Ergod. Th. & Dynam. Sys., (2024), **44**, 2700–2736 © The Author(s), 2024. Published by Cambridge University Press.
doi:10.1017/etds.2023.116

Joint partial equidistribution of Farey rays in negatively curved manifolds and trees

JOUNI PARKKONEN† and FRÉDÉRIC PAULIN‡

† *Department of Mathematics and Statistics, P.O. Box 35, 40014 University of Jyväskylä, Finland*
(e-mail: jouni.t.parkkonen@jyu.fi)
‡ *Laboratoire de mathématique d'Orsay, UMR 8628 CNRS, Université Paris-Saclay, 91405 Orsay Cedex, France*
(e-mail: frederic.paulin@universite-paris-saclay.fr)

(*Received* 6 *January* 2023 *and accepted in revised form* 25 *September* 2023)

Abstract. We prove a joint partial equidistribution result for common perpendiculars with given density on equidistributing equidistant hypersurfaces, towards a measure supported on truncated stable leaves. We recover a result of Marklof on the joint partial equidistribution of Farey fractions at a given density, and give several analogous arithmetic applications, including in Bruhat–Tits trees.

Key words: equidistribution, negative curvature, geodesic flow, common perpendiculars, Farey fractions
2020 Mathematics Subject Classification: 37D40, 53C22, 11N45 (Primary); 20G20, 28A33, 51M10, 57K32, 20E08 (Secondary)

1. *Introduction*

In this paper, we study geometric equidistribution results on negatively curved manifolds with applications to arithmetic problems. Let N be a complete connected Riemannian manifold with pinched negative sectional curvature at most -1. Let m_{BM} be its Bowen–Margulis measure, which, when finite and renormalized and when the sectional curvature has bounded derivative, is the probability measure of maximal entropy for the geodesic flow on T^1N. When, for instance, N has finite volume, it is well known that the conditional measure of m_{BM} on the image $\mathrm{g}^t W$ of a closed strong unstable leaf W by the geodesic flow g^t at time t equidistributes towards m_{BM} as $t \to +\infty$. See, for instance, the works of Dani, Eskin and McMullen [**EM**, Theorem 7.1], Margulis, Kleinbock and Margulis [**KlM**, Proposition 2.2.1], Ratner, Sarnak [**Sar**, Theorem 1], as well as [**PaP2**, Theorem 1] and [**BPP**, Theorem 10.2] for generalizations. Given an increasing family $(\mathcal{F}_t)_{t\in\mathbb{R}}$ of finite subsets $\mathcal{F}_t$ of points on $\mathrm{g}^t W$ for all $t \in \mathbb{R}$, it is natural to study the limiting distribution properties of $\mathcal{F}_t$ as $t \to +\infty$. If $\mathcal{F}_t$ is denser and denser in $\mathrm{g}^t W$, it is expected that $\mathcal{F}_t$ will also equidistribute to m_{BM}. If $\mathcal{F}_t$ is too sparse in $\mathrm{g}^t W$, the limiting distribution is expected to be purely punctual. A threshold seems to occur when $\mathcal{F}_t$ has a constant density in $\mathrm{g}^t W$, possibly yielding equidistribution of partial nature.

In this paper we take $\mathcal{F}_t$ to be the image by g^t of the subset of W of initial tangent vectors of the common perpendiculars to another cusp neighbourhood, having a length bound chosen in order to have a constant density at each time t. We prove that $\mathcal{F}_t$ then equidistributes towards the conditional measure of m_{BM} on a truncated weak stable leaf. This type of partial equidistribution result seems to be quite original in hyperbolic dynamical systems. For instance, we recover the case $n = 2$ of a theorem by Marklof [**Mar2**, Theorem 6], as well as [**Lut**, Theorem 6.1]. We actually prove a joint partial equidistribution result, for more general families, give a version of our results for tree quotients, and give several arithmetic applications.

More precisely, let $\widetilde{M}$ be a complete simply connected Riemannian manifold with pinched negative sectional curvature at most -1, and let Γ be a non-elementary discrete subgroup of $\mathrm{Isom}(\widetilde{M})$, with critical exponent δ_Γ (see, for instance, [**BH**]). Let D be a non-empty proper closed convex subset of $\widetilde{M}$ and let H be a horoball of $\widetilde{M}$ such that the families $\mathcal{D}^- = (\gamma D)_{\gamma\in\Gamma}$ and $\mathcal{D}^+ = (\gamma H)_{\gamma\in\Gamma}$ are locally finite (modulo stabilizers) in $\widetilde{M}$.

Let us introduce the measures that come into play in this paper, referring to §2 and [**BPP**] for further explanations. We denote by $\|\mu\|$ the total mass of a measure μ.

