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RAMANUJAN SERIES WITH A SHIFT

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Communicated by B. Sims

In memory of Jon Borwein,
the main driver and master of experimental mathematics

Abstract

We consider an extension of the Ramanujan series with a variable x. If we let $x = x_0$, we call the resulting series 'Ramanujan series with the shift x_0'. Then we relate these shifted series to some q-series and solve the case of level 4 with the shift $x_0 = 1/2$. Finally, we indicate a possible way towards proving some patterns observed by the author corresponding to the levels $\ell = 1, 2, 3$ and the shift $x_0 = 1/2$.

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1. Shift and upside-down transformations

A shift is a transformation that consists in applying the substitution $n \to n + x_0$ inside a series, and we say that x_0 is the shift. For example, the series

$$\sum_{n=0}^{\infty} z^n \frac{(\frac{1}{2})_n (\frac{1}{s})_n (\frac{s-1}{s})_n}{(1)_n^3} (a + bn) \tag{1.1}$$

shifted by x_0 becomes

$$\sum_{n=0}^{\infty} z^{n+x_0} \frac{(\frac{1}{2} + x_0)_n (\frac{1}{s} + x_0)_n (\frac{s-1}{s} + x_0)_n}{(1 + x_0)_n^3} (a + b(n + x_0))$$

multiplied by a factor which does not depend on n (we will ignore that factor). An upside-down transformation consists of the substitution $n \to -n$. That is,

$$\sum_{n=1}^{\infty} z^{-n} \frac{(\frac{1}{2})_{-n} (\frac{1}{s})_{-n} (\frac{s-1}{s})_{-n}}{(1)_{-n}^3} (a - bn), \tag{1.2}$$

understanding the rising factorials in the way indicated below:

$$(a)_{-n} \to \frac{(-1)^n}{(1-a)_n} \quad \text{if } a \neq 1 \text{ and } (1)_{-n} \to \frac{n(-1)^n}{(1)_n}.$$

These substitutions are justified because they preserve formally the recurrence equation $\Gamma(x+1) = x\Gamma(x)$; see the duality property [9, Ch. 7] and the application shown in [7, Section 4], and see [8] for the analytic interpretation. If $|z| > 0$, we understand the 'divergent' series (1.1) as its analytic continuation, and, if $|z| < 0$, we interpret the 'divergent' series (1.2) in the same way. While in [8] we have studied the 'upside-down' transformation, in this paper we consider the transformation with a shift. In [8] we prove that the upside-down transformation modifies the value of the modular variable q. Here we will see that a shift does not modify it.

The following kind of series for $1/\pi$:

$$\sum_{n=0}^{\infty} z^n \frac{(\frac{1}{2})_n(\frac{1}{s})_n(\frac{s-1}{s})_n}{(1)_n^3}(a+bn) = \frac{1}{\pi}, \tag{1.3}$$

where $s \in \{2,3,4,6\}$, can be parametrized with a modular function $z = z_\ell(q)$ and two modular forms $b = b_\ell(q)$ and $a = a_\ell(q)$ of weight two. It is known that the level of these functions is $\ell = 1,2,3,4$ for $s = 6,4,3,2$, respectively, and that for $q = \pm e^{-\pi\sqrt{r}}$ with $r \in \mathbb{Q}^+$ the values of z, b, a are algebraic reals (the sign + corresponds to series of positive terms and the sign − to alternating series). In these cases the series (1.3) are called Ramanujan-type series, in honour of the Indian genius Srinivasa Ramanujan who gave 17 examples of them. If we want to consider algebraic complex solutions, then we let $q = e^{2\pi i\tau}$, where τ is a quadratic irrational with $\mathrm{Im}(\tau) > 0$. In this paper we are mainly interested in the evaluations of (1.5) for those special values of q and $x = 1/2$. We will use the following theorems.

Theorem 1.1. *Let*

$$F_\ell(x,z) = \sum_{n=0}^{\infty} z^{n+x}\frac{(\frac{1}{2}+x)_n(\frac{1}{s}+x)_n(\frac{s-1}{s}+x)_n}{(1+x)_n^3}, \qquad \frac{F_\ell(x,z)}{F_\ell(0,z)} = \phi(q)$$

and

$$G_\ell(x,z) = \sum_{n=0}^{\infty} z^{n+x}\frac{(\frac{1}{2}+x)_n(\frac{1}{s}+x)_n(\frac{s-1}{s}+x)_n}{(1+x)_n^3}(a+b(n+x)), \tag{1.4}$$

where $z = z(q)$, $b = b(q)$ and $a = a(q)$ are the functions mentioned before. Then

$$G_\ell(x,q) = \frac{1}{\pi}\Big(\phi(q) - \ln|q|\, q\frac{d\phi(q)}{dq}\Big). \tag{1.5}$$

Proof. It is a particular case of [8, Proposition 2]. □

Theorem 1.2. *The following identity holds:*

$$\Big(q\frac{d}{dq}\Big)^3\phi(q) = \frac{x^3z^x}{\sqrt{1-z}}\Big(\frac{q}{z}\frac{dz}{dq}\Big)^2 = x^3F_\ell^2(0,q)\sqrt{1-z}\,\sqrt{z}.$$

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PROOF. The differential operator

$$\mathcal{D} = \left(z\frac{d}{dz}\right)^3 - z\left(z\frac{d}{dz} + \frac{1}{2}\right)\left(z\frac{d}{dz} + \frac{1}{s}\right)\left(z\frac{d}{dz} + \frac{s-1}{s}\right)$$

annihilates $F_\ell(0,z)$, and in [6] we proved that $\mathcal{D}F_\ell(x,z) = x^3 z^x$. As $F(0,q)$ is a modular form such that $\mathcal{D}F(0,z) = 0$, we can apply [11, Lemma 1], and, as

$$\mathcal{D} = (1-z)\left(z\frac{d}{dz}\right)^3 + \cdots,$$

$$\left(q\frac{d}{dq}\right)^3 \frac{F_\ell(x,q)}{F_\ell(0,q)} = \frac{\mathcal{D}F_\ell(x,z)}{F_\ell(0,z)(1-z)}\left(\frac{q}{z}\frac{dz}{dq}\right)^3.$$

Finally, using [6, Equation (2.34)], we complete the proof. □

2. Ramanujan series of level 4 with a shift

This is motivated by the evaluations found in [8] by observing that when $s = 2$, a shift of $x = 1/2$ of a convergent Ramanujan-type series is equivalent to the upside-down shift of a related 'divergent' Ramanujan-type series.

