countable family F of collections of discrete refinements of H such that the sum of the elements of F covers space—is overcome in the proof of Theorem 3 with the use of a well-ordering

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technique and König's Lemma **(10)** or **(13**, p. 47**)**. A related difficulty is overcome in the proof of Theorem 4 by a technique similar to one that Arhangel'skiĭ uses in **(4**, proof of Theorem 1**)**. It is here that we use the requirement that every closed point set be an inner limiting set (G_δ-set).

Prior to characterizing developability, we shall prove two theorems concerning bases of countable order. Theorem 1 states a remarkable property associated with this concept. Theorem 2, closely related in method of proof to Theorem 1, gives an alternative way of describing essentially T_1-spaces with a base of countable order that is similar in form to the definition of *development*. Theorem 2 is used in proving Theorem 3; these theorems provide information about the relation of developable spaces to those with a base of countable order.

We conclude the paper with applications to the metrization problem. Since developability is involved in the Alexandroff–Urysohn metrization theorem, our characterization theorems 3 and 4 yield two metrization theorems. We use the Alexandroff–Urysohn theorem and a result of Bing **(6)** on the paracompactness of collectionwise normal developable spaces to state and prove Theorems 5 and 6 below. The latter imply certain well-known metrization theorems, as is shown in the final section.

Definitions and Notation. For concepts not defined below, we refer the reader to **(9)** or **(11)**. Rather than speaking of a topological space (S, τ) we shall speak of a topological space S (or just of *space*) and understand that a certain topology τ for S underlies the discussion. Where it is contextually clear, we use the conventions that *point* means *element of* S and *open set* means *element of* τ. Recall that a *monotonic* collection of sets is a collection such that for every two of its elements one is a subset of the other **(13)**. A collection of sets will be called *perfectly decreasing* if and only if it contains a proper subset of each of its elements. Recall that a *development* for S is a sequence $G_1, G_2, \ldots$ of collections of open sets covering S such that for each point P of any open set D there exists an integer n such that all elements of G_n which contain P are subsets of D**(6)**. Also recall that a *developable space* is defined to be a space with a development **(6)**. The statement that a space is *essentially* T_1 means that for any points P and Q either $\{\bar{P}\} = \{\bar{Q}\}$ or $\{\bar{P}\}$ does not intersect $\{\bar{Q}\}$. The sum of all elements of a collection H of sets will be denoted by H^*.

2. Bases of countable order. By a *base of countable order* **(4)** for a topological space S is meant a base B for S such that if T is a perfectly decreasing subcollection of B, and P is a point common to all elements of T, then T is a base for S at P.

THEOREM 1. *If for every point P of a topological space S some open subset of S contains P and has a base of countable order, then every subspace of S has a base of countable order* (*in the relative topology*).

Proof. It will first be shown that S has a base of countable order. Suppose for every point P there exists a collection B_P of open sets such that B_P^* contains P and B_P is a base of countable order for B_P^*. There exists a well-ordered collection Q such that (1) for some P in S, B_P is the first element of Q, (2) if σ is a proper initial segment of Q there exists a point P belonging to no element of σ^* such that B_P is the first element of Q following every element of σ, and (3) Q^* covers S. (We leave this and similar constructions, which involve the relatively straightforward use of well-ordering and transfinite induction, to the reader.)

For each q in Q let W_q denote the collection to which X belongs if and only if X belongs to q and contains a point belonging to no element of Q^* contained in a predecessor of q. Let V denote the sum of all W_q. If Y is a point of an open set D, there is a first q in Q such that q^* contains Y. Some subset R of D contains Y and belongs to q and therefore R belongs to W_q. So V is a base for S.

Suppose T is a perfectly decreasing monotonic subcollection of V, and P is a point common to every element of T. There is a first q in Q intersecting T. If t belongs to $q \cdot T$ there exists a proper subset t' of t that belongs to T. There is an element q' of Q such that $W_{q'}$ contains t'. But q' does not follow q; for if it did, then t' would contain a point not in q^*. Since q' does not precede q, $q' = q$. Thus $q \cdot T$ is perfectly decreasing. Since q is a base of countable order for q^*, $q \cdot T$ is a base for S at P. So V is a base of countable order for S.