Let $(\mu_x)_{x\in\widetilde{M}}$ be a Patterson density for Γ and let m_{BM} be the associated Bowen–Margulis measure on $\Gamma\backslash T^1\widetilde{M}$. When $\widetilde{M}$ is a symmetric space and Γ has finite covolume, then (up to a scalar multiple) μ_x is the unique probability measure on $\partial_\infty\widetilde{M}$ invariant under the stabilizer of x in the isometry group of $\widetilde{M}$, and m_{BM} is the Liouville measure, which is finite and mixing. Let W be the strong stable leaf in $T^1\widetilde{M}$ whose image in $\widetilde{M}$ is ∂H, and let $\mu^{0+}_{\mathcal{D}^+,t_0}$ be the conditional measure of m_{BM} on the truncated weak stable leaf $\Gamma\bigcup_{s\geq t_0}\mathrm{g}^s W$. The measure $\mu^{0+}_{\mathcal{D}^+,t_0}$ is finite and non-zero, for instance, when H is centred at a bounded parabolic fixed point of Γ. Let $\sigma^+_{\mathcal{D}^-}$ be the outer skinning measure of $\mathcal{D}^-$; see, for instance, [**PaP2**], as well as [**OS1**, **OS2**] when $\widetilde{M}$ is geometrically finite with constant curvature, and when D is a ball, horoball or complete totally geodesic submanifold. When D is a horoball, $\sigma^+_{\mathcal{D}^-}$ is the conditional measure of m_{BM} on the strong unstable leaf in $\Gamma\backslash T^1\widetilde{M}$ having a lift to $T^1\widetilde{M}$ whose image in $\widetilde{M}$ is ∂D.

For every $\gamma \in \Gamma$ such that $d(D, \gamma H) > 0$, let $v_\gamma \in T^1\widetilde{M}$ be the outgoing normal vector of D pointing towards the point at infinity of γH.

THEOREM 1.1. *Let $t_0 \in \mathbb{R}$. Assume that m_{BM} is finite and mixing for the geodesic flow on $\Gamma\backslash T^1\widetilde{M}$, and that $\sigma^+_{\mathcal{D}^-}$ and $\mu^{0+}_{\mathcal{D}^+,t_0}$ are finite and non-zero. Then for the weak-star convergence of measures on $(\Gamma\backslash T^1\widetilde{M})^2$, we have*

$$\lim_{t\to+\infty} \|m_{\mathrm{BM}}\|\, e^{-\delta_\Gamma t} \sum_{\substack{\gamma\in\Gamma_D\backslash\Gamma/\Gamma_H \\ 0<d(D,\gamma H)\leq t-t_0}} \Delta_{\Gamma v_\gamma} \otimes \Delta_{\mathrm{g}^t \Gamma v_\gamma} = \sigma^+_{\mathcal{D}^-} \otimes \mu^{0+}_{\mathcal{D}^+,t_0}.$$

See Theorem 3.3 for a more general version, as well as a version for quotients of trees by discrete groups of automorphisms. See §3 for a proof, after some preliminary work in §2, in particular on the truncated weak stable leaves and their measures. The proof starts by using the joint equidistribution result of common perpendiculars from [**PaP5**], but the statement of Theorem 1.1 is only apparently similar to Eq. (12) in that paper and new ideas and techniques are required. One of these ideas is a new subdivision scheme along the geodesic flow that allows good control of the exponential growth. One of the techniques is an important regularity study of the splitting of the weak stable leaves and of the dynamics on the unstable horospheres.

As a consequence of our main result (Theorem 1.1), we recover the case $n = 2$ of a theorem by Marklof [**Mar2**, Theorem 6] on the joint partial equidistribution of Farey points chosen with constant average density on an equidistributing horocycle on the modular curve $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}^2_{\mathbb{R}}$; see Corollary 4.1. In the present case (in contrast to other distribution results in number theory), the restriction to a fixed denominator of the Farey fractions in [**Mar2**] is only marginally stronger, by the growth properties of the horospheres. The relationship between Farey fractions and hyperbolic geometry (and, in particular, with the divergent geodesics) is not new, probably going back to Ford. See, for instance, the works of Athreya and Cheung [**AC**], Sarnak, Series, Sullivan, and the references in [**HeP**, **PaP7**]. We also recover [**Lut**, Theorem 6.1], originally proved for hyperbolic surfaces.

In §4, we give several generalizations of Marklof's result, including the three-dimensional real hyperbolic version below. See Corollary 4.2 for a more general statement, and §§4.3 and 4.4 for distribution results for Farey points with constant average density on closed horospheres in complex and quaternionic hyperbolic orbifolds. It might be that it is possible to obtain these applications using purely homogeneous dynamics techniques, along the lines of the cross-section method of Marklof [**Mar2**] and Athreya and Cheung [**AC**]. But no such results appear in the literature yet. We believe that covering all our examples might require a lot of work, even starting from the three-dimensional real hyperbolic case with a large class number of the imaginary quadratic field, as the cross-sections, as well as other fundamental domain issues, are considerably more complicated for general arithmetic lattices in rank-one real Lie groups than for $\mathrm{SL}_2(\mathbb{Z})$. Furthermore, the case of groups over local fields with positive characteristic is likely to require major innovations by homogeneous dynamics methods.

