Sets of Semi-Commutative Matrices: Part II[1]

By J. WILLIAMSON, Johns Hopkins University.

(*Received 7th July*, 1932. *Received in revised form 8th November*, 1932.
Read 4th November, 1932.)

§ 2. In this section we extend the definition of an E-set, so that it includes sets of the type

(19) $$E_i E_j = \omega E_j E_i; \quad i < j; \quad i, j = 1, 2, \ldots, q,$$

where the only restriction on the E_i is that they be non-singular. We now consider matrices of the type

(20) $$A = \Sigma\, a(e_i)\, E(e_i), \;\; a(e_i) = a(e_1, e_2, \ldots, e_q), \quad E(e_i) = E_1^{e_1} E_2^{e_2} \ldots E_q^{e_q},$$

where each e_i takes independently the values $0, 1, \ldots, n-1$, while the $a(e_i)$ are either complex numbers or else matrices of order r, the product $a(e_i)\,E(e_i)$, in the latter case, being interpreted as the direct product of the two matrices $a(e_i)$ and $E(e_i)$. We shall call the n^q matrices $a(e_i)\,E(e_i)$ the *terms* of A, $a(e_i)$ the *coefficient* of $E(e_i)$, and the set of integers $e_1, e_2, \ldots, e_q$ the *exponents* of $E(e_i)$. We first prove

THEOREM 3. *If the matrices* $E_1, E_2, \ldots, E_q = E_{2p}$ *form a set of matrices, of order* n^p, *satisfying* (19), *then a matrix* A *of the form* (20) *is zero if, and only if, each* $a(e_i)$ *is zero.*

If each $a(e_i)$ is zero, A must be zero, so we have only to show that, if A is zero, every coefficient $a(e_i)$ is zero. Now, corresponding to each term $a(e_i)\,E(e_i)$ there exists a set of q equations

(21) $$E_j\, E(e_i)\, E_j^{-1} = \omega^{d_j}\, E(e_i), \qquad j = 1, 2, \ldots, q,$$

where

(22) $$d_j \equiv -e_1 - e_2 - \ldots - e_{j-1} + e_{j+1} + \ldots + e_q, \qquad (\text{mod } n).$$

---

[1] This is the continuation of a paper by the same author, pp. 179-188 of this volume. The numbering of sections, equations, and theorems follows on after that of the previous paper.

But, since q is even, the congruences (22) possess a unique solution; in fact

$$e_f \equiv d_{f-1} - d_{f-2} + \ldots + (-1)^f d_1 - d_{f+1} + d_{f+2} - \ldots + (-1)^f d_q,$$

where

$$0 \leqq e_f \leqq n-1,$$

and the solutions of (22), for two sets of q integers d_j, incongruent modulo n, are distinct. Moreover, corresponding to each term $a(e_i)\,E(e_i)$, we can define a set of q matrices by means of the recursion formula[1]

$$(23) \qquad A_j = \sum_{k=0}^{n-1} \omega^{-kd_j} E_j^k A_{j-1} E_j^{-k}, \qquad j = 1, 2, \ldots, q,$$

where $A_0 = A$. We notice that, if

$$A_{j-1} = \Sigma\, b(e_i)\, E(e_i),$$

then

$$A_j = n\, \Sigma'\, b(e_i)\, E(e_i),$$

where the accent means that the summation extends only over those terms of A_{j-1} whose exponents satisfy the j^{th} of the congruences (22). Accordingly

$$A_q = n^q\, \Sigma''\, a(e_i)\, E(e_i),$$

where the summation now extends only over those terms of A whose exponents satisfy all of the congruences (22), and, as there is only one such term,

$$A_q = n^q\, a(e_i)\, E(e_i).$$

Now if A is zero, A_q must be zero, and as $E(e_i)$ is non-singular, $a(e_i)$ must be zero. Thus the theorem is proved.

