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HOMOLOGICAL PROPERTIES OF CERTAIN BANACH MODULES OVER GROUP ALGEBRAS

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Abstract Homological properties of several Banach left $L^1(G)$-modules have been studied by Dales and Polyakov and recently by Ramsden. In this paper, we characterize some homological properties of $L_0^\infty(G)$ and $L_0^\infty(G)^*$ as Banach left $L^1(G)$-modules, such as flatness, injectivity and projectivity.

Keywords: amenability; Banach module; flatness; injectivity; projectivity; locally compact group

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1. Introduction and preliminaries

Throughout this paper, G denotes a locally compact group with the identity element e, the modular function Δ and a fixed left Haar measure λ. As usual, let $L^1(G)$ denote the group algebra of G as defined in [**5**] equipped with the norm $\|\cdot\|_1$ and the convolution product $*$ of functions on G defined by

$$(\phi * \psi)(x) = \int_G \phi(y)\psi(y^{-1}x)\,\mathrm{d}\lambda(y)$$

for all $\phi, \psi \in L^1(G)$ and locally almost all $x \in G$. Also, let $L^\infty(G)$ denote the Banach space as defined in [**5**] equipped with the essential supremum norm $\|\cdot\|_\infty$. Then $L^\infty(G)$ is the dual bimodule of the Banach $L^1(G)$-bimodule $L^1(G)$ under the pairing

$$\langle f, \phi\rangle = \int_G f(x)\phi(x)\,\mathrm{d}\lambda(x)$$

for all $\phi \in L^1(G)$ and $f \in L^\infty(G)$. The left and right module actions of $L^1(G)$ on $L^\infty(G)$ are given by the formulae

$$\phi \cdot f = f * \tilde{\phi} \quad \text{and} \quad f \cdot \phi = \frac{1}{\Delta}\tilde{\phi} * f$$

for all $f \in L^\infty(G)$ and $\phi \in L^1(G)$, where $\tilde{\phi}(x) = \phi(x^{-1})$ for all $x \in G$. Let $L_0^\infty(G)$ denote the closed subspace of $L^\infty(G)$ consisting of all $g \in L^\infty(G)$ that vanish at infinity: that is, for each $\varepsilon > 0$, there is a compact subset K of G for which $\|g\chi_{G\setminus K}\|_\infty < \varepsilon$, where $\chi_{G\setminus K}$ denotes the characteristic function of $G \setminus K$ on G. Then $L_0^\infty(G)$ is a closed submodule of the Banach $L^1(G)$-bimodule $L^\infty(G)$; in fact, for each $g \in L_0^\infty(G)$ and $\phi \in L^1(G)$, we have $\phi \cdot g, g \cdot \phi \in C_0(G)$, the space of all continuous functions on G vanishing at infinity. Hence, the dual space $L_0^\infty(G)^*$ of $L_0^\infty(G)$ is also a Banach $M(G)$-bimodule with the dual actions

$$\langle \phi \cdot m, g\rangle = \left\langle m, \frac{1}{\Delta}\tilde{\phi} * g\right\rangle \quad \text{and} \quad \langle m \cdot \phi, g\rangle = \langle m, g * \tilde{\phi}\rangle$$

for all $\phi \in L^1(G)$ and $m \in L_0^\infty(G)^*$. For an extensive study of $L_0^\infty(G)$ and $L_0^\infty(G)^*$, see [**7**] (see also [**6**] for the compact group case).

Homological properties of several Banach left $L^1(G)$-modules have recently been studied by Dales and Polyakov [**2**] and by Ramsden [**8**]. However, homological properties of the Banach left $L^1(G)$-modules $L_0^\infty(G)$ and $L_0^\infty(G)^*$ have not been studied so far. Our aim in this paper is to characterize some properties of $L_0^\infty(G)^*$ and $L_0^\infty(G)$ as Banach left $L^1(G)$-modules, such as flatness, injectivity and projectivity in terms of G.