Let K be an imaginary quadratic number field, with ring of integers $\mathcal{O}_K$ and discriminant different from -4 and -3 in order to simplify the statement in this introduction. Let $G = \mathrm{PSL}_2(\mathbb{C})$, let Γ be the Bianchi group $\mathrm{PSL}_2(\mathcal{O}_K)$, let

$$H = \left\{\mathfrak{n}_-(r) = \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix} : r \in \mathbb{C}\right\} \quad \text{and} \quad \text{for all } t \in \mathbb{R}, \text{ let} \quad \Phi^t = \begin{bmatrix} e^{-t/2} & 0 \\ 0 & e^{t/2} \end{bmatrix}.$$

Let $M = \left\{\left[\begin{smallmatrix} e^{-i\,\theta/2} & 0 \\ 0 & e^{i\,\theta/2} \end{smallmatrix}\right] : \theta \in \mathbb{R}\right\}$. We endow the compact abelian groups $\mathbb{C}/\mathcal{O}_K$ and $(H \cap \Gamma)\backslash H$ with their probability Haar measures dx and $d\mu_{(H\cap\Gamma)\backslash H}$. For every $t \in \mathbb{R}$, we consider the set $\mathcal{F}_t$ of *complex Farey fractions of height at most* $e^{t/2}$, defined by

$$\mathcal{F}_t = \left\{\frac{p}{q} \bmod \mathcal{O}_K : p, q \in \mathcal{O}_K, p\mathcal{O}_K + q\mathcal{O}_K = \mathcal{O}_K, 0 < |q| \leq e^{t/2}\right\}.$$

https://doi.org/10.1017/etds.2023.116 Published online by Cambridge University Press

COROLLARY 1.2. *Let* $f : (\mathbb{C}/\mathcal{O}_K) \times (\Gamma\backslash G/M) \to \mathbb{R}$ *be a continuous function with compact support. Then for every* $t_0 \in \mathbb{R}$*, we have*

$$\lim_{t\to+\infty} \frac{1}{\operatorname{Card} \mathcal{F}_{t-t_0}} \sum_{r\in\mathcal{F}_{t-t_0}} f(r, \Gamma\mathfrak{n}_-(r)\Phi^t M)$$
$$= 2\, e^{2\, t_0} \int_{s=t_0}^{+\infty} \int_{y\,\in\,(H\cap\Gamma)\backslash H} \int_{x\,\in\,\mathbb{C}/\mathcal{O}_K} f(x, \Gamma\, {}^t y^{-1}\Phi^s M)\, dx\, d\mu_{(H\cap\Gamma)\backslash H}(y)\, e^{-2\, s}\, ds.$$

We now give a joint partial equidistribution result for arithmetic points with given density on an expanding horosphere in an arithmetic quotient of a non-archimedean simple Lie group (see Corollary 4.7 for a more general version). Let $R = \mathbb{F}_q[Y]$ be the ring of polynomials over a finite field $\mathbb{F}_q$ with one indeterminate Y, and let $\widehat{K} = \mathbb{F}_q((Y^{-1}))$ be the valued field of formal Laurent series in Y^{-1} over $\mathbb{F}_q$ with $|Y^{-1}| = 1/q$. Let $G = \mathrm{PGL}_2(\widehat{K})$, let $\Gamma = \mathrm{PGL}_2(R)$, let

$$H = \left\{\mathfrak{n}_-(r) = \begin{bmatrix} 1 & r \\ 0 & 1\end{bmatrix} : r \in \widehat{K}\right\} \quad \text{and} \quad \text{for all } n \in \mathbb{Z}, \text{ let} \quad \Phi^n = \begin{bmatrix} 1 & 0 \\ 0 & Y^n\end{bmatrix}.$$

Let $\Gamma_H = N_G(H) \cap \Gamma$ and $M = \{[\begin{smallmatrix} 1 & 0 \\ 0 & u\end{smallmatrix}] : u \in \widehat{K}, |u| = 1\}$. We endow $\Gamma_H\backslash H$ with the induced measure $d\mu_{\Gamma_H\backslash H}$ of a Haar measure of H, normalized to be a probability measure. For every $n \in \mathbb{Z}$, we consider the set $\mathcal{F}_n$ of *non-archimedean Farey fractions of height at most* q^n, defined by

$$\mathcal{F}_n = \Gamma_H\backslash\left\{\mathfrak{n}_-\left(\frac{P}{Q}\right) :\ P, Q \in R,\ PR + QR = R,\ 0 \leq \deg Q \leq n\right\}.$$