COROLLARY. *Under the hypotheses of Theorem* 3, *the* n^{2p} *matrices* $E(e_i)$ *form a basis for the algebra of matrices of order* n^p, *and, in the more general case, every matrix of order* $n^p r$ *can be written uniquely in the form* (20).

For, by Theorem 3, the n^{2p} matrices $E(e_i)$ are linearly independent with respect to the field of complex numbers, and so form a basis for the algebra of matrices of order n^p. The second part of the corollary is now an immediate consequence.

---

[1] In this formula E_j is written for the matrix eE_j where e is the unit matrix of order r.

Before proceeding to determine the coefficients $a(e_i)$ of the terms of A in (20), we consider the matrices of the type (20) which satisfy the equations

(24) $$AE_i = \omega E_i A, \qquad i = 1, 2, \ldots . q,$$

and also those which satisfy the equations

(25) $$AE_i = E_i A, \qquad i = 1, 2, \ldots . q.$$

If a matrix A satisfies (24), then A must consist solely of terms whose exponents e_i satisfy the congruences (22) where each d_j has the value unity. Accordingly, by (23), A reduces to the single term $a E_{q+1}$, where

$$E_{q+1} = E_1^{-1} E_2 E_3^{-1} E_4 \ldots . E_{q-1}^{-1} E_q.$$

Hence, if $a_1, a_2, \ldots, a_s$ form a maximal set of matrices of order r satisfying (19), the matrices

(26) $$e E_1, a_j E_{q+1}, \; (i = 1, 2, \ldots, 2p; \; j = 1, 2, \ldots, s;)$$

where e is the unit matrix of order r, form a maximal E-set of matrices of order $n^p r$. In particular, if $r \not\equiv 0 \pmod{n}$, then $s = 1$ and consequently *a maximal E-set of matrices of order* $n^p r$, $r \not\equiv 0$ *mod* n, *contains exactly* $2p + 1$ *matrices.*

Similarly, if A satisfies the equations (25), A must consist of the single term aE. But the matrices $(eE_j)^n$, where $j = 1, 2, \ldots, q$, all satisfy (25) and therefore $(eE_j)^n = a_j E$.

In particular, if $r = 1$, we see that *the* n^{th} *power, but no lower power, of every matrix of a maximal E-set of matrices, of order* n^p, *is a scalar matrix.* Thus, if $r = 1$, by multiplication with suitably chosen scalar matrices, we can always take the members of a maximal E-set to be n^{th} roots of the unit matrix. When this is done we shall say that the set is *normalised.*

In determining the coefficients $a(e_i)$ of A in (20) we first show that, if $E(e_i) \not\equiv E$, the trace of $E(e_i)$ is zero. For by (21), we have, denoting $w^{-d_j} E(e_i) E_j^{-1}$ by Q_j,

$$E(e_i) = E_j Q_j = \omega^{d_j} Q_j E_j,$$

where $j = 1, 2, \ldots, q$. But since the trace of a product of two matrices is the same as the trace of the product of the matrices in reverse order, we obtain

$$\text{trace}\,[E(e_i)] = \text{trace}\,[E_j Q_j] = \omega^{d_j}\,\text{trace}\,[Q_j E_j] = \text{trace}\,[Q_j E_j].$$

Accordingly the trace of $E(e_i)$ is zero, unless $d_j \equiv 0 \pmod{n}$, for $j = 1, 2, \ldots. q$, that is, unless $E(e_i) = E$. Now the matrix $A\,E(e_i)^{-1}$ has $a(e_i)$ as coefficient of E, so that, if $a(e_i)$ is a complex number,