2.1. Ramanujan series of level 4 with the shift 1/2.

THEOREM 2.1. *Case $s = 2$ ($\ell = 4$). Let*

$$F_4(x,q) = \sum_{n=0}^{\infty} \frac{(\frac{1}{2}+x)_n^3}{(1+x)_n^3} z_4^{n+x}, \quad G_4(x,q) = \sum_{n=0}^{\infty} \frac{(\frac{1}{2}+x)_n^3}{(1+x)_n^3}[a_4 + b_4(n+x)] z_4^{n+x}.$$

The following identities hold:

$$\phi(q) = 8\sqrt{q}\sum_{n=0}^{\infty} \sigma_3(2n+1)\frac{(-q)^n}{(2n+1)^3}, \quad F_4\left(\frac{1}{2},q\right) = F_4(0,q)\phi(q) \tag{2.1}$$

and

$$G_4\left(\frac{1}{2},q\right) = \frac{8\sqrt{q}}{\pi}\left(\sum_{n=0}^{\infty} \sigma_3(2n+1)\frac{(-q)^n}{(2n+1)^3} - \frac{\ln|q|}{2}\sum_{n=0}^{\infty} \sigma_3(2n+1)\frac{(-q)^n}{(2n+1)^2}\right), \tag{2.2}$$

where $q = \pm e^{-\pi\sqrt{r}}$, and $\sigma_3(n)$ is the sum of the cubes of the divisors of n.

PROOF. Applying Theorem 1.2,

$$\left(q\frac{d}{dq}\right)^3 \phi(q) = \frac{1}{8}F_4^2(0,q)\sqrt{1-z_4}\sqrt{z_4}, \quad \phi(q) = \frac{F_4(1/2,q)}{F_4(0,q)}.$$

But, for $s = 2$, we know that

$$F_4(0,q) = \theta_3^4(q), \quad z_4(q) = 4\lambda(q)(1-\lambda(q)), \quad \lambda(q) = \frac{\theta_2^4(q)}{\theta_3^4(q)}.$$

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Using the identity $\theta_3^4(q) = \theta_2^4(q) + \theta_4^4(q)$,

$$\left(q\frac{d}{dq}\right)^3 \phi(q) = \frac{1}{4}\theta_2^2(q)\theta_4^2(q)[\theta_4^4(q) - \theta_2^4(q)] = \sqrt{q} f(q).$$

Then, with *The On-Line Encyclopedia of Integer Sequences (OEIS)* [10], we could identify the coefficient of $(-q)^n$ in the expansion of $f(q)$ as $\sigma_3(2n+1)$. Hence,

$$\left(q\frac{d}{dq}\right)^3 \phi(q) = \sqrt{q}\sum_{n=0}^{\infty} \sigma_3(2n+1)(-q)^n,$$

which proves (2.1). Finally, we only have to apply Theorem 1.1 to arrive at (2.2). □

The following identity is known:

$$\sqrt{q}\sum_{n=0}^{\infty} \sigma_3(2n+1)(-q)^n = \frac{i}{240}[E_4(\sqrt{-q}) - 9E_4(-q) + 8E_4(q^2)],$$

where $E_4(q)$ is the Eisenstein series

$$E_4(q) = \frac{45}{\pi^4}\sum_{(n,m)\neq(0,0)} \frac{1}{(n+\tau m)^4}, \qquad q = e^{2\pi i\tau}.$$

Hence,

$$\left(q\frac{d}{dq}\right)^3 \phi(q) = \frac{i}{240}[E_4(\sqrt{-q}) - 9E_4(-q) + 8E_4(q^2)].$$

We use it to prove the following theorem.

Theorem 2.2. *If $q = -e^{-\pi\sqrt{r}}$, where $r \in \mathbb{Q}^+$ (case of alternating series),*

$$G_4\left(\frac{1}{2}, z\right) = i\frac{r^{3/2}}{\pi^2}\left[\frac{1}{16}S\left(1, 0, \frac{r}{16}; 2\right) - \frac{9}{16}S\left(1, 0, \frac{r}{4}; 2\right) + \frac{1}{2}S(1, 0, r; 2)\right], \tag{2.3}$$

and, if $q = e^{-\pi\sqrt{r}}$ (case of series of positive terms),

$$G_4\left(\frac{1}{2}, z\right) = \frac{r^{3/2}}{\pi^2}\left[\frac{1}{16}S\left(1, 1, \frac{r}{16} + \frac{1}{4}; 2\right) - \frac{9}{16}S\left(1, 1, \frac{r}{4} + \frac{1}{4}; 2\right) - \frac{1}{2}S\left(1, 1, r + \frac{1}{4}; 2\right)\right], \tag{2.4}$$

where

$$S(A, B, C; t) = \sum_{(n,m)\neq(0,0)} \frac{1}{(An^2 + Bnm + Cm^2)^t}$$

is the Epstein zeta function [3].