Now suppose that S' is a subspace of S. There exists a sequence $H_1, H_2, \ldots$ of well-ordered subcollections of V covering S' and satisfying these conditions: (1) For each n and h in H_n there exists a point $P_{n,h}$ belonging to $h \cdot S'$ such that no element of H_n precedes h and contains $P_{n,h}$. (2) If $n < k$, the first element h of H_k containing the point P of S' is a subset of the first element h' of H_n doing so, and if some proper open subset of h' contains P, then h is a proper subset of h'. Let B denote the collection to which an element belongs if and only if it is the common part of S' and some element of $H_1 + H_2 + \ldots$.

I. Suppose that $R = D \cdot S'$, where D is open in S and P is a point of R. For each n let h_n denote the first element of H_n that contains P. From condition (2) above it follows that each h_{n+1} is a subset of h_n and that if some $h_n = h_{n+1}$, then $h_n \cdot S'$ is a subset of R. If there exists no n such that $h_n = h_{n+1}$, then the collection of all sets h_n is a perfectly decreasing monotonic subcollection of V each element of which contains P. Since V is a base of countable order for S, some h_n is a subset of D. Hence $h_n \cdot S'$ is a subset of R, and B is a base for S'.

II. Suppose that T is a perfectly decreasing monotonic subcollection of B and that P belongs to every element of T. If N is a finite set of integers such that for each n in N, H_n has a first element h_n for which $h_n \cdot S'$ belongs to T, then there exists some k in N such that $h_k \cdot S'$ is a subset of all the sets $h_n \cdot S'$. For some i and h in H_i, $h \cdot S'$ belongs to T and is a proper subset of $h_k \cdot S'$. If i belongs to N, then either h is h_i or h_i precedes h. If h_i precedes h, then h_i does not contain the point $P_{i,h}$ of $h \cdot S'$, and since T is monotonic, $h_i \cdot S'$ is a subset of $h \cdot S'$. Thus, in either case, $h_k \cdot S'$ is a subset of $h \cdot S'$. Since this involves a contradiction, i does not belong to N. From these considerations and condition (2) it follows that for each n there is a first element h_n of H_n that includes an element of T. Moreover, if $n < k$, there exists some $i > k$ and some h in H_i such that $h \cdot S'$ is an element of T which is a proper subset of $h_n \cdot S'$ and $h_k \cdot S'$. By condition (2), h is a subset of the first h' in H_n to contain $P_{i,h}$. Since h_n is the first element of H_n to include an element of T and $P_{i,h}$ is in h_n, $h' = h_n$. It follows that h is a proper subset of h_n. Similarly, h_k is the first element of H_k to contain $P_{i,h}$. Condition (2) implies that h_k is a proper subset of h_n. So the collection of all sets h_n is a perfectly decreasing monotonic subcollection of V all elements of which contain P. If $R = D \cdot S'$ where D is open in S and R contains P, it follows that some h_n is a subset of D. Hence a subset of h_n belonging to T is a subset of R. Thus B is a base of countable order for S'.

THEOREM 2. *An essentially T_1 topological space S has a base of countable order if and only if there exists a sequence $G_1, G_2, \ldots$ of bases for S such that if P is a point and $g_1, g_2, \ldots$ is a sequence such that for each n, g_n belongs to G_n, g_n contains P, and g_{n+1} is a subset of g_n, then the collection of all g_n's is a base for S at P.*