$$\text{trace}\,[A\,E(e_i)^{-1}] = \text{trace}\,[a(e_i)\,E] = n^p\,a(e_i),$$

or

$$a(e_i) = n^{-p}.\,\text{trace}\,[A\,E(e_i)^{-1}]. \tag{27}$$

Formula (27) must be somewhat modified when $a(e_i)$ is a matrix of order r. Thus, if the direct product $a(e_i)\,E(e_i)$ is written as a matrix whose elements are matrices of order n^p, it takes the form $(a_{jk}\,E(e_i))$, where a_{jk} is the element in the j^{th} row and k^{th} column of $a(e_i)$. Accordingly the matrix $A\,E(e_i)^{-1}$ has the form (b_{jk}) where each matrix b_{jk} is a matrix of order n^p. Then by a proof similar to that of the simpler case, it follows that formula (26) must be replaced by

$$a_{jk} = n^{-p}.\,\text{trace}\,[b_{jk}],$$

where j and k take the values $1, 2, \ldots. r$.

If the matrices E_i form an E-set, so do the matrices $G_i = B^{-1}E_iB$, where B is any non-singular matrix, and the two sets are said to be *similar*. Conversely we shall now prove

THEOREM 4. *If E_i and G_i, $i = 1, 2, \ldots, 2m$, are any two normalised E-sets of matrices of order $n^p r$, then the two E-sets E_i and G_i are similar.*

We shall prove this theorem by showing that the set E_i and the set G_i are both similar to the same E-set. We know that there exists a non-singular matrix A, such that $A^{-1}E_iA = F_i$, where F_s is defined by (3) when $s = 1$ and by (9) when $s > 1$. Let D denote the *diagonal block matrix*

$$\text{diag}\,(F_{12},\ F_{22}F_{12},\ F_{32}F_{22}F_{12},\ \ldots,\ F_{n-1,2}\,F_{n-2,2}\ldots.\ F_{12},\ e);$$

this means that, when D is written as a matrix of matrices, all the component matrices are zero, except those in the principal diagonal, which are the matrices F_{12}, $F_{22}F_{12}$, etc. Then it is easily verified that

$$D^{-1}F_1D = F_1 = e\cdot\Omega_1, \qquad D^{-1}F_2D = e\cdot\Omega_2,$$

where Ω_1 and Ω_2 are defined by (7). It now follows from (26) that

$$D^{-1}F_sD = A_{s-2}\cdot\Omega_1^{-1}\Omega_2, \quad s = 3, 4, \ldots, 2m,$$

where the $2m$ matrices A_s form an E-set of matrices of order t/n. If $m = 1$, we need proceed no further, since E_1 and E_2 have been shown to be similar to $e \cdot \Omega_1$ and $e \cdot \Omega_2$ respectively. If $m > 1$, we apply the same process to the matrices A_s and show that the set A_s is similar to the set

$$e' \cdot \Omega_1,\ e' \cdot \Omega_2,\ B_{s-2} \cdot \Omega_1^{-1}\,\Omega_2, \qquad s = 3, 4, \ldots, 2(m-1),$$

where e' is the unit matrix of order t/n^2 and the matrices B_s form an E-set of matrices of order t/n^2. Thus, if $m = 2$, the set E_1, E_2, E_3, E_4 is similar to the set $e \cdot \Omega_1$, $e \cdot \Omega_2$, $(e' \cdot \Omega_1) \cdot \Omega_1^{-1}\,\Omega_2$, $(e' \cdot \Omega_2) \cdot \Omega_1^{-1}\,\Omega_2$. If, however, $m > 2$, we proceed as before with the matrices B_s and finally, in m steps, arrive at a standard E-set, expressed in terms of the matrices Ω_1 and Ω_2, similar to the set E_i. In the same manner it can be shown that the set G_i is similar to the same standard E-set, so that the two sets E_i and G_i are similar. As an immediate consequence we have the following corollary:

Two maximal normalised E-sets of matrices of order t, where t is divisible by n, are similar.