Proof. If $q = -e^{-\pi\sqrt{r}}$, then $-q = e^{-\pi\sqrt{r}}$, and the value of τ corresponding to $-q$ is $\tau = i\sqrt{r}/2$. If we define

$$U_{n,m}(r) = \frac{1}{(n + m\frac{i\sqrt{r}}{4})^4} - \frac{9}{(n + m\frac{i\sqrt{r}}{2})^4} + \frac{8}{(n + mi\sqrt{r})^4},$$

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then

$$E_4(\sqrt{-q}) - 9E_4(-q) + 8E_4(q^2) = \sum_{n,n\neq(0,0)} U_{n,m}(r),$$

and, taking into account that $dq/q = \pi\, dr/(2\sqrt{r})$,

$$\phi(r) = \frac{3i}{16\pi^5} \sum_{(n,m)\neq(0,0)} \mathrm{Re}\Big[\int \frac{\pi\, dr}{2\sqrt{r}} \int \frac{\pi\, dr}{2\sqrt{r}} \int \frac{\pi\, dr}{2\sqrt{r}}\, U_{n,m}(r) + \pi\sqrt{r} \int \frac{\pi\, dr}{2\sqrt{r}} \int \frac{\pi\, dr}{2\sqrt{r}}\, U_{n,m}(r)\Big],$$

where we have taken the real part inside the summation because for alternating series $\phi(r)$ has to be a purely imaginary number. Integrating and simplifying, we obtain (2.3). The proof of (2.4) is completely similar. □

2.2. Examples of Ramanujan-type series with $s = 2$ (level $\ell = 4$) shifted by 1/2. For $r = 4$, using the known values

$$S(1,0,1;2) = \frac{2\pi^2}{3} L_{-4}(2), \quad S(1,0,4;2) = \frac{7\pi^2}{24} L_{-4}(2),$$

see the method and the tables of [3] or [1], and the obvious relation $S(1,0,\frac{1}{4};2) = 16S(1,0,4;2)$, we get from (2.3)

$$\sum_{n=0}^{\infty} \frac{(1)_n^3}{(\frac{3}{2})_n^3} \Big(-\frac{1}{8}\Big)^{n+1/2} \Big(\frac{4}{2\sqrt{2}} + \frac{6}{2\sqrt{2}} n\Big) = \frac{8i}{\pi^2}\Big(\frac{3}{2} S(1,0,4;2) - \frac{9}{16} S(1,0,1;2)\Big) = \frac{i}{2} L_{-4}(2),$$

where $L_{-4}(2) = G$ (the Catalan constant). Hence,

$$\sum_{n=0}^{\infty} (-1)^n \frac{(1)_n^3}{(\frac{3}{2})_n^3} \frac{2+3n}{8^n} = 2G.$$

Below, we show two more examples:

$$\sum_{n=0}^{\infty} \frac{(1)_n^3}{(\frac{3}{2})_n^3} \Big(\frac{42\sqrt{5}+30}{32} n + \frac{26\sqrt{5}+14}{32}\Big) \frac{1}{2^{6n+3}} \Big(\frac{\sqrt{5}-1}{2}\Big)^{8n+4} = \frac{\pi^2}{240},$$

which corresponds to the value $q = e^{-\pi\sqrt{15}}$, and

$$\sum_{n=0}^{\infty} (-1)^n \frac{(1)_n^3}{(\frac{3}{2})_n^3} \Big(\frac{5\sqrt{2}-6}{4} n + \frac{4\sqrt{2}-5}{4}\Big) \Big(\frac{\sqrt{2}-1}{2}\Big)^{3n} = 2L_{-4}(2) - \frac{\sqrt{2}}{2} L_{-8}(2),$$

which corresponds to the value $q = -e^{-\pi\sqrt{8}}$. Observe that in [8] we arrive at the results by relating 'divergent' series to convergent ones by means of the 'upside-down' transformation. In addition, observe that for the levels $\ell = 1, 2, 3$ the two transformations (shift and 'upside-down') lead to completely different series.

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2.3. Some q-series corresponding to $s = 2$ ($\ell = 4$) with other shifts. We have proved the following identity:

$$\left(q\frac{d}{dq}\right)^3 \frac{F_4(x,q)}{F_4(0,q)} = x^3 F_4^2(0,q)\sqrt{1-z_4}\, z_4^x, \quad \phi(q) = \frac{F_4(x,q)}{F_4(0,q)}.$$

Hence, if we define

$$f(x,q) = F_4^2(0,q)\sqrt{1-z_4}\, z_4^x = \theta_3^8(q)(1-2\lambda(q))[4\lambda(q)(1-\lambda(q))]^x,$$

$$\phi(q) = x^3 \int_0^q \int_0^q \int_0^q f(x,q)\,\frac{dq}{q}\,\frac{dq}{q}\,\frac{dq}{q}.$$

Finally,

$$\sum_{n=0}^{\infty} z^{n+x} \frac{(\frac{1}{2}+x)_n^3}{(1+x)_n^3}(a+b(n+x)) = \frac{1}{\pi}\left(\phi(q) - \ln|q|\, q\frac{d\phi(q)}{dq}\right).$$