COROLLARY 1.3. *Let* $f : (\Gamma_H\backslash H) \times (\Gamma\backslash G/M) \to \mathbb{R}$ *be a continuous function with compact support. Then for every* $n_0 \in \mathbb{Z}$*, we have*

$$\lim_{n\to+\infty} \frac{1}{\operatorname{Card} \mathcal{F}_{n-n_0}} \sum_{r\in\mathcal{F}_{n-n_0}} f(r, \Gamma\, r\, \Phi^{2n} M)$$
$$= (1 - q^{-2})\, q^{2n_0} \sum_{m=n_0}^{+\infty} \int_{x,y\,\in\,\Gamma_H\backslash H} f(x, \Gamma\, {}^t y^{-1}\Phi^{2m} M)\, d\mu_{\Gamma_H\backslash H}(x)\, d\mu_{\Gamma_H\backslash H}(y)\, q^{-2m}.$$

2. *Background and definitions*

Let X be either a complete simply connected Riemannian manifold with pinched negative sectional curvature at most -1 or a proper geodesically complete $\mathbb{R}$-tree. Let Γ be a non-elementary discrete group of isometries of X. We refer to [**Rob**] or [**BPP**, Chs. 2 and 3], with potential 0 throughout this paper, for background information on the data (X, Γ). In particular, see §3.3 of [**BPP**] for the definitions of the boundary at infinity $\partial_\infty X$ of X and the critical exponent $\delta_\Gamma > 0$ of Γ.

We refer to [**BPP**, §2.2] for the following definitions. We denote by $\check{\mathcal{G}}X$ the Bartels–Lück space of generalized geodesics in X (that is, of continuous maps $\mathbb{R}\to X$

https://doi.org/10.1017/etds.2023.116 Published online by Cambridge University Press

that are isometric on a closed interval of $\mathbb{R}$ with non-empty interior and locally constant outside it), endowed with the distance d defined by

$$\text{for all } \ell, \ell' \in \check{\mathcal{G}}X, \text{ let } \quad d(\ell, \ell') = \int_{-\infty}^{+\infty} d(\ell(t), \ell'(t))\, e^{-2|t|}\, dt. \tag{1}$$

It contains the closed subspace $\mathcal{G}X$ of (true) geodesic lines and the closed subspaces $\mathcal{G}_{\pm,0}X$ of (positive/negative) geodesic rays, that is, of generalized geodesics that are isometric on exactly $\pm[0, +\infty[$ (which we identify with their restriction to $\pm[0, +\infty[$). We denote by $\ell \mapsto \ell_\pm$ the two endpoint maps from $\check{\mathcal{G}}X$ to $X \cup \partial_\infty X$. Let $(\mathrm{g}^t)_{t\in\mathbb{R}}$ be the (continuous-time) geodesic flow on $\check{\mathcal{G}}X$, which preserves $\mathcal{G}X$. Let

$$\mathcal{G}_\pm X = \mathcal{G}X \cup \bigcup_{t\in\mathbb{R}} \mathrm{g}^t \mathcal{G}_{\pm,0}$$

be the closed subspaces of generalized geodesics that are isometric at least on an interval $\pm[a, +\infty[$ for some $a \in \mathbb{R}$, so that $\mathcal{G}_- X \cap \mathcal{G}_+ X = \mathcal{G}X$. The Bartels–Lück space is important in order to allow the positive geodesic rays pushed by the geodesic flow at large positive times to converge to geodesic lines.

We denote by $\widetilde{m}_{\mathrm{BM}}$ the Bowen–Margulis measure of Γ on $\mathcal{G}X$ and by m_{BM} the Bowen–Margulis measure on $\Gamma\backslash\mathcal{G}X$ associated with any choice of Patterson–Sullivan density $(\mu_x)_{x\in X}$; see, for instance, [**Rob**] or [**BPP**, §4.2] with potential 0.

Given a proper closed convex subset D of X, we refer to [**BPP**, §2.4] for the definition of their inner/outer normal bundles $\partial^1_\pm D$, which are contained in $\mathcal{G}_{\pm,0}X$. We refer to [**BPP**, §7.1] again with potential 0 (see also [**PaP2**] in the manifold case) for the definition of the outer/inner skinning measures $\widetilde{\sigma}^\pm_D$ on $\partial^1_\pm D$. Given a measurable map f, we denote by f_* the pushforward map of measures. Recall that, for every $\gamma \in \Gamma$, we have

$$\gamma_*(\widetilde{\sigma}^\pm_D) = \widetilde{\sigma}^\pm_{\gamma D}. \tag{2}$$