§ 3. *Groups of periodic collineations.* The matrices in any E-set consisting of $2m$ members generate, under multiplication, a group of order n^{2m}, if two matrices, which differ from each other only by a scalar factor, are considered to represent the same element of the group. Such a group is simply isomorphic with a group of collineations in a space of one dimension less than the order of the matrices in the E-set. Since the n^{th} power of each matrix is a scalar matrix, the corresponding collineations are periodic, of period a divisor of n, while the fact that any two matrices of the group are *semi-commutative* means that the two corresponding collineations are *commutative.* A group of collineations will be said to be periodic of period n, if at least one of its members has an actual period n. We shall now determine the structure of all maximal groups of commutative periodic collineations, of period n, in a space of $t-1$ dimensions[1].

If T_1 and T_2 are two members of a group of commutative collineations of period n in a space of $t-1$ dimensions, T_1 and T_2 determine uniquely two matrices E_1 and E_2 of order t, satisfying the two equations

$$E_1^n = E_2^n = E, \tag{28}$$

$$E_1 E_2 = k E_2 E_1. \tag{29}$$

[1] This problem was solved for $n = 2$ by E. Study, *Göttinger Nachrichten* (1912), 452-479.

But it follows from (28) that $E_1^n E_2 = E_2 E_1^n$, and from (29) that $E_1^n E_2 = k^n E_2 E_1^n$. Hence k is an n^{th} root of unity. Accordingly, if $T_1, T_2, \ldots, T_f$ are the members of a commutative group of periodic collineations of period n, the elements $E_1, E_2, \ldots, E_f$ of the corresponding group of matrices must satisfy the two conditions

$$(30) \qquad E_i^n = \lambda_i E, \quad E_i E_j = \omega^{r_{ij}} E_j E_i,$$

where ω is a primitive n^{th} root of unity and the r_{ij} are positive integers. If $E_1, E_2, \ldots, E_s$ are generators of the group, $\lambda_1, \lambda_2, \ldots, \lambda_s$ may all have the value unity but then the values of λ_j for $j > s$, are determined. We shall call such a group an E-group and notice that to every E-group there corresponds a group of periodic commutative collineations and *vice versa.*

We shall require the following lemmas.

LEMMA 1. *In every E-group there exist two matrices E_1 and E_2, such that $E_1 E_2 = \rho E_2 E_1$, while $E_1 E_k = \rho^{k_1} E_k E_1$ and $E_2 E_k = \rho^{k_2} E_k E_2$ for every other matrix E_k in the group, where ρ is a primitive m^{th} root of unity, m a divisor of n, and k_1, k_2 are integers.*

If $r = r_{12}$ is a minimum value for the exponents r_{ij} of ω in (30), then r_{1k} and r_{2k} are integral multiples of r. For, if $r_{1j} = wr + t$, where $0 \leqq t < r$, then

$$E_1 E_2^{-w} E_j = \rho^t E_2^{-w} E_j E_1;$$

since $E_2^{-w} E_j$ belongs to the group, t must be zero, as otherwise r would not be a minimum value of r_{ij}. Similarly it can be shown that r_{2j} must be an integral multiple of r. But $\omega^r = \rho$ where ρ is a primitive m^{th} root of unity and m is a divisor of n; accordingly the lemma is proved.

LEMMA 2. *In every maximal E-group, in which not every pair of matrices is commutative, there exist two matrices E_i and E_j, such that $E_i E_j = \rho E_j E_i$, $E_i^m = E_j^m = E$, where ρ is a primitive m^{th} root of unity.*

By lemma 1 there exist in the E-group two matrices E_1 and E_2 such that $E_1 E_2 = \rho E_2 E_1$ and $E_1^n = E_2^n = E$. Accordingly $E_1 = E_2^{-1} \rho E_1 E_2$, so that the latent roots of E_1 are the same as the latent roots of ρE_1. As each latent root of E_1 is an n^{th} root of unity, the latent roots of E_1 can be arranged into sets $\omega_i, \omega_i \rho, \ldots, \omega_i \rho^{m-1}$ $(i = 1, 2, \ldots, t/m)$ where ω_i is an n^{th} root of unity. If $\omega_i = \rho^s \omega_k$ for any integral value of s, the i^{th} of these sets coincides with the k^{th}, so that two sets either coincide or else have no member in common. Let the set $\omega_i, \omega_i \rho, \ldots, \omega_i \rho^{m-1}$ be repeated exactly t_i times; then, if R_1 is the diagonal matrix $(1, \rho, \rho^2, \ldots, \rho^{m-1})$ and $K_i = \omega_i e_i \cdot R_1$, where e_i is the unit matrix of order t_i, the latent roots of E_1 are the same as the latent roots of the diagonal block matrix