For $m = 2, 3, 4, 6, 8, 12, 24$, the function

$$h_m(q) = 64^{-1/m} f(1/m, q^m) = 16^{-1/m}\theta_3^8(q^m)(1-2\lambda(q^m))[\lambda(q^m)(1-\lambda(q^m))]^{1/m}$$

has integer coefficients. Below we write the cases $m = 2, 3, 4, 6, 8$:

$$\begin{aligned}
h_2(q) &= q - 28q^3 + 126q^5 - 344q^7 + 757q^9 - 1332q^{11} + 2198q^{13} - 3528q^{15} + 4914q^{17} \\
&\quad - 6860q^{19} + 9632q^{21} - 12168q^{23} + 15751q^{25} - 20440q^{27} + 24390q^{29} - \cdots, \\
h_3(q) &= q - 24q^4 + 20q^7 + 0q^{10} - 70q^{13} + 192q^{16} + 56q^{19} + 0q^{22} - 125q^{25} - 480q^{28} \\
&\quad - 308q^{31} + 0q^{34} + 110q^{37} + 0q^{40} - 520q^{43} + 0q^{46} + 57q^{49} + 1680q^{52} + \cdots, \\
h_4(q) &= q + 22q^5 - 27q^9 - 18q^{13} - 94q^{17} + 0q^{21} + 359q^{25} - 130q^{29} + 0q^{33} + 214q^{37} \\
&\quad - 230q^{41} - 594q^{45} - 343q^{49} + 518q^{53} + 0q^{57} + 830q^{61} - 396q^{65} + \cdots, \\
h_6(q) &= q - 20q^7 - 70q^{13} - 56q^{19} - 125q^{25} - 308q^{31} + 110q^{37} - 520q^{43} + 57q^{49} \\
&\quad + 0q^{55} + 182q^{61} - 880q^{67} + 1190q^{73} - 884q^{79} + 0q^{85} - 1400q^{91} + \cdots, \\
h_8(q) &= q - 19q^9 - 90q^{17} - 125q^{25} - 200q^{33} - 522q^{41} - 343q^{49} + 360q^{57} + 0q^{65} \\
&\quad - 430q^{73} + 145q^{81} + 1026q^{89} + 1910q^{97} - 270q^{113} + 3669q^{121} + 1368q^{129} \\
&\quad - 2250q^{137} + 0q^{145} + 1710q^{153} + 0q^{161} - 2197q^{169} + 920q^{177} + \cdots.
\end{aligned}$$

The cases $m = 2, 3, 4, 6$ are in OEIS [10], while the cases 8, 12, 24 are not yet in it. We observe that the coefficient of q^k multiplied by the coefficient of q^j, where $k-1$ and $j-1$ are multiples of m, equals the coefficient of q^{kj} when k and j are coprime.

3. The q-series for Ramanujan series shifted by 1/2. Cases $s = 4, 6, 3$

In [4], we conjecture the value of (1.4) in cases when z, b, a are algebraic numbers and $x = 1/2$. The observed results corresponding to $s = 4$, $s = 6$ and $s = 3$ involve Napierian logarithms in case of alternating series and arc tangent values in case of series of positive terms. We rewrite those conjectures, together with all the other cases corresponding to rational values of z, in the tables of this paper. Some few but representative examples are in my thesis [5, pages 44–46]. Notice that in [4] and in [5, pages 44–46], there are also examples of shifted series corresponding to Ramanujan-like series for $1/\pi^2$ and $1/\pi^3$. However, we do not know how to get q-series for those shifted series.

3.1. The q-series for Ramanujan series with $s = 4$ ($\ell = 2$) and the shift $1/2$.

THEOREM 3.1. *Case $s = 4$ ($\ell = 2$). Let*

$$\begin{aligned} f(q) &= \frac{1}{2\sqrt{q}} \int \eta^4(q)\eta^4(q^2)\Big(1 - \frac{128}{64 + \eta^{24}(q)\eta^{-24}(q^2)}\Big)\frac{dq}{q} \\ &= 1 - 44q + 1126q^2 - 27096q^3 + 640909q^4 - 15036548q^5 + 351245038q^6 - \cdots \end{aligned}$$

and

$$F_2(x, q) = \sum_{n=0}^{\infty} \frac{(\frac{1}{2}+x)_n(\frac{1}{4}+x)_n(\frac{3}{4}+x)_n}{(1+x)_n^3} z_2^{n+x},$$

$$G_2(x, q) = \sum_{n=0}^{\infty} \frac{(\frac{1}{2}+x)_n(\frac{1}{4}+x)_n(\frac{3}{4}+x)_n}{(1+x)_n^3} [a_2 + b_2(n+x)] z_2^{n+x},$$

where z, a, b depend on q and $G(0, q) = 1/\pi$. Then the following identities hold:

$$F_2\Big(\frac{1}{2}, q\Big) = 16F_2(0, q)\sqrt{q} \sum_{n=0}^{\infty} c_n \frac{q^n}{(2n+1)^2}$$

and

$$G_2\Big(\frac{1}{2}, q\Big) = \frac{16\sqrt{q}}{\pi}\Big(\sum_{n=0}^{\infty} c_n \frac{q^n}{(2n+1)^2} - \frac{\ln|q|}{2} \sum_{n=0}^{\infty} c_n \frac{q^n}{2n+1}\Big),$$

where c_n is the coefficient of q^n in $f(q)$.

PROOF. In this case we know that [2, Table 1]

$$z_2(q) = 4x_2(q)(1 - x_2(q)), \quad F_2(0, q) = 8\frac{\eta^8(q^2)}{\eta^4(q)} \frac{1}{\sqrt{x_2(q)}}, \quad x_2(q) = \frac{64}{64 + \eta^{24}(q)\eta^{-24}(q^2)},$$

where $\eta(q)$ is the Dedekind η function:

$$\eta(q) = q^{1/24} \prod_{n=1}^{\infty}(1 - q^n).$$

Hence, for

$$\Big(q\frac{d}{dq}\Big)^3 \phi(q) = \frac{1}{8} F_2^2(0, q)\sqrt{z_2(q)}\sqrt{1 - z_2(q)},$$

$$\left(q\frac{d}{dq}\right)^3 \phi(q) = \eta^4(q)\eta^4(q^2)\left(1 - \frac{128}{64+\eta^{24}(q)\eta^{-24}(q^2)}\right) = \sqrt{q}\, g(q),$$

where

$$g(q) = 1 - 3\times 44\, q + 5\times 126\, q^2 - 7\times 27096\, q^3 \\ + 9\times 640909\, q^4 - 11\times 15036548\, q^5 + \cdots .$$

Hence, the theorem holds. □

CONJECTURE 3.1. The coefficient of q^n in $g(q)$ is divisible by $2n+1$, which is equivalent to assuming that all the coefficients c_n of $f(q)$ are integer numbers. In addition, $c_n \equiv 1 \bmod p^2$ when $2n+1$ is a prime number p.