2. Projectivity of $L_0^\infty(G)$ and $L_0^\infty(G)^*$

Let E and F be two Banach spaces and denote by $B(E,F)$ the Banach space of all bounded operators from E into F. An operator $T \in B(E,F)$ is called *admissible* if $T \circ S \circ T = T$ for some $S \in B(F,E)$. In the case where A is a Banach algebra and E and F are Banach left A-modules, ${}_AB(E,F)$ denotes the closed linear subspace of $B(E,F)$ of all left A-module morphisms. An operator $T \in {}_AB(E,F)$ is a *retraction* if there exists $S \in {}_AB(F,E)$ with $T \circ S = I_F$, the identity operator on F; in this case, F is called a *retract* of E.

A Banach left A-module P is called *projective* if, for Banach left A-modules E and F, each admissible epimorphism $T \in {}_AB(E,F)$ and each $S \in {}_AB(P,F)$, there exists $R \in {}_AB(P,E)$ such that $T \circ R = S$.

Our first result characterizes projectivity of $L_0^\infty(G)$ as a Banach left $L^1(G)$-module.

Theorem 2.1. *Let G be a locally compact group. Then $L_0^\infty(G)$ is a projective Banach left $L^1(G)$-module if and only if G is finite.*

Proof. It is well known that $L^\infty(G)$ is a projective Banach left $L^1(G)$-module if and only if G is finite [**2**, Theorem 3.3]. We therefore only need to recall that G is compact if there is a projective Banach left $L^1(G)$-module E with $C_0(G) \subseteq E \subseteq L^\infty(G)$ [**2**, Theorem 3.1]. □

We now describe projectivity of $L_0^\infty(G)^*$ as a Banach left $L^1(G)$-module.

Theorem 2.2. *Let G be a locally compact group. Then $L_0^\infty(G)^*$ is a projective Banach left $L^1(G)$-module if and only if G is discrete.*

Proof. It is clear that if G is discrete, then $L_0^\infty(G)$ is the space of all functions on G vanishing at infinity, and so $L_0^\infty(G)^* = L^1(G)$. So, the 'if' part follows from the fact that $L^1(G)$ is always a projective Banach left $L^1(G)$-module [**2**, Theorem 2.4].

To prove the converse, suppose that $L_0^\infty(G)^*$ is a projective Banach left $L^1(G)$-module. We will show that G is discrete.

On the one hand, $C_0(G)$ is a closed submodule of the Banach $L^1(G)$-bimodule $L_0^\infty(G)$, and its dual $C_0(G)^*$ is a projective Banach right $L^1(G)$-module with the dual actions $\langle \phi \cdot \mu, g\rangle = \langle \mu, g \cdot \phi\rangle$ for all $\phi \in L^1(G)$ and $\mu \in C_0(G)^*$ if and only if G is discrete [**2**, Theorem 2.6]. On the other hand, each retraction of a projective Banach left $L^1(G)$-module is projective [**3**]. We therefore only need to prove that $C_0(G)^*$ is a retraction of the Banach left $L^1(G)$-module $L_0^\infty(G)^*$.

To this end, let $\mathcal{P}\colon L_0^\infty(G)^* \to C_0(G)^*$ be the restriction map, so that $\mathcal{P}$ is a left $L^1(G)$-module morphism. Now, let u be an extension of the Dirac measure δ_e at e from $C_0(G)$ to a bounded functional on $L_0^\infty(G)$, and define the map $\mathcal{Q}\colon C_0(G)^* \to L_0^\infty(G)^*$ by $\mathcal{Q}(\mu)(g) = \langle u, \mu g\rangle$ for all $\mu \in C_0(G)^*$ and $g \in L_0^\infty(G)$, where

$$(\mu g)(x) = \int_G g(xy)\,\mathrm{d}\mu(y)$$

for locally almost all $x \in G$. Since $\mathcal{Q}(\mu)(g) = \langle \mu, g\rangle$ when $\mu \in L^1(G)$ or $g \in C_0(G)$, it follows that $\mathcal{Q}$ is a right inverse for $\mathcal{P}$. Moreover, $\mathcal{Q}$ is a left $L^1(G)$-module morphism; indeed, for $\phi \in L^1(G)$, $\mu \in C_0(G)^*$ and $g \in L_0^\infty(G)$, we have $\phi \cdot \mu \in L^1(G)$ and $g \cdot \phi \in C_0(G)$, and so