$$(31) \qquad F_1 = (K_1, K_2, \ldots, K_q), \quad t_1 + t_2 + \ldots t_q = t/m.$$

Moreover, if $i \neq j$, no latent root of K_i is the same as a latent root of K_j, or differs from a latent root of K_j by an integral power of ρ. Accordingly there exists a non-singular matrix D such that $D^{-1} E_k D = F_k$, where F_1 is defined by (31). If, now, F_k is written as (F_{ij}), where i and j take the values 1, 2, ..., q, F_{ij} being a matrix of $t_i m$ rows and $t_j m$ columns, then it follows from the equation $F_1 F_k = \rho^d F_k F_1$ that $K_i F_{ij} = \rho^d F_{ij} K_j$. But, if $i \neq j$, since K_i and $\rho^d K^j$ have no latent root in common, F_{ij} is the zero matrix, so that F_k is a diagonal block matrix $(F_{11}, F_{22}, \ldots, F_{qq})$. In particular F_2 is the diagonal block matrix $(M_1, M_2, \ldots M_q)$, where $K_i M_i = \rho M_i K_i$, i taking the values 1, 2, ... q.

By methods similar to those used in the proof of theorem 4, we can find a non-singular matrix G such that

$$G^{-1} K_i G = K_i, \; G^{-1} M_i G = N_i = B_i \cdot R_2, \; G^{-1} F_{ii} G = F'_{ii},$$

where B_i is a diagonal matrix whose elements are n^{th} roots of unity, and R_2 is the square matrix of order m,

$$R_2 = \begin{bmatrix} 0 & 0 & 0 & . & . & . & 0 & 1 \\ 1 & 0 & 0 & . & . & . & 0 & 0 \\ 0 & 1 & 0 & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & 0 & 0 \\ 0 & 0 & 0 & . & . & . & 1 & 0 \end{bmatrix}.$$

Moreover G can be chosen in such a manner that $B_i \cdot R_2$ becomes a diagonal block matrix $(S_1, S_2, \ldots, S_w)$, where each matrix S_j is of the form $\omega_j e'_j \cdot R_2$ and no latent root of S_i is the same as a latent root of S_j or differs from a latent root of S_j by a power of ρ. Accordingly, since $F'_{ii} N_i = \rho^p N_i F'_{ii}$, by a proof similar to the above, F'_{ii} must also be a diagonal block matrix. We have thus shown that the matrices E_i, of the original E-group, can be reduced by the same similarity transformation to the diagonal block matrices $T_i = (T_{i1}, T_{i2}, \ldots, T_{ih})$, where $T_{1j} = \omega_j e_j \cdot R_1$, $T_{2j} = e_j \omega'_j \cdot R_2$, e_j being a unit matrix of some order. But the matrices $T'_1 = (T'_{11}, T'_{12}, \ldots, T'_{1h})$ and $T'_2 = (T'_{21}, T'_{22}, \ldots, T'_{2h})$, where $T'_{1j} = e_j \cdot R_1$ and $T'_{2j} = e_j \cdot R_2$, are members of any maximal E-group, in which the matrices T_1 and

T_2 lie. For, from the equations $T_k T_1 = \rho^{k_1} T_1 T_k$ and $T_k T_2 = \rho^{k_2} T_2 T_k$ it follows immediately that $T_k T'_1 = \rho^{k_1} T'_1 T_k$ and $T_k T'_2 = \rho^{k_2} T'_2 T_k$. Since $(T'_1)^m = (T'_2)^m = E$ and $T'_1 T'_2 = \rho T'_2 T'_1$, we may take, for the matrices E_i and E_j, the matrices in the original E-group, which are similar to T_1 and T_2 respectively. Thus the lemma is proved.