Given w in $\mathcal{G}_+X$ or $\mathcal{G}_-X$ respectively, we refer to [**BPP**, §2.3] for the definitions of its strong stable leaf $W^+(w)$ or strong unstable leaf $W^-(w)$, of its (weak) stable leaf $W^{0+}(w)$ or (weak) unstable leaf $W^{0-}(w)$, and of its stable horoball $HB_+(w)$ or unstable horoball $HB_-(w)$. The *antipodal (or time reversal) map* $\iota : \check{\mathcal{G}}X \to \check{\mathcal{G}}X$ defined by $\ell \mapsto \{t \mapsto \ell(-t)\}$ is an involution satisfying $\iota(\mathcal{G}_+X) = \mathcal{G}_-X$ and,

$$\text{for all } w \in \mathcal{G}_+X, \quad \iota\, W^+(w) = W^-(\iota\, w).$$

Let $w \in \mathcal{G}_+X$. We refer to [**BPP**, §2.4] for the definition of the canonical homeomorphism $N^+_w : W^+(w) \to \partial^1_- HB_+(w)$ that associates to a geodesic line $\ell \in W^+(w)$ the unique (negative) geodesic ray $\rho \in \partial^1_- HB_+(w)$ such that $\ell_- = \rho_-$. We also denote, by an abuse of notation, $N^+_w(\ell) = \ell_{|\,]-\infty,0]}$. The homeomorphism N^+_w relates the inner skinning measure $\widetilde{\sigma}^-_{HB_+(w)}$ of $HB_+(w)$ to the conditional $\mu_{W^+(w)}$ on the strong stable leaf $W^+(w)$ of w of the Bowen–Margulis measure $\widetilde{m}_{\mathrm{BM}}$ as follows (see [**BPP**, end of p. 162]): for $\ell \in W^+(w)$, we have

$$d\mu_{W^+(w)}(\ell) = d\,((N^+_w)^{-1})_*\widetilde{\sigma}^-_{HB_+(w)}(\ell) = d\,\widetilde{\sigma}^-_{HB_+(w)}(\ell_{|\,]-\infty,0]}). \tag{3}$$

https://doi.org/10.1017/etds.2023.116 Published online by Cambridge University Press

Recall that we have a homeomorphism

$$h_w : W^+(w) \times \mathbb{R} \to W^{0+}(w), \quad (\ell, s) \mapsto \mathrm{g}^s \ell.$$

For every isometry γ of X, for all $t, s \in \mathbb{R}$ and $\ell \in W^+(w)$, we have

$$\gamma h_w(\ell, s) = h_{\gamma w}(\gamma \ell, s) \quad \text{and} \quad \mathrm{g}^t \circ h_w(\ell, s) = h_{\mathrm{g}^t w}(\mathrm{g}^t \ell, s).$$

The homeomorphism h_w writes the conditional measure $\mu_{W^{0+}(w)}$ on the stable leaf $W^{0+}(w)$ of w of the Bowen–Margulis measure $\widetilde{m}_{\mathrm{BM}}$ as a twisted product measure of the measure $\mu_{W^+(w)}$ on $W^+(w)$ and the Lebesgue measure on $\mathbb{R}$ (see [**BPP**, Eq. (7.12)] with potential 0): for all $s \in \mathbb{R}$ and $\ell \in W^+(w)$, we have

$$d\mu_{W^{0+}(w)}(\mathrm{g}^s \ell) = e^{-\delta_\Gamma s} d\mu_{W^+(w)}(\ell)\, ds. \tag{4}$$

Note that for every $\gamma \in \Gamma$, we have

$$\gamma_* \mu_{W^{0+}(w)} = \mu_{W^{0+}(\gamma w)}. \tag{5}$$

Since the Lebesgue measure is atomless, for every Borel subset Ω^+ of $W^+(w)$, the boundary of $h_w(\Omega^+ \times [a, b])$ has measure 0 for $\mu_{W^{0+}(w)}$ if and only if the boundary of Ω^+ has measure 0 for $\mu_{W^+(w)}$.

For all $w \in \mathcal{G}_+ X$ and $s \in \mathbb{R}$, let

$$\mathrm{g}^s{}_| : \partial^1_- HB_+(w) \to \partial^1_- HB_+(\mathrm{g}^s w)$$

be the homeomorphism that associates to $\rho \in \partial^1_- HB_+(w)$ the unique $\rho' \in \partial^1_- HB_+(\mathrm{g}^s w)$ such that $\rho_- = \rho'_-$, or equivalently such that we have $\rho(t) = \rho'(t - s)$ for every $t \in \mathbb{R}$ such that $t \leq \min\{0, s\}$. Note that $\mathrm{g}^s W^+(w) = W^+(\mathrm{g}^s w)$ and that the following diagram is commutative:

$$\begin{array}{ccc} W^+(w) & \xrightarrow{N^+_w} & \partial^1_- HB_+(w) \\ \mathrm{g}^s \downarrow & & \downarrow \mathrm{g}^s{}_| \\ W^+(\mathrm{g}^s w) & \xrightarrow{N^+_{\mathrm{g}^s w}} & \partial^1_- HB_+(\mathrm{g}^s w). \end{array} \tag{6}$$

Let us now introduce the truncated (weak) stable leaves in $\mathcal{G}X$. The projections on the second factor of the limiting measures of our upstairs empirical joint distributions will have as support the union of a locally finite family of truncated stable leaves. For every $\sigma \in \mathbb{R} \cup \{-\infty\}$, the *$\sigma$-stable leaf* of $w \in \mathcal{G}_+ X$ is

$$W^{0+}_\sigma(w) = \bigcup_{t \geq \sigma} \mathrm{g}^t W^+(w),$$

so that $W^{0+}_{-\infty}(w)$ equals $W^{0+}(w)$.