$$\mathcal{Q}(\phi \cdot \mu)(g) = \langle \phi \cdot \mu, g\rangle = \langle \mu, g \cdot \phi\rangle = \mathcal{Q}(\mu)(g \cdot \phi) = \phi \cdot \mathcal{Q}(\mu)(g).$$

This shows that $C_0(G)^*$ is a retraction of $L_0^\infty(G)^*$. □

3. Injectivity of $L_0^\infty(G)$ and $L_0^\infty(G)^*$

A Banach left A-module I is called *injective* if, for Banach left A-modules E and F, each admissible monomorphism $T \in {}_A B(E,F)$ and each $S \in {}_A B(E,I)$, there exists $R \in {}_A B(F,I)$ such that $R \circ T = S$. Similar definitions apply for Banach right A-modules.

To study injectivity of the Banach left $L^1(G)$-module $L_0^\infty(G)$, we require two essential lemmas. But first, let E be a Banach left $L^1(G)$-module and recall that a map $T \in B(L^1(G), E)$ has compact *support* if there is a compact subset K of G such that $T(\phi) = 0$ in E for all $\phi \in L^1(G)$ with $\phi\chi_K = 0$ in $L^1(G)$.

Lemma 3.1. *Let G be a locally compact group that is σ-compact and non-compact. Let $\varrho\colon B(L^1(G), C_0(G)) \to L_0^\infty(G)$ be a continuous linear operator that is also a left $L^1(G)$-module morphism. If $T \in B(L^1(G), C_0(G))$ has compact support, then $\varrho(T) = 0$.*

Proof. Let $(e_\gamma)_{\gamma\in\Gamma}$ be a bounded left approximate identity for $L^1(G)$. For each $\gamma \in \Gamma$, define the map $\varrho_\gamma\colon B(L^1(G), C_0(G)) \to C_0(G)$ by

$$\varrho_\gamma(T) = \varrho(T) \cdot e_\gamma$$

for all $T \in B(L^1(G), C_0(G))$. Then $\varrho_\gamma\colon B(L^1(G), C_0(G)) \to C_0(G)$ is a continuous linear operator that is also a left $L^1(G)$-module morphism. If $T \in B(L^1(G), C_0(G))$ has compact support, then $\varrho_\gamma(T) = 0$ for all $\gamma \in \Gamma$ [**2**, Lemma 3.5]. It follows that

$$\varrho(T)(\phi) = \lim_\gamma \varrho(T)(e_\gamma * \phi) = \lim_\gamma (\varrho(T) \cdot e_\gamma)(\phi) = \lim_\gamma \varrho_\gamma(T)(\phi)$$

for all $\phi \in L^1(G)$. Therefore, $\varrho(T) = 0$ as required. □

Lemma 3.2. *Let G be a locally compact non-compact group. Then $L_0^\infty(G)$ is not complemented in $L^\infty(G)$.*

Proof. Since G is a locally compact non-compact group, there exists a sequence $(x_n)_{n\geqslant 1}$ of disjoint elements of G and a compact symmetric neighbourhood U of e such that the sets $x_n U$ for all $n \geqslant 1$ are pairwise disjoint [**5**, 11.43 (e)]. Choose a compact symmetric neighbourhood V of e with $VV \subset U$, and set $V_n := x_n V$ for $n \geqslant 1$. Then for any compact subset K of G, there exists a natural number $N_K \geqslant 1$ such that $V_n \cap K = \emptyset$ for all $n \geqslant N_K$.