If all the matrices in an E-group of matrices of order t are commutative, the group must be simply isomorphic with a subgroup of the group of order n^{t-1}, whose component matrices are all diagonal matrices with n^{th} roots of unity as their elements, the first element in each matrix being unity. Thus *the only type of maximal E-group, in which all the matrices are commutative, is one of order* n^{t-1}; in this case every matrix can be reduced simultaneously by a similarity transformation to diagonal form.

If, however, all the matrices in a maximal E-group are not commutative, the minimum value r of r_{ij} in (30) is less than n, so that, by lemmas 1 and 2, there exist in the E-group two matrices E_1 and E_2, such that $E_1 E_2 = \rho E_2 E_1$ and $E_1^m = E_2^m = E$, where ρ is a primitive m^{th} root of unity. Then, by Theorem 4, E_1 and E_2 are similar to the matrices $e' \cdot R_1$ and $e' \cdot R_2$, where e' is the unit matrix of order t/m; R_1 and R_2 are then obtained from Ω_1 and Ω_2 respectively by replacing ω by ρ and n by m. Moreover $R_1 R_2 = \rho R_2 R_1$; if $E_1 E_k = \rho^{d_1} E_k E_1$ and $E_2 E_k = \rho^{d_2} E_k E_2$, then E_k is similar to the matrix $A_k \cdot R_1^{e_1} R_2^{e_2}$, where e_1 and e_2 are determined uniquely from d_1 and d_2 by congruences similar to (22). Accordingly the matrices A_k must form an E-group of matrices of order t/m, which must also be maximal since the original E-group is maximal. Thus the original E-group is the direct product of one maximal E-group of matrices of order t/m and another of matrices of order m. If we denote the group of order m^2, generated by R_1 and R_2, by $G(m)$, we may say that the original E-group is of type $H \times G(m)$, where H is a maximal E-group of matrices of order t/m. Thus the problem of determining all E-groups of matrices of order t, is reduced to that of determining all E-groups of matrices of order t/m.

But the matrices in a maximal E-group are either all commutative, in which case H is of order $n^{t/m-1}$, or else H is the direct product of a group $G(m_1)$ and a group H_1, where H_1 is a maximal E-group of matrices of order t/mm_1. Thus, by repeated applications of this process, we are led to the conclusion that *if* $m_1, m_2, \ldots, m_k$ *are* k *divisors (not necessarily distinct) of* n, *and if*

$$t = m_1 m_2 \ldots m_k s \tag{32}$$

https://doi.org/10.1017/S0013091500027310 Published online by Cambridge University Press

where s is a positive integer, then there exists a maximal E-group $G(m_1, m_2 \ldots . m_k)$ *of matrices of order t, which is the direct product of a group* $G(m_1)$ *of order* m_1^2, *a group* $G(m_2)$ *of order* m_2^2, ..., *a group* $G(m_k)$ *of order* m_k^2 *and a group H of order* n^{s-1}, *so that the order of* $G(m_1, m_2, \ldots, m_k)$ *is* $m_1^2 m_2^2 \ldots m_k^2 n^{s-1}$. *Moreover every maximal E-group of matrices of order t is simply isomorphic to a group* $G(m_1, m_2, \ldots, m_k)$ *for some set* m_i *of divisors of n which satisfy* (32).

In the group $G(m_1, m_2, \ldots, m_k)$ the s^{n-1} matrices in the subgroup H are permutable with every matrix in the group, while no other matrix in $G(m_1, m_2, \ldots, m_k)$ has this property. Moreover, since the matrices in H can all be reduced simultaneously to diagonal form, it follows that the matrices in any E-group, simply isomorphic to $G(m_1, m_2, \ldots, m_k)$, can be reduced simultaneously by a similarity transformation to diagonal block matrices, whose blocks are matrices of order t/s.