3.2. The q-series for Ramanujan series with $s = 3$ ($\ell = 3$) shifted by 1/2.

THEOREM 3.2. *Let*

$$f(q) = \frac{1}{2\sqrt{q}} \int \eta^2(q^3)\eta^6(q)\left(1 + 9\frac{\eta^3(q^9)}{\eta^3(q)}\right)\left(1 - \frac{54}{27+\eta^{12}(q)\eta^{-12}(q^3)}\right)\frac{dq}{q} \\ = 1 - 17\,q + 126\,q^2 - 832\,q^3 + 5329\,q^4 - 33516\,q^5 \\ + 209054q^6 - 1298142q^7 + \cdots$$

and

$$F_3(x,q) = \sum_{n=0}^{\infty} \frac{(\frac12+x)_n(\frac13+x)_n(\frac23+x)_n}{(1+x)_n^3} z_3^{n+x},$$

$$G_3(x,q) = \sum_{n=0}^{\infty} \frac{(\frac12+x)_n(\frac13+x)_n(\frac23+x)_n}{(1+x)_n^3} [a_3 + b_3(n+x)] z_3^{n+x}.$$

The following identities hold:

$$F_3\left(\frac12, q\right) = 16F_3(0,q)\sqrt{q}\sum_{n=0}^{\infty} c_n \frac{q^n}{(2n+1)^2}$$

and

$$G_3\left(\frac12, q\right) = \frac{16\sqrt{q}}{\pi}\left(\sum_{n=0}^{\infty} c_n \frac{q^n}{(2n+1)^2} - \frac{\ln|q|}{2}\sum_{n=0}^{\infty} c_n \frac{q^n}{2n+1}\right),$$

where c_n is the coefficient of q^n in $f(q)$.

PROOF. In this case we know that [2, Table 1]

$$z_3(q) = 4x_3(q)(1 - x_3(q)), \quad \text{where } x_3(q) = \frac{27}{27+\eta^{12}(q)\eta^{-12}(q^3)},$$

and

$$F_3^2(0,q) = 27\eta^8(q^3)\left(1 + 9\frac{\eta^3(q^9)}{\eta^3(q)}\right)\left(1 + \frac{1}{27}\frac{\eta^{12}(q)}{\eta^{12}(q^3)}\right).$$

Hence, for

$$\left(q\frac{d}{dq}\right)^3\phi(q) = \frac{1}{8}F_3^2(0,q)\sqrt{z_3(q)}\sqrt{1-z_3(q)},$$

$$\left(q\frac{d}{dq}\right)^3\phi(q) = \eta^2(q^3)\eta^6(q)\left(1+9\frac{\eta^3(q^9)}{\eta^3(q)}\right)\left(1-\frac{54}{27+\eta^{12}(q)\eta^{-12}(q^3)}\right) = \sqrt{q}\,g(q),$$

where

$$g(q) = 1 - 3\times 17\,q + 5\times 126\,q^2 - 7\times 832\,q^3 + 9\times 5329\,q^4 - 11\times 33516\,q^5 + \cdots.$$

Hence, the theorem holds. □

Conjecture 3.2. The coefficient of q^n in $g(q)$ is divisible by $2n+1$ which is equivalent to assuming that all the coefficients c_n of $f(q)$ are integer numbers. In addition, $c_n \equiv 1 \bmod p^2$ when $2n+1$ is a prime number p.

3.3. The q-series for Ramanujan series with $s = 6$ ($\ell = 1$) shifted by $1/2$. In this case we know that [2, Table 1]

$$F_1^2(0,q) = E_4(q), \quad z_1(q) = 1728\,\frac{\eta^{24}(q)}{E_4^3(q)},$$

where $E_4(q)$ is the Eisenstein series

$$E_4(q) = 1 + 240\sum_{n=0}^{\infty}\sigma_3(n)q^n = 1 + 240\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n}.$$

Proceeding in the same way as in the other cases,

$$\begin{aligned}\left(q\frac{d}{dq}\right)^3\phi(q) &= 3\sqrt{3}\frac{\eta^{12}(q)}{E_8(q)}\sqrt{E_4^3(q)-1728\eta^{24}(q)}\\ &= 3\sqrt{3}\,\frac{\eta^{12}(q)}{E_4^2(q)}E_6(q) = 3\sqrt{3}\,\eta^{12}(q)\frac{E_6(q)}{E_8(q)}\\ &= 3\sqrt{3}\sqrt{q}(1 - 3\times 332q + 5\times 81126q^2 - 7\times 19147288q^3\\ &\quad + 9\times 4472942221q^4 - 11\times 1040187455460q^5 + \cdots),\end{aligned}$$

where $E_6(q)$ and $E_8(q)$ are the Eisenstein series

$$E_6(q) = 1 - 504\sum_{n=0}^{\infty}\sigma_5(n)q^n = 1 - 504\sum_{n=1}^{\infty}\frac{n^5q^n}{1-q^n}$$

and

$$E_8(q) = 1 + 480\sum_{n=0}^{\infty}\sigma_7(n)q^n = 1 + 480\sum_{n=1}^{\infty}\frac{n^7q^n}{1-q^n}.$$