Now, let $I\colon l^\infty \to L^\infty(G)$ and $R\colon L_0^\infty(G) \to c_0$ be the linear maps defined by

$$I((\alpha_n)) = \sum_{n=1}^\infty \alpha_n \chi_{V_n}$$

for all $(\alpha_n) \in l^\infty$, and by

$$R(g) = \left(\frac{1}{\lambda(V_n)} \int_{V_n} g(x)\, \mathrm{d}\lambda(x)\right)_{n\geqslant 1}$$

for all $g \in L_0^\infty(G)$. Clearly, both maps are continuous. Next, suppose on the contrary that there exists a continuous linear projection $P\colon L^\infty(G) \to L_0^\infty(G)$. If $Q\colon l^\infty \to c_0$ is the composition $R \circ P \circ I$, then $I((\alpha_n)) \in L_0^\infty(G)$ for all $(\alpha_n) \in c_0$, and we have

$$\begin{aligned} Q((\alpha_n)) &= R\biggl(\sum_{m=1}^\infty \alpha_m \chi_{V_m}\biggr) \\ &= \left(\frac{1}{\lambda(V_n)} \int_{V_n} \biggl(\sum_{m=1}^\infty \alpha_m \chi_{V_m}(x)\biggr) \mathrm{d}\lambda(x)\right) \\ &= (\alpha_n). \end{aligned}$$

So, $Q\colon l^\infty \to c_0$ is a projection which contradicts the fact that c_0 is not complemented in l^∞ (see for example, [**4**, Theorem 0.1.16]). □

Let A be a Banach algebra and let E be a Banach left A-module. Then $B(A, E)$ is a Banach left A-module with $(a \cdot T)(b) = T(ba)$ for all $a, b \in A$ and $T \in B(A, E)$. Define the left A-module morphism $\Pi\colon E \to B(A, E)$ by the formula $\Pi(\xi)(a) = a \cdot \xi$ for $\xi \in E$ and $a \in A$. Before we state the following result from [**2**, Proposition 1.7], let us recall that E is called faithful if $A \cdot \xi \neq \{0\}$ for all $\xi \in E \setminus \{0\}$.

Proposition 3.3. *Let A be a Banach algebra and let E be a faithful Banach left A-module. Then E is an injective Banach left A-module if and only if there exists a left A-module morphism $\rho\colon B(A, E) \to E$ with $\rho \circ \Pi = I_E$.*

We are now ready to characterize injectivity of the Banach left $L^1(G)$-module $L_0^\infty(G)$.

Theorem 3.4. *Let G be a locally compact group. Then $L_0^\infty(G)$ is an injective Banach left $L^1(G)$-module if and only if G is compact.*

Proof. The 'if' part follows from the fact that $L^\infty(G)$ is always an injective Banach left $L^1(G)$-module [**2**, Theorem 2.4].

For the converse, suppose on the contrary that $L_0^\infty(G)$ is an injective Banach left $L^1(G)$-module but G is not compact. In view of Proposition 3.3, there exists a left $L^1(G)$-module morphism $\rho_G\colon B(L^1(G), C_0(G)) \to L_0^\infty(G)$ such that $\rho_G \circ \Pi_G = I_{L_0^\infty(G)}$, where $\Pi_G\colon L_0^\infty(G) \to B(L^1(G), C_0(G))$ is the canonical embedding defined by $\Pi_G(g)(\phi) = \phi \cdot g$ for all $g \in L_0^\infty(G)$ and $\phi \in L^1(G)$. As in the proof of Lemma 3.4 in [**2**], there exists an open, non-compact and σ-compact subgroup H of G and a linear isometric operator $Q\colon B(L^1(H), C_0(H)) \to B(L^1(G), C_0(G))$ with the following properties:

(1) Q is a left $L^1(H)$-module morphism,

(2) $Q \circ \Pi_H = \Pi_G \circ I$ on $C_0(H)$,

where $I\colon L_0^\infty(H) \to L_0^\infty(G)$ and $\Pi_H\colon L_0^\infty(H) \to B(L^1(H), C_0(H))$ are the natural embeddings. An argument similar to the proof of Lemma 3.4 of [**2**] shows that $Q \circ \Pi_H = \Pi_G \circ I$ on $L_0^\infty(H)$. Now, let $R\colon L_0^\infty(G) \to L_0^\infty(H)$ be the restriction map and note that the linear operator $\rho_H := R \circ \rho_G \circ Q\colon B(L^1(H), C_0(H)) \to L_0^\infty(H)$ is a continuous left $L^1(H)$-module morphism. Moreover,

$$\rho_H \circ \Pi_H = R \circ \rho_G \circ Q \circ \Pi_H = R \circ \rho_G \circ \Pi_G \circ I = R \circ I = I_{L_0^\infty(H)}.$$

Now, choose a sequence (K_n) of compact subsets of H with $K_n \subsetneq \operatorname{int} K_{n+1}$ such that $H = \bigcup_{n=1}^\infty K_n$, and let $P\colon L^\infty(H) \to B(L^1(H), C_0(H))$ be the continuous map given by the formulae

$$P(f)(\phi) = \sum_{n=1}^\infty (\chi_{K_n \setminus K_{n-1}} \phi) \cdot (\chi_{K_n} f)$$

for all $f \in L^\infty(H)$ and $\phi \in L^1(H)$. We show that the map $\rho_H \circ P$ is a projection of $L^\infty(H)$ onto $L_0^\infty(H)$.

To prove this fact, let $L_{00}^\infty(G)$ be the space of all $g \in L_0^\infty(G)$ with $\|g\chi_{G\setminus K}\|_\infty = 0$ for some compact subset K of G, and note that $L_0^\infty(G)$ is the $\|\cdot\|_\infty$-closure of $L_{00}^\infty(G)$. So, it suffices to show that $(\rho_H \circ P)$ is the identity on $L_{00}^\infty(H)$. Take $h \in L_{00}^\infty(H)$ and choose $m \geqslant 1$ such that h vanishes outside K_m almost everywhere. Define $T_0 = P(h) - \Pi_H(h)$ and note that T_0 has compact support K_m. Then $\rho_H(T_0) = 0$ by Lemma 3.1. Therefore,

$$(\rho_H \circ P)(h) = \rho_H(T_0) + \rho_H(\Pi_H(h)) = h.$$

That is, $\rho_H \circ P$ is a projection of $L^\infty(H)$ onto $L_0^\infty(H)$, which contradicts Lemma 3.2. □

Let φ_G be the *augmentation character* on $L^1(G)$ that is defined by

$$\varphi_G(\phi) = \int_G \phi(x)\,\mathrm{d}\lambda(x)$$

for all $\phi \in L^1(G)$. Let E be a Banach left $L^1(G)$-module. Following [**2**], a functional $\Lambda \in E^*$ is called *augmentation invariant* whenever $\langle \Lambda, \phi \cdot \xi \rangle = \varphi_G(\phi)\langle \Lambda, \xi \rangle$ for all $\xi \in E$, $\phi \in L^1(G)$. In the case where there exists a non-zero augmentation-invariant functional in E^*, then E is said to be *augmentation invariant*.

Recall that a locally compact group G is called *amenable* if $L^\infty(G)$ is an augmentation-invariant Banach left $L^1(G)$-module. The class of amenable groups includes all compact groups and all abelian locally compact groups; however, the discrete free group $\mathbb{F}_2$ on two generators is not amenable (see [**9**] for more details). Here, we characterize locally compact groups G for which $L_0^\infty(G)$ or its dual is augmentation invariant.

Proposition 3.5. *Let G be a locally compact group. The following then hold.*

(i) *The Banach left $L^1(G)$-module $L_0^\infty(G)$ is augmentation invariant if and only if G is compact.*

(ii) *The Banach left $L^1(G)$-module $L_0^\infty(G)^*$ is always augmentation invariant.*

Proof. (i) We need only note that, if G is non-compact, zero is the only augmentation-invariant functional in $L_0^\infty(G)^*$ [**5**, 17.19(c)].