LEMMA 2.1. *Let $w \in \mathcal{G}_+ X$ and $s \in \mathbb{R}$.*

(1) *The homeomorphism $N^+_{\mathrm{g}^s w} : W^+(\mathrm{g}^s w) \to \partial^1_- HB_+(\mathrm{g}^s w)$ is uniformly bicontinuous, uniformly in s.*
(2) *The homeomorphism $h_w : W^+(w) \times \mathbb{R} \to W^{0+}(w)$ is uniformly bicontinuous.*
(3) *The homeomorphism $\mathrm{g}^s{}_| : \partial^1_- HB_+(w) \to \partial^1_- HB_+(\mathrm{g}^s w)$ is uniformly bicontinuous, uniformly on s varying in a compact subset of $\mathbb{R}$.*

https://doi.org/10.1017/etds.2023.116 Published online by Cambridge University Press

Proof. (1) For all $\ell, \ell' \in W^+(\mathrm{g}^s w)$, by equation (1), we have

$$d(N^+_{\mathrm{g}^s w}(\ell), N^+_{\mathrm{g}^s w}(\ell')) = \int_{-\infty}^{0} d(\ell(t), \ell'(t))\, e^{2\,t}\, dt + \int_0^{+\infty} d(\ell(0), \ell'(0))\, e^{-2\,t}\, dt$$
$$\leq d(\ell, \ell') + \tfrac{1}{2}\, d(\ell(0), \ell'(0)).$$

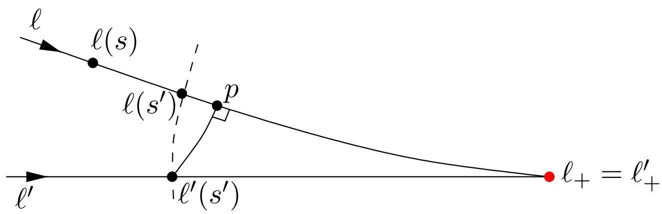
Since the footpoint map $\pi : \check{\mathscr{G}}X \to X$ defined by $\ell \mapsto \ell(0)$ is $\frac{1}{2}$-Hölder continuous (see [**BPP**, Proposition 3.2]), this proves that $N^+_{\mathrm{g}^s w}$ is uniformly continuous (actually $\frac{1}{2}$-Hölder continuous), uniformly in s.

Conversely, note that by convexity, for all $\ell, \ell' \in W^+(\mathrm{g}^s w)$, since $\ell_+ = \ell'_+$, we have $d(\ell(t), \ell'(t)) \leq d(\ell(0), \ell'(0))$ for every $t \geq 0$. Hence,

$$\begin{aligned} d(\ell, \ell') &= \int_{-\infty}^{0} d(\ell(t), \ell'(t))\, e^{2\,t}\, dt + \int_0^{+\infty} d(\ell(t), \ell'(t))\, e^{-2\,t}\, dt \\ &\leq \int_{-\infty}^{0} d(\ell(t), \ell'(t))\, e^{2\,t}\, dt + \int_0^{+\infty} d(\ell(0), \ell'(0))\, e^{-2\,t}\, dt \\ &= d(N^+_{\mathrm{g}^s w}(\ell), N^+_{\mathrm{g}^s w}(\ell')). \end{aligned}$$

Therefore, $(N^+_{\mathrm{g}^s w})^{-1}$ is 1-Lipschitz, hence uniformly continuous, uniformly in s.

(2) Again since the footpoint map is $\frac{1}{2}$-Hölder continuous, there exists a constant $c > 0$ such that for every $\epsilon \in\,]0, 1]$, for all $s, s' \in \mathbb{R}$ and $\ell, \ell' \in W^+(w)$, if $d(\mathrm{g}^s \ell, \mathrm{g}^{s'} \ell') \leq \epsilon$, then $d(\ell(s), \ell'(s')) \leq c\, \epsilon^{1/2}$. We may assume that $s \leq s'$.