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Hence,

$$G_1(q) = \frac{3\sqrt{3}}{\pi}\Big[\int \frac{dq}{q}\int \frac{dq}{q}\int \frac{dq}{q}\,\eta^{12}(q)\frac{E_6(q)}{E_8(q)} - \ln|q|\int \frac{dq}{q}\int \frac{dq}{q}\eta^{12}(q)\frac{E_6(q)}{E_8(q)}\Big].$$

For the first single integral,

$$\int \eta^{12}(q)\frac{E_6(q)}{E_8(q)}\frac{dq}{q} = 2\sqrt{q}f(q) = 2\sqrt{q}(1 - 332q + 81126q^2 - 19147288q^3 + 4472942221q^4 - 1040187455460q^5 + \cdots),$$

and, again, we observe that the coefficients c_n of q^n in $f(q)$ are all integer numbers and that $c_n \equiv 1 \bmod p^2$ when $2n+1$ is a prime number p.

4. Examples of conjectured formulas, $\ell = 1, 2, 3$

In this section we show several examples of evaluation of some Ramanujan-type series with a shift $x_0 = 1/2$. More examples are in the tables. For discovering the conjectured results, we have used techniques of experimental mathematics, for example the integer relation algorithms and the function identify. For level $\ell = 2$ and $q = -e^{-\pi\sqrt{13}}$:

$$\sum_{n=0}^{\infty} \frac{(1)_n(\frac{3}{4})_n(\frac{5}{4})_n}{(\frac{3}{2})_n^3}\Big(\frac{153}{72} + \frac{260}{72}n\Big)\frac{(-1)^n}{18^{2n+1}} \overset{?}{=} 2\ln 3 - 3\ln 2.$$

For level $\ell = 2$ and $q = e^{-\pi\sqrt{58}}$:

$$\sum_{n=0}^{\infty} \frac{(1)_n(\frac{3}{4})_n(\frac{5}{4})_n}{(\frac{3}{2})_n^3}\Big(\frac{4\times 14298}{9801\sqrt{2}} + \frac{4\times 26390}{9801\sqrt{2}}n\Big)\frac{1}{99^{4n+2}} \overset{?}{=} \frac{13}{2}\pi - 16\arctan\frac{\sqrt{2}}{2} - 24\arctan\frac{\sqrt{2}}{3}.$$

It is interesting to observe that the last result can also be written with logarithms as

$$-13i\ln\frac{1+i}{1-i} + 8i\ln\frac{\sqrt{2}+i}{\sqrt{2}-i} + 12i\ln\frac{3+\sqrt{2}i}{3-\sqrt{2}i}$$

and observe in addition that $(\sqrt{2}+i)(\sqrt{2}-i) = 3$ and $(3+\sqrt{2}\,i)(3-\sqrt{2}\,i) = 11$, which are divisors of 99. For level $\ell = 3$ and $q = -e^{-\pi\sqrt{25/3}}$:

$$\sum_{n=0}^{\infty} \frac{(1)_n(\frac{5}{6})_n(\frac{7}{6})_n}{(\frac{3}{2})_n^3}\Big(\frac{11}{24} + \frac{3}{4}n\Big)\frac{(-1)^n}{80^n} \overset{?}{=} 9\ln 3 - 2\ln 2 - 5\ln 5.$$

For level $\ell = 1$ and $q = e^{-\pi\sqrt{8}}$:

$$\sum_{n=0}^{\infty} \frac{(1)_n(\frac{2}{3})_n(\frac{4}{3})_n}{(\frac{3}{2})_n^3}\Big(\frac{136}{125} + \frac{224}{125}n\Big)\Big(\frac{3}{5}\Big)^{3n} \overset{?}{=} \pi - 4\arctan\frac{1}{2}.$$

In the tables we show all the examples corresponding to rational values of z. We finally give an example of an irrational series. For level $\ell = 2$ and $q = -e^{-\pi\sqrt{21}}$:

$$\sum_{n=0}^{\infty} \frac{(1)_n(\frac{3}{4})_n(\frac{5}{4})_n}{(\frac{3}{2})_n^3}[(756+448\sqrt{3})n+(429+256\sqrt{3})]\frac{(-1)^n}{(42+24\sqrt{3})^{2n}}$$
$$\stackrel{?}{=} 2\times(42+24\sqrt{3})^2\times\ln\left[\frac{42+24\sqrt{3}}{81}\right]^2.$$

Of course our conjectured evaluations agree with the numerical approximations obtained from the corresponding $G_\ell(1/2, q)$.

In Table 1, we show the Ramanujan-type series for $1/\pi$ with rational values of z in the case $s = 4$ (level 2) and, in Table 2, we have written the corresponding conjectured values of $G_2(1/2, q)$. In Tables 3 and 5, we show the Ramanujan-type series for $1/\pi$ with rational values of z in the cases $s = 3$ (level 3) and $s = 6$ (level 1), respectively, and in Tables 4 and 6, we have written the corresponding conjectured values of $G_3(1/2, q)$ and $G_1(1/2, q)$.