(ii) Let $\{C_\alpha\}$ be the family of all compact subsets of G directed by upward inclusion. Then (χ_{C_α}) is a bounded approximate identity for the C^*-algebra $L_0^\infty(G)$. Define $\Lambda \in L_0^\infty(G)^{**}$ to be a weak* cluster point of (χ_{C_α}). We show that Λ is an augmentation-invariant functional in $L_0^\infty(G)^{**}$: that is,

$$\langle \Lambda, \phi \cdot m \rangle = \varphi_G(\phi)\langle \Lambda, m \rangle$$

for all $\phi \in L^1(G)$ and $m \in L_0^\infty(G)^*$. To see this, recall from [**7**, Proposition 2.6] that m can be approximated in the norm topology by functionals with compact carrier in $L_0^\infty(G)^*$, i.e. functionals n for which there is a compact subset C of G with $\langle n, g \rangle = \langle n, g\chi_C \rangle$ for $g \in L_0^\infty(G)$. We may thus assume that there is α_0 with $\langle m, g \rangle = \langle m, g\chi_{C_{\alpha_0}} \rangle$ for $g \in L_0^\infty(G)$; by the norm density of functions with compact support in $L^1(G)$, we may also assume that $\phi = \phi\chi_{C_{\alpha_0}}$ almost everywhere. Choose $\alpha_1 \geqslant \alpha_0$ with $C_{\alpha_0}^2 \subseteq C_{\alpha_1}$ and note that, for every $\alpha \geqslant \alpha_1$ and $x \in C_{\alpha_0}$,

$$\left(\frac{1}{\Delta}\tilde{\phi} * \chi_{C_\alpha}\right)(x) = \varphi_G(\phi)\chi_{C_\alpha}(x);$$

indeed, for each $\psi \in L^1(G)$, we have

$$\begin{aligned}\left\langle \left(\frac{1}{\Delta}\tilde{\phi} * \chi_{C_\alpha}\right)\chi_{C_{\alpha_0}}, \psi \right\rangle &= \int_G \int_G \frac{1}{\Delta(y)}\phi(y^{-1})\chi_{C_\alpha}(y^{-1}x)\chi_{C_{\alpha_0}}(x)\psi(x)\,\mathrm{d}\lambda(y)\,\mathrm{d}\lambda(x)\\ &= \int_G \int_G \phi(y)\chi_{C_\alpha}(yx)\chi_{C_{\alpha_0}}(x)\psi(x)\,\mathrm{d}\lambda(y)\,\mathrm{d}\lambda(x)\\ &= \int_G \int_G \phi(y)\chi_{C_{\alpha_0}}(x)\psi(x)\,\mathrm{d}\lambda(y)\,\mathrm{d}\lambda(x)\\ &= \varphi_G(\phi)\langle \chi_{C_{\alpha_0}}, \psi \rangle.\end{aligned}$$

It follows that

$$\begin{aligned}\langle \Lambda, \phi \cdot m\rangle &= \lim_{\alpha}\langle \phi \cdot m, \chi_{C_\alpha}\rangle \\ &= \lim_{\alpha}\left\langle m, \frac{1}{\Delta}\tilde{\phi} * \chi_{C_\alpha}\right\rangle \\ &= \lim_{\alpha}\left\langle m, \left(\frac{1}{\Delta}\tilde{\phi} * \chi_{C_\alpha}\right)\chi_{C_{\alpha_0}}\right\rangle \\ &= \varphi_G(\phi)\lim_{\alpha}\langle m, \chi_{C_\alpha}\rangle \\ &= \varphi_G(\phi)\langle \Lambda, m\rangle,\end{aligned}$$

which completes the proof. □

A Banach left A-module F is called *flat* if F^* is an injective Banach right A-module.