Since $\ell_+ = \ell'_+$, by the convexity of the horoballs and by the fact that closest point maps on non-empty closed convex subsets do not increase the distances, with p the closest point to $\ell'(s')$ on $\ell([s, +\infty[)$, we have $p \in \ell([s', +\infty[)$ and

$$|s - s'| = d(\ell(s), \ell(s')) \leq d(\ell(s), p) \leq d(\ell(s), \ell'(s')) \leq c\, \epsilon^{1/2}.$$

Let us fix $T > 0$ and let us assume that $s \in [-T, T]$. By [**BPP**, Eq. (2.8)], we have $d(\mathrm{g}^{s'-s} \ell', \ell') \leq |s - s'|$. By the change of variable $t \mapsto t + s$ in equation (1), we have

$$d(\ell, \mathrm{g}^{s'-s} \ell') \leq e^{2|s|} d(\mathrm{g}^s \ell, \mathrm{g}^{s'} \ell').$$

Therefore,

$$d(\ell, \ell') \leq d(\ell, \mathrm{g}^{s'-s} \ell') + d(\mathrm{g}^{s'-s} \ell', \ell') \leq e^{2T}\, \epsilon + c\, \epsilon^{1/2}.$$

https://doi.org/10.1017/etds.2023.116 Published online by Cambridge University Press

Conversely, for all $\epsilon \in]0, 1]$, $T > 0$, $s, s' \in [-T, T]$ and $\ell, \ell' \in W^+(w)$, assume that $\max\{|s - s'|,\ d(\ell, \ell')\} \le \epsilon$. Then by similar arguments, we have

$$d(\mathrm{g}^s \ell, \mathrm{g}^{s'} \ell') \le d(\mathrm{g}^s \ell, \mathrm{g}^s \ell') + d(\mathrm{g}^s \ell', \mathrm{g}^s \mathrm{g}^{s'-s} \ell') \le e^{2T} (d(\ell, \ell') + |s' - s|) \le 2\, e^{2T}\, \epsilon.$$

This proves assertion (2) of Lemma 2.1.

(3) Let $T > 0$ and $s \in [-T, T]$. By Assertion (1), by the commutativity of diagram (6) and by the invertibility of g^s, we only have to prove that $\mathrm{g}^s : \mathcal{G}X \to \mathcal{G}X$ is uniformly continuous, uniformly in $s \in [-T, T]$. As already seen, for all $\ell, \ell' \in \mathcal{G}X$, we have $d(\mathrm{g}^s \ell, \mathrm{g}^{s'} \ell') \le e^{2T}\ d(\ell, \ell')$, hence the result follows. □

We refer to [**BPP**, §7.2] for the following definitions. Let $\mathcal{D}^- = (D_i)_{i \in I^-}$ be a locally finite (in the sense that we will explain below) Γ-equivariant family of non-empty proper closed convex subsets of X and let $\mathcal{D}^+ = (H_j)_{j \in I^+}$ be a locally finite Γ-equivariant family of (closed) horoballs in X. Let $\sim_+$ be the equivalence relation on I^+ defined by $j \sim_+ j'$ if and only if $H_{j'} = H_j$ and there exists $\gamma \in \Gamma$ such that $j' = \gamma j$. Let $\sim_-$ be the similarly defined equivalence relation on I^-. By locally finite, we mean that for every compact subset K of X, the quotients sets $\{i \in I^- : K \cap D_i \neq \emptyset\}/_{\sim_-}$ and $\{j \in I^+ : K \cap H_j \neq \emptyset\}/_{\sim_+}$ are finite.

For all $j \in I^+$ and $s \in \mathbb{R}$, let $H_{j,s}$ be the horoball contained in H_j consisting of points at a distance at least s from the complement of H_j if $s \ge 0$, and otherwise, let $H_{j,s}$ be the closed $(-s)$-neighbourhood of H_j, which is the horoball containing H_j consisting of the points that are at distance at most $-s$ from H_j.

For every $j \in I^+$, let w_j be any geodesic ray starting from the boundary of the horoball H_j and converging to the point at infinity of H_j, so that $HB_+(w_j) = H_j$. We denote

$$W_j^+ = W^+(w_j),\quad W_j^{0+} = W^{0+}(w_j),\quad W_{\sigma,j}^{0+} = W_\sigma^{0+}(w_j),$$
$$N_j^+ = N_{w_j}^+,\quad \mu_j^+ = \mu_{W^+(w_j)},\quad h_j = h_{w_j}\quad \text{and}\quad \mu_j^{0+} = \mu_{W^{0+}(w_j)}.$$

Using the homeomorphism h_j from $W_j^+ \times \mathbb{R}$ to W_j^{0+} defined by $(\ell, s) \mapsto \mathrm{g}^s \ell$ and the homeomorphism $N_j^+ : W_j^+ \to \partial_-^1 H_j$ defined by $\ell \mapsto \ell_{|\,]-\infty,0]}$, for all $s \in \mathbb{R}$ and $\ell \in W_j^+$, we thus have by equations (4) and (3),

$$d\mu_j^{0+}(\mathrm{g}^s \ell) = e^{-\delta_\Gamma s}\ d\, \widetilde{\sigma}_{H_j}^-(\ell_{|\,]-\infty,0]})\, ds. \tag{7}$$