Proof. Suppose S has a base V of countable order. Let $H_1, H_2, \ldots$ be as in the above proof, where $S = S'$. For each n, let

$$G_n = H_n + H_{n+1} + \ldots.$$

Clearly each G_n is a base for S. If P and $g_1, g_2, \ldots$ are as above, then for each n there is a first h_n in H_n that includes a term of $g_1, g_2, \ldots$. For every n, there exists $i > n + 1$ and h in H_i such that h is a term of $g_1, g_2, \ldots$ that is a subset of h_n and h_{n+1}. The first elements of H_n and H_{n+1} to contain $P_{i,h}$ are h_n and h_{n+1}, respectively. It follows from condition (2) that h_n includes h_{n+1}. If $h_n = h_{n+1}$ there exists no proper open subset of h_n containing $P_{i,h}$. Since S is essentially T_1, and P belongs to h_n, $h_n = \{\overline{P_{i,h}}\} = \{\bar{P}\}$. So h_n is a subset of every open set containing P. If no $h_n = h_{n+1}$, the collection of all h_n's is a base for S at P. Since each h_n includes a term of $g_1, g_2, \ldots$, the collection of these terms is a base for S at P.

Suppose $G_1, G_2, \ldots$ is a sequence as in the statement of Theorem 2. There exists a sequence $H_1, H_2, \ldots$ of well-ordered collections covering S such that (1) each H_n is a subcollection of G_n, (2) each element h of H_n contains a point belonging to no predecessor of h in H_n, and (3) if $n < k$ the first element of H_k containing the point P is a subset of the first element of H_n doing so. Clearly, $H_1 + H_2 + \ldots$ is a base for S. If T is a perfectly decreasing monotonic subcollection of $H_1 + H_2 + \ldots$ and P belongs to each element of T, then, by an argument similar to that of II in the proof of Theorem 1, for each n there is a first element h_n of H_n that includes an element of T and each h_{n+1} is a subset of h_n. Since each h_n belongs to G_n and contains P, the collection of all h_n's is a base for S at P, and so T is also such a base.

Remarks. 1. The conditions on $G_1, G_2, \ldots$ in Theorem 2 are closely related to some considered by Aronszajn **(5)**.

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2. The above proof shows that in the statement of Theorem 2 one may require that for each n, G_{n+1} is a subset of G_n. We shall refer to this result as Theorem 2′.

3. The existence of a sequence $G_1, G_2, \ldots$ as in Theorem 2, with the stipulation that g_{n+1} is a subset of g_n removed, is a necessary and sufficient condition for a topological space to be developable.

3. The concepts of θ-base and θ-refinability. In this section we define the basic concepts used in the characterizations of Section 4. The notion of θ-base generalizes those of σ locally finite and σ point-finite base. The concept of θ-refinability is a generalization of both paracompactness and metacompactness. To say that a collection W of point sets is *finite at a point* P will mean that P is in W^* and only a finite number of elements of W contain P.

DEFINITION 1. *A collection W of point sets is σ-distributively point-finite if and only if there exists a countable family C of subcollections of W such that C^* is W and for every point P of W^* there exists an F in C which is finite at P.*

DEFINITION 2. *A topological space S is θ-refinable if and only if for every covering H of S whose elements are open, there is a countable family C such that each F in C is a collection of open sets which is a refinement of H covering S and for every point P of S there exists an F in C which is finite at P.*

By a *θ-refinement* of a collection H of open sets covering S is meant a collection C^* such that C and H have the properties indicated in Definition 2. Note that a θ-refinement C^* of H is a σ-distributively point-finite refinement of H in which *each* element of the countable family C covers space.

Remark. The following equivalence is useful in proving Theorem 3:

A topological space S is θ-refinable if and only if for every collection H of open sets covering S there exists a countable collection K of closed point sets such that $K^* = S$, and for each M in K there exists a refinement V of H such that V is a collection of open sets covering S which is finite at each point of M. A proof of necessity may be given using the fact that if F is a collection of open sets covering S and there is a point P contained in k but not in $k + 1$ elements of F, then the set of all points contained in not more than k elements of F is closed.

DEFINITION 3. *A θ-base for S is a base for S which is the sum of a countable family C of collections of open sets such that if D is an open set and P is a point of D, there is an F in C such that the collection of all elements of F containing P is finite and contains a subset of D.*

The following theorems are stated to make more precise the sense in which θ-refinability generalizes metacompactness.

THEOREM (i). *In a T_1 θ-refinable topological space, a closed point set is bicompact* **(2)** *if and only if it is compact* **(8)**.