Theorem 3.6. *Let G be a locally compact group. The following assertions are then equivalent.*

(a) *G is amenable.*

(b) *$L_0^\infty(G)$ is a flat Banach left $L^1(G)$-module.*

(c) *$L_0^\infty(G)^*$ is an injective Banach left $L^1(G)$-module.*

Proof. It is shown in [**8**, Theorem 3.4.2] that if an augmentation-invariant Banach left $L^1(G)$-module E is the dual left module of a Banach right $L^1(G)$-module, then E is injective as a Banach left $L^1(G)$-module if and only if G is amenable. In fact, this result was proved for faithful Banach left $L^1(G)$-modules in [**2**]. Now, the equivalence of (a) and (c) follows from Proposition 3.5 together with the fact that $L_0^\infty(G)^*$ is the dual left module of the Banach right $L^1(G)$-module $L_0^\infty(G)$. Similarly, G is amenable if and only if $L_0^\infty(G)^*$ is an injective Banach right $L^1(G)$-module. The proof is therefore complete. □

We conclude this work with a result on the flatness of the Banach left $L^1(G)$-module $L_0^\infty(G)^*$. First, we state the following proposition communicated to us by Ramsden.

Proposition 3.7. *Let A be a Banach algebra with a bounded approximate identity and let E be a Banach left A-module. Then E is flat as a Banach left A-module if and only if the closed submodule $A \cdot E$ is flat. In particular, the quotient module $E/A \cdot E$ is always flat.*

Proof. First we show that the quotient module $F := E/A \cdot E$ is flat. To that end, let (e_γ) be a bounded right approximate identity for A. For each γ, define $\rho_\gamma\colon F \to (A^\flat \hat{\otimes} F)^{**}$ by $\rho_\gamma(\xi) = (e^\flat - e_\gamma) \otimes \xi$ for all $\xi \in F$, where $A^\flat$ is the algebra formed by adjoining an identity $e^\flat$ to A. Regard (ρ_γ) as a bounded net in $B(F, (A^\flat \hat{\otimes} F)^{**}) = (F \hat{\otimes} (A^\flat \hat{\otimes} F)^*)^*$. Since $A \cdot F = \{0\}$, the weak* cluster point ρ of this net is a left A-module morphism such that $\pi^{**} \circ \rho = i_F$, where $i_F\colon F \to F^{**}$ is the natural embedding into the second dual and $\pi\colon A^\flat \hat{\otimes} F \to F$ is the canonical map defined by $\pi(b \otimes \xi) = b \cdot \xi$

for all $b \in A^\flat$ and $\xi \in F$. This is equivalent to F being flat as a Banach left A-module (see [**3**, Exercise VII.2.8] or [**9**, Lemma 4.3.22]).

Now, since A has a bounded approximate identity, it follows from [**1**, Corollary 2.9.26] that the short exact sequence $0 \to A \cdot E \to E \to F \to 0$ of Banach left A-modules is weakly admissible: that is, the adjoint of the quotient map $q\colon E \to E/F$ has a bounded left inverse. Since F is flat, the result follows from [**3**, Proposition VII.1.17]. □

Theorem 3.8. *Let G be a locally compact group. Then $L_0^\infty(G)^*$ is a flat Banach left $L^1(G)$-module.*

Proof. For each $\phi \in L^1(G)$, let ϕ also denote the functional in $L_0^\infty(G)^*$ defined by

$$\langle \phi, g\rangle = \int_G \phi(x)g(x)\,\mathrm{d}\lambda(x)$$

for all $g \in L_0^\infty(G)$, and recall from [**7**] that $\phi \cdot m \in L^1(G)$ for all $m \in L_0^\infty(G)^*$. Now, let u be a weak* cluster point of an approximate identity (e_γ) in $L^1(G)$ bounded by 1. Then, for every $\phi \in L^1(G)$, using the weak* continuity of the map $k \mapsto \phi \cdot k$ on $L_0^\infty(G)^*$, we conclude that $\phi \cdot e_\gamma \to \phi \cdot u$ in the weak* topology of $L_0^\infty(G)^*$. It follows that $\phi \cdot u = \phi$. It follows that $L^1(G) \cdot L_0^\infty(G)^* = L^1(G)$. The result therefore follows from Proposition 3.7 and the fact that $L^1(G)$ is always a flat Banach left $L^1(G)$-module [**2**, Theorem 2.4]. □

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