We now proceed to show that two different sets m_i of divisors of n, both of which satisfy (32) with the same value for s, do not determine two groups $G(m_1, m_2, \ldots, m_k)$ which are necessarily distinct.

To do this we consider a group G which is the direct product of two groups $G(m_i)$ and $G(m_j)$. If w is the greatest common divisor of m_i and m_j, so that $m_i = wg$ and $m_j = wf$, where g and f are relatively prime, the least common multiple of m_i and m_j is $wfg = m$. Then, if ρ is a primitive m^{th} root of unity, ρ^f is a primitive m_i^{th} root of unity, and ρ^g a primitive m_j^{th} root. Accordingly in $G(m_i)$ there exist two matrices E_1 and E_2 and in $G(m_j)$ two matrices F_1 and F_2, such that $E_1 E_2 = \rho^f E_2 E_1$, $F_1 F_2 = \rho^g F_2 F_1$, $E_i F_j = F_j E_i$, where i and j take the values 1, 2. Now, since f and g are relatively prime, there exist two integers α and β satisfying the equation $\alpha f + \beta g = 1$. Hence the two matrices $E_1 F_1$ and $E_2^\alpha F_2^\beta$, which both lie in G, satisfy the condition

$$(E_1 F_1)(E_2^\alpha F_2^\beta) = \rho (E_2^\alpha F_2^\beta)(E_1 F_1).$$

Accordingly, by our previous results, G must be the direct product of a group $G(m)$ and some other group, which must necessarily be $G(w)$. Thus the integers m_i in $G(m_1, m_2, \ldots, m_k)$ can always be chosen in such a way that, if m_i and m_j are any two of them, then either m_i is a divisor of m_j or else m_j is a divisor of m_i. Moreover, if the group $G(r) \times G(s)$ is simply isomorphic with the group $G(r') \times G(s')$, where s is a divisor of r and s' of r', then $r = r'$ and $s = s'$. For,

https://doi.org/10.1017/S0013091500027310 Published online by Cambridge University Press

if not, we may suppose $r > r'$; then in $G(r) \times G(s)$ there are at least two elements of order r, while in $G(r') \times G(s')$ every element is of order not exceeding r'; and this is impossible. Hence we have the following result:

Every maximal E-group of matrices of order t is simply isomorphic to one and only one group $G(m_1, m_2, \ldots, m_k)$, where m_i is a divisor of m_{i-1}, and $i = 2, 3, \ldots, k$.

It should be noted that m_i is a *divisor* of m_{i-1}, not a *proper divisor*, so that the case in which m_i coincides with m_{i-1} is not excluded.

As an alternative form of the last result we have the following: *Every maximal E-group of matrices of order t is simply isomorphic to one and only one group $G(m_1, m_2, \ldots, m_k)$, where each m_i is a power of a prime.*

For, if a and b are two relatively prime integers whose product is m, it is easily shown that the group $G(m)$ is the direct product of two groups $G(a)$ and $G(b)$. Therefore, if $m = p_1^{q_1} p_2^{q_2} \ldots p_h^{q_h}$, where $p_1, p_2, \ldots, p_h$ are the distinct prime factors of m, $G(m)$ is the direct product of h groups $G(p_i^{q_i})$.

In conclusion we state our results as a theorem on groups of commutative collineations of period n:

THEOREM 5. *Every maximal group of commutative periodic collineations of period n in a space of $t-1$ dimensions is simply isomorphic to an E-group of type $G(m_1, m_2, \ldots, m_k)$, where the m_i form a set of divisors of n satisfying* (32), *and such that m_i is a divisor of m_{i-1}, $i = 2, 3, \ldots, k$. Corresponding to each group $G(m_1, m_2, \ldots, m_k)$, satisfying the above conditions, there is one and only one projectively distinct collineation group.*