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THEOREM (ii). *If H is a collection of open sets covering a θ-refinable topological space S, there exists a σ-distributively point-finite refinement of H that is a collection of open sets covering S minimally.*

THEOREM (iii). *Every T_1 collectionwise normal θ-refinable space is a paracompact T_2-space.*

(The references cited immediately below contain proofs which can be modified so as to obtain these theorems.)

There is a technical unity to these theorems revolving around the σ point-*finite* conditions involved in their hypotheses so that, roughly speaking, there are proofs for each of them which are almost proofs for the other two. Theorems (i) and (ii) generalize results of Arens and Dugundji **(3)** for metacompact spaces; Theorem (iii) extends a result of Michael **(12)** on metacompact spaces. It is this technical unity as regards method of proof which, from our viewpoint, gives Theorem (i) a natural place in the above sequence of theorems, for the following more general theorem holds true.

THEOREM (iv). *If S is a T_1-space and F is a countable family of collections of open sets such that every point P belongs to an element of a member of F that does not have an uncountable number of elements containing P, then every closed and compact subset of S is covered by a finite subcollection of F^*.*

The *countability* of certain coverings at certain points may be used in obtaining a proof of this theorem.

4. Developable spaces. The characterizations of developable spaces are given here.

THEOREM 3. *A topological space is developable if and only if it is essentially T_1, θ-refinable, and has a base of countable order.*

Proof. Suppose S is essentially T_1, θ-refinable, and has a base of countable order. Let $G_1, G_2, \ldots$ be a sequence satisfying the conditions of Theorem 2′; see Remark 2 following Theorem 2. With the use of the remark following Definition 2, it may be shown that there exists a sequence $H_1, H_2, \ldots$ and, for every positive integer n, sequences $V_{n1}, V_{n2}, \ldots$ and $M_{n1}, M_{n2}, \ldots$ such that for each n:

(1) H_n is a subcollection of G_n covering S.

(2) The sum of the collection of the V_{ni}'s is a θ-refinement of H_n.

(3) $M_{n1}, M_{n2}, \ldots$ is a sequence of closed point sets such that

$$S = M_{n1} + M_{n2} + \ldots .$$

(4) For each i, V_{ni} is finite at every point of M_{ni}.

(5) Each element h of H_n contains a point $P_{n,h}$ such that if $i < n$ and $j < n$, then if $P_{n,h}$ belongs to M_{ij}, h is a subset of every element of V_{ij} that contains $P_{n,h}$ and if $P_{n,h}$ does not belong to M_{ij}, then h does not intersect M_{ij}.

If $H_1, H_2, \ldots$ is not a development of S, there exists an open set D containing a point P such that each H_n has an element containing P that is not a subset of D. There exists a sequence $n_1, n_2, \ldots$ of positive integers such that (1) $n_1 = 1$ and (2) if $i > 1$, n_i is the first positive integer such that if k denotes $\sum_{j=1}^{i-1} n_j$, then M_{k,n_i} contains P. For each i, let w_i denote $\sum_{j=1}^{i} n_j$. If $i > 1$ and h is an element of H_{w_i} containing P, then h intersects M_{w_{i-1},n_i}. Moreover $w_{i-1} < w_i$ and $n_i < w_i$. So $P_{w_i,h}$ belongs to M_{w_{i-1},n_i}. Hence h is a subset of every element of V_{w_{i-1},n_i} that contains $P_{w_i,h}$. It follows that there is a finite collection Q_{i-1} to which an element belongs if and only if it belongs to V_{w_{i-1},n_i}, contains P, and is not a subset of D. For each $i > 1$ there exists a finite subcollection K_{i-1} of $H_{w_{i-1}}$ such that every element of K_{i-1} contains an element of Q_{i-1} and every element of Q_{i-1} is a subset of some element of K_{i-1}. For each $i > 1$, if k belongs to K_i, then k is a subset of an element of Q_{i-1} and is therefore a subset of an element of K_{i-1}. So by König's Lemma there exists a sequence $k_1, k_2, \ldots$ such that for each $i > 1$, k_{i-1} belongs to K_{i-1} and k_i is a subset of k_{i-1}. Since each k_i belongs to G_{w_i}, since $w_i \geqslant i$, and since G_{n+1} is a subset of G_n for each n, it follows that each k_i belongs to G_i. But this involves a contradiction, for the collection of all k_i's is a base for S at P. Thus the conditions stated are sufficient for the space to be developable.