TABLE 1. Ramanujan series with $s = 4$ ($\ell = 2$).

q	a	b	$z<0$	q	a	b	$z>0$
$-e^{-\pi\sqrt{5}}$	$\frac{3}{8}$	$\frac{20}{8}$	$-\frac{1}{4}$	$e^{-\pi\sqrt{4}}$	$\frac{2}{9}$	$\frac{14}{9}$	$\frac{32}{81}$
$-e^{-\pi\sqrt{7}}$	$\frac{8}{9\sqrt{7}}$	$\frac{65}{9\sqrt{7}}$	$-\frac{16^2}{63^2}$	$e^{-\pi\sqrt{6}}$	$\frac{1}{2\sqrt{3}}$	$\frac{8}{2\sqrt{3}}$	$\frac{1}{9}$
$-e^{-\pi\sqrt{9}}$	$\frac{3\sqrt{3}}{16}$	$\frac{28\sqrt{3}}{16}$	$-\frac{1}{48}$	$e^{-\pi\sqrt{10}}$	$\frac{4}{9\sqrt{2}}$	$\frac{40}{9\sqrt{2}}$	$\frac{1}{81}$
$-e^{-\pi\sqrt{13}}$	$\frac{23}{72}$	$\frac{260}{72}$	$-\frac{1}{18^2}$	$e^{-\pi\sqrt{18}}$	$\frac{27}{49\sqrt{3}}$	$\frac{360}{49\sqrt{3}}$	$\frac{1}{7^4}$
$-e^{-\pi\sqrt{25}}$	$\frac{41\sqrt{5}}{288}$	$\frac{644\sqrt{5}}{288}$	$-\frac{1}{5\times72^2}$	$e^{-\pi\sqrt{22}}$	$\frac{19}{18\sqrt{11}}$	$\frac{280}{18\sqrt{11}}$	$\frac{1}{99^2}$
$-e^{-\pi\sqrt{37}}$	$\frac{1123}{3528}$	$\frac{21460}{3528}$	$-\frac{1}{882^2}$	$e^{-\pi\sqrt{58}}$	$\frac{4412}{9801\sqrt{2}}$	$\frac{105560}{9801\sqrt{2}}$	$\frac{1}{99^4}$

TABLE 2. Some conjectured values of $G_2(1/2, q)$.

q	$-iG_2(\frac{1}{2}, q)$	q	$G_2(\frac{1}{2}, q)$
$-e^{-\pi\sqrt{5}}$	$\ln 2$	$e^{-\pi\sqrt{4}}$	$\frac{\pi}{2} - 2\arctan\frac{1}{2\sqrt{2}}$
$-e^{-\pi\sqrt{7}}$	$\ln(88+13\sqrt{7}) - 4\ln 3$	$e^{-\pi\sqrt{6}}$	$\frac{\pi}{6}$
$-e^{-\pi\sqrt{9}}$	$\frac{3}{2}\ln 3 - 2\ln 2$	$e^{-\pi\sqrt{10}}$	$\frac{\pi}{2} + 4\arctan\frac{1}{2\sqrt{2}}$
$-e^{-\pi\sqrt{13}}$	$2\ln 3 - 3\ln 2$	$e^{-\pi\sqrt{18}}$	$-\frac{\pi}{6} + 4\arctan\frac{1}{4\sqrt{3}}$
$-e^{-\pi\sqrt{25}}$	$9\ln 2 - 2\ln 3 - \frac{5}{2}\ln 5$	$e^{-\pi\sqrt{22}}$	$-\frac{\pi}{2} + 4\arctan\frac{7}{5\sqrt{11}}$
$-e^{-\pi\sqrt{37}}$	$\ln 2 + 10\ln 3 - 6\ln 7$	$e^{-\pi\sqrt{58}}$	$\frac{13\pi}{2} - 16\arctan\frac{1}{\sqrt{2}} - 24\arctan\frac{\sqrt{2}}{3}$

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TABLE 3. Ramanujan series for $s = 3$ ($\ell = 3$).

q	a	b	$z<0$	q	a	b	$z>0$
$-e^{-\pi\sqrt{9/3}}$	$\frac{\sqrt{3}}{4}$	$\frac{5\sqrt{3}}{4}$	$-\frac{9}{16}$	$e^{-\pi\sqrt{8/3}}$	$\frac{1}{3\sqrt{3}}$	$\frac{6}{3\sqrt{3}}$	$\frac{1}{2}$
$-e^{-\pi\sqrt{17/3}}$	$\frac{7}{12\sqrt{3}}$	$\frac{51}{12\sqrt{3}}$	$-\frac{1}{16}$	$e^{-\pi\sqrt{16/3}}$	$\frac{8}{27}$	$\frac{60}{27}$	$\frac{2}{27}$
$-e^{-\pi\sqrt{25/3}}$	$\frac{\sqrt{15}}{12}$	$\frac{9\sqrt{15}}{12}$	$-\frac{1}{80}$	$e^{-\pi\sqrt{20/3}}$	$\frac{8}{15\sqrt{3}}$	$\frac{66}{15\sqrt{3}}$	$\frac{4}{125}$
$-e^{-\pi\sqrt{41/3}}$	$\frac{106}{192\sqrt{3}}$	$\frac{1230}{192\sqrt{3}}$	$-\frac{1}{2^{10}}$				
$-e^{-\pi\sqrt{49/3}}$	$\frac{26\sqrt{7}}{216}$	$\frac{330\sqrt{7}}{216}$	$-\frac{1}{3024}$				
$-e^{-\pi\sqrt{89/3}}$	$\frac{827}{1500\sqrt{3}}$	$\frac{14151}{1500\sqrt{3}}$	$-\frac{1}{500^2}$				

TABLE 4. Some conjectured values of $G_3(1/2, q)$.