The reader may easily verify that every developable space is essentially T_1. If a topological space S has the property that for every open covering H of S there exists a countable family F of discrete collections of closed point sets such that F^* covers S and refines H, then S is θ-refinable. Every developable space has this property **(6)**. That every developable space has a base of countable order is a corollary of Theorem 2.

Remark. That a developable space S has a base of countable order also follows easily from the fact that every collection of open sets covering S has a refinement which is a collection of open sets covering S minimally. Theorem (ii) implies this.

LEMMA. *Let S be a topological space. Suppose* (1) *M is a closed subset of S,* (2) *H is a collection of open sets covering M,* (3) *k is a positive integer such that some point P of M belongs to k, but not to $k+1$, elements of H, and J is the set of all such points P, and* (4) *every closed point set is an inner limiting set. Then there exists a countable family F of discrete collections of closed point sets such that* (1) *J is the sum of the sets belonging to members of F and* (2) *if δ belongs to a member of F, then δ is included in an open set that is a subset of every element of H that intersects δ.*

Outline of proof. For each P in M let M_P denote the set of all points P' of M such that P and P' lie in the same sets of H. Let Q denote the collection of all sets M_P, for points P belonging to J.

I. Suppose every point P of M is in at least k elements of H. Then if P

belongs to J, M_P is closed, and if P and P' are points of J, $M_{P'} = M_P$ or $M_{P'}$ does not intersect M_P. If P is not in M, it is in an open set that does not intersect an element of Q. If P is a point of M, there is an open set D containing P that is a subset of k elements of H if P belongs to J and that is a subset of $k + 1$ elements of H if P does not belong to J. Thus if P belongs to J, M_P is the only element of Q that intersects D and if P does not belong to J, no element of Q intersects D. So Q is a discrete collection. It may be seen that $\{Q\}$ is a family F as above.

II. Suppose that for some point P of M, there are not k elements of H containing P, and let C denote the set of all these P's. There exists a sequence $D_1, D_2, \ldots$ of open sets having C as their common part such that D_{i+1} is a subset of D_i and does not include J. For each n let Δ_n denote the collection to which δ belongs if and only if for some q in Q not a subset of D_n, δ is the set of all points of q not in D_n. For reasons similar to some involved in I above, each Δ_n is a discrete collection of closed subsets of J such that each δ in Δ_n is contained in an open set that is a subset of every element of H that intersects δ. Moreover, $J = \Delta_1 + \Delta_2 + \ldots$. So the collection of all Δ_n's is a family F as above.

THEOREM 4. *A topological space S is developable if and only if S has a θ-base and every closed subset of S is an inner limiting set.*

Proof. Suppose that S has a θ-base B and that every closed set is an inner limiting set. Let V_1, $V_2, \ldots$ denote the elements of a countable family which determines B according to Definition 3. We may assume that each V_n is finite at some point. By the above lemma, there exist a sequence $\Delta_1, \Delta_2, \ldots$ and a collection Ω of ordered pairs of positive integers such that (1) if (m, n) belongs to Ω, then Δ_n is a refinement of V_m that is a discrete collection of closed point sets such that each δ in Δ_n is included in an open set that is a subset of every element of V_m that intersects δ, (2) every positive integer is the second term of some element of Ω, and (3) if V_m is finite at P, there exists n such that (m, n) belongs to Ω and P belongs to an element of Δ_n. For each (m, n) in Ω there exists a collection $H_{m,n}$ to which h belongs if and only if (i) for some δ in Δ_n, h is an open subset of every element of V_m that intersects δ and h intersects no element of Δ_n except δ or (ii) h is an open set that intersects no element of Δ_n.

By condition (1) above, each $H_{m,n}$ covers S. Moreover, if P is a point of an open set D, there exists some m such that V_m is finite at P and some element v of V_m contains P and is a subset of D. By condition (3) above there exists some n such that (m, n) belongs to Ω and P belongs to an element δ of Δ_n. If h belongs to $H_{m,n}$ and contains P, then, by definition of $H_{m,n}$, h is a subset of every element of V_m that intersects δ and is therefore a subset of v. Hence h is a subset of D. There exists a sequence $G_1, G_2, \ldots$ such that every term of this sequence is one of the collections $H_{m,n}$ and every collection $H_{m,n}$ is a term of this sequence. It follows that G_1, $G_2, \ldots$ is a development of S.

From considerations similar to some involved in the proof of Theorem 2 it may be seen that every developable topological space has a θ-base. Closed point sets are inner limiting sets in developable spaces.

5. Applications to the metrization problem. In this section we state and prove two metrization theorems. The proofs will be based on the characterization theorems 3 and 4 above and the original theorem of Alexandroff and Urysohn, who based their proof on a result of Chittenden **(7)**. A statement of their theorem is included here for completeness; we note that it is given in terms of a condition on a development.

THEOREM A (Alexandroff–Urysohn). *A T_1-space is metrizable if and only if it has a development $G_1, G_2, \ldots$ such that, for each n, the sum of any two elements of G_{n+1} having a point in common is included in a member of G_n* **(1)**.

We state the following theorem of Bing for use in the proofs of our theorems.

THEOREM B (Bing). *A T_1 regular developable space (equivalently, a Moore space) is metrizable if and only if it is collectionwise normal* **(6)**.

Proof. This follows from the restatement of the Alexandroff–Urysohn theorem given in the Introduction and the theorem of Bing that a developable collectionwise normal space is paracompact.

THEOREM 5. *A collectionwise normal T_1-space is metrizable if and only if it is θ-refinable and has a base of countable order.*

Proof of sufficiency. Theorem 3 implies that the space is developable; an application of Theorem B completes the proof.

THEOREM 6. *A collectionwise normal T_0-space is metrizable if and only if it has a θ-base and every closed subset of the space is an inner limiting set.*

Proof of sufficiency. Theorem 4 implies that the space is developable, and a T_0 essentially T_1-space is T_1. Theorem B, again, may be used to complete the proof.

These theorems generalize, in turn, Theorems C and D below of Arhangel'skiĭ **(4)**.

THEOREM C. *A paracompact Hausdorff space is metrizable if and only if it has a base of countable order.*

THEOREM D. *A T_1-space with a σ point-finite base is metrizable if and only if it is perfectly normal and collectionwise normal.*

Since a paracompact Hausdorff space is collectionwise normal, T_1, and θ-refinable, Theorem C follows from Theorem 5. A σ point-finite base is a θ-base and, by definition, every perfectly normal space has the property that every closed point set is an inner limiting set. Thus Theorem D is a consequence of Theorem 6.

We now state four well-known theorems and indicate how they follow from Theorems 5 and 6.

THEOREM E (Bing). *A regular T_1-space is metrizable if it has a σ-discrete base* **(6)**.

THEOREM F (Nagata–Smirnov). *A regular T_1-space is metrizable if it has a σ locally finite base* **(14, 16)**.

THEOREM G (Urysohn). *A regular T_1-space is metrizable if it has a countable base* **(17; 9**, p. 125**)**.

THEOREM H (Smirnov). *A paracompact T_2-space is metrizable if it is locally metrizable* **(16)**.

The hypotheses of Theorems E, F, and G all clearly imply that the spaces have σ locally finite bases and, therefore, θ-bases. Regular spaces with σ locally finite bases are collectionwise normal as may be seen by an argument modelled on the proof that such spaces are normal **(9**, p. 127**)**. These spaces clearly have the property that closed point sets are inner limiting sets. Thus the hypothesis of Theorem 6 is satisfied and the theorems follow.

The requirement of local metrizability in Theorem H implies that the space has a local base of countable order at each point. By Theorem 1, the space has a base of countable order. The result now follows from Theorem 5.

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