q	$-iG_3(\frac{1}{2}, q)$	q	$G_3(\frac{1}{2}, q)$
$-e^{-\pi\sqrt{9/3}}$	$\frac{\sqrt{3}}{4}(3\ln 3 - 2\ln 2)$	$e^{-\pi\sqrt{8/3}}$	$\frac{\sqrt{3}}{4}(3\pi - 12\arctan\frac{\sqrt{2}}{2})$
$-e^{-\pi\sqrt{17/3}}$	$\frac{3\sqrt{3}}{4}(2\ln 2 - \ln 3)$	$e^{-\pi\sqrt{16/3}}$	$\frac{\sqrt{3}}{4}(5\pi - 24\arctan\frac{\sqrt{2}}{2})$
$-e^{-\pi\sqrt{25/3}}$	$\frac{\sqrt{3}}{4}(9\ln 3 - 2\ln 2 - 5\ln 5)$	$e^{-\pi\sqrt{20/3}}$	$\frac{\sqrt{3}}{4}(-3\pi + 12\arctan\frac{\sqrt{5}}{2})$
$-e^{-\pi\sqrt{41/3}}$	$\frac{3\sqrt{3}}{4}(8\ln 2 - 5\ln 3)$		
$-e^{-\pi\sqrt{49/3}}$	$\frac{\sqrt{3}}{4}(7\ln 7 - 10\ln 2 - 6\ln 3)$		
$-e^{-\pi\sqrt{89/3}}$	$\frac{3\sqrt{3}}{4}(6\ln 5 - 6\ln 2 - 5\ln 3)$		

TABLE 5. Ramanujan series for $s = 6$ ($\ell = 1$).

q	a	b	$z<0$	q	a	b	$z>0$
$-e^{-\pi\sqrt{7}}$	$\frac{8}{5\sqrt{15}}$	$\frac{63}{5\sqrt{15}}$	$-\frac{4^3}{5^3}$	$e^{-\pi\sqrt{8}}$	$\frac{3}{5\sqrt{5}}$	$\frac{28}{5\sqrt{5}}$	$\frac{3^3}{5^3}$
$-e^{-\pi\sqrt{11}}$	$\frac{15}{32\sqrt{2}}$	$\frac{154}{32\sqrt{2}}$	$-\frac{3^3}{8^3}$	$e^{-\pi\sqrt{12}}$	$\frac{6}{5\sqrt{15}}$	$\frac{66}{5\sqrt{15}}$	$\frac{4}{5^3}$
$-e^{-\pi\sqrt{19}}$	$\frac{25}{32\sqrt{6}}$	$\frac{342}{32\sqrt{6}}$	$-\frac{1}{8^3}$	$e^{-\pi\sqrt{16}}$	$\frac{20}{11\sqrt{33}}$	$\frac{252}{11\sqrt{33}}$	$\frac{2^3}{11^3}$
$-e^{-\pi\sqrt{27}}$	$\frac{279}{160\sqrt{30}}$	$\frac{4554}{160\sqrt{30}}$	$-\frac{9}{40^3}$	$e^{-\pi\sqrt{28}}$	$\frac{144\sqrt{3}}{85\sqrt{85}}$	$\frac{2394\sqrt{3}}{85\sqrt{85}}$	$\frac{4^3}{85^3}$
$-e^{-\pi\sqrt{43}}$	$\frac{526\sqrt{15}}{80^2}$	$\frac{10836\sqrt{15}}{80^2}$	$-\frac{1}{80^3}$				
$-e^{-\pi\sqrt{67}}$	$\frac{10177\sqrt{330}}{3\times 440^2}$	$\frac{261702\sqrt{330}}{3\times 440^2}$	$-\frac{1}{440^3}$				
$-e^{-\pi\sqrt{163}}$	$\frac{27182818\sqrt{10005}}{3\times 53360^2}$	$\frac{1090280268\sqrt{10005}}{3\times 53360^2}$	$-\frac{1}{53360^3}$				

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TABLE 6. Some conjectured values of $G_1(1/2, q)$.

q	$-iG_1(\frac{1}{2}, q)$	q	$G_1(\frac{1}{2}, q)$
$-e^{-\pi\sqrt{7}}$	$\frac{3\sqrt{3}}{8} \ln \frac{3^3}{5}$	$e^{-\pi\sqrt{8}}$	$\frac{3\sqrt{3}}{8}(\pi - 4 \arctan \frac{1}{2})$
$-e^{-\pi\sqrt{11}}$	$\frac{3\sqrt{3}}{8} \ln 2$	$e^{-\pi\sqrt{12}}$	$\frac{3\sqrt{3}}{8}(-\pi + 8 \arctan \frac{1}{2})$
$-e^{-\pi\sqrt{19}}$	$\frac{3\sqrt{3}}{8} \ln \frac{2^5}{3^3}$	$e^{-\pi\sqrt{16}}$	$\frac{3\sqrt{3}}{8}(3\pi - 4 \arctan \frac{\sqrt{2}}{3} - 12 \arctan \frac{\sqrt{2}}{2})$
$-e^{-\pi\sqrt{27}}$	$\frac{3\sqrt{3}}{8} \ln \frac{3^3 \times 5}{2^7}$	$e^{-\pi\sqrt{28}}$	$\frac{3\sqrt{3}}{8}(3\pi - 16 \arctan \frac{1}{2} - 8 \arctan \frac{1}{4})$
$-e^{-\pi\sqrt{43}}$	$\frac{3\sqrt{3}}{8} \ln \frac{2^2 \times 3^9}{5^7}$		
$-e^{-\pi\sqrt{67}}$	$\frac{3\sqrt{3}}{8} \ln \frac{2^{13} \times 11^5}{3^3 \times 5^{11}}$		
$-e^{-\pi\sqrt{163}}$	$\frac{3\sqrt{3}}{8} \ln \frac{3^{21} \times 5^{13} \times 29^5}{2^{38} \times 23^{11}}$		

5. Conclusion

It may be that discovering explicit formulas (as we have done in (2.2) for the case $s = 2$ and $x = 1/2$) for the coefficients c_n could be a useful step towards proving the patterns observed by the author. The final step would be evaluating the q-series at $q = \pm \exp(-\pi\sqrt{r})$. The analogous patterns observed for shifted Ramanujan-like series for $1/\pi^k$ with $k \geq 2$ (see [4] and [5, pages 44–46]) are further beyond the ideas of this paper.

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