For all $j \in I^+$ and $s_0 \in \mathbb{R}$, since H_j is the s_0-neighbourhood of H_{j,s_0} if $s_0 \ge 0$ and since H_{j,s_0} is the $(-s_0)$-neighbourhood of H_j if $s_0 \le 0$, by [**BPP**, Eq. (7.7)] (see also [**PaP2**, Proposition 4 (iii)] in the manifold case), for every $\ell \in W_j^+$, we have

$$d\, \widetilde{\sigma}_{H_j}^-(\ell_{|\,]-\infty,0]}) = e^{\delta_\Gamma s_0}\ d\, \widetilde{\sigma}_{H_{j,s_0}}^-((\mathrm{g}^{s_0}\ell)_{|\,]-\infty,0]}). \tag{8}$$

For every $t_0 \in \mathbb{R}$ fixed, we also define

$$\widetilde{\sigma}_{\mathcal{D}^-}^+ = \sum_{i \in I^-/_{\sim_-}} \widetilde{\sigma}_{D_i}^+ \quad \text{and} \quad \widetilde{\mu}_{\mathcal{D}^+,t_0}^{0+} = \sum_{j \in I^+/_{\sim_+}} \mu_j^{0+}{}_{|W_{t_0,j}^{0+}}. \tag{9}$$

Since the Γ-equivariant family $(H_j)_{j \in I^+}$ is locally finite and since $t_0 > -\infty$, the two measures $\widetilde{\sigma}_{\mathcal{D}^-}^+$ and $\widetilde{\mu}_{\mathcal{D}^+,t_0}^{0+}$ on $\check{\mathcal{G}}X$ are locally finite. This is the reason why it is important

https://doi.org/10.1017/etds.2023.116 Published online by Cambridge University Press

to restrict the (weak) stable leaves W^{0+}_j to their upper parts $W^{0+}_{t_0,j}$. These two measures are also Γ-equivariant by equations (2) and (5) (and by the Γ-equivariance of the families $\mathcal{D}^\pm$). Hence (see, for instance, [**PaPS**, §2.8] for the definition of the induced measure when Γ may have torsion), they induce locally finite measures $\sigma^+_{\mathcal{D}^-}$ and $\mu^{0+}_{\mathcal{D}^+,t_0}$ on $\Gamma\backslash\breve{\mathcal{G}}X$.

3. *Joint partial equidistribution of common perpendiculars to shrinking horoballs at a given density*

In this section, we prove, as an application of [**BPP**, Theorem 11.3], a joint partial equidistribution theorem for pairs consisting of a common perpendicular between a locally convex subset and a quotient horoball on the one hand and its image by the geodesic flow at a large time on the other hand. This gives a generalized geometric version in negative curvature (including variable curvature and in any dimension) of the case $n = 2$ of [**Mar2**, Theorem 6] and [**Lut**, Theorem 6.1] for hyperbolic surfaces.

With the notation of §2 (at its beginning and after the proof of Lemma 2.1), under the finiteness and mixing assumption on the Bowen–Margulis measure and the finiteness and non-vanishing assumption on the skinning measures, the image $\mathrm{g}^t\Gamma\partial^1_+ D_i$ by the geodesic flow at time $t \geq 0$ of the image in $\Gamma\backslash\mathcal{G}X$ of the outer normal bundle of D_i (endowed with its skinning measure) equidistributes as $t \to +\infty$ towards the Bowen–Margulis measure in $\Gamma\backslash\mathcal{G}X$. For a proof, we refer to [**PaP2**, Theorem 1] in the manifold case and to [**BPP**, Theorem 10.2 with potential 0] in general. We will take on $\mathrm{g}^t\Gamma\partial^1_+ D_i$ sufficiently many images by g^t and Γ of common perpendiculars from D_i to H_j in order to have a constant density with respect to the skinning measure on $\Gamma\backslash\mathcal{G}X$ of the t-neighborhood of ΓD_i.

For all $i \in I^-$ and $j \in I^+$ such that the point at infinity of H_j is not contained in $\partial_\infty D_i$ (or equivalently such that $\partial_\infty D_i \cap \partial_\infty H_j = \emptyset$), let $\rho_{i,j}$ be the unique geodesic ray in $\partial^1_+ D_i$ such that $\rho_{i,j}(+\infty)$ is the point at infinity of H_j, and let $\lambda_{i,j} = d(D_i, H_j)$.

THEOREM 3.1. *Let X be either a proper geodesically complete $\mathbb{R}$-tree or a complete simply connected Riemannian manifold with pinched negative sectional curvature at most -1. Let Γ be a non-elementary discrete group of isometries of X. Let $\mathcal{D}^- = (D_i)_{i\in I^-}$ be a locally finite Γ-equivariant family of non-empty proper closed convex subsets of X and let $\mathcal{D}^+ = (H_j)_{j\in I^+}$ be a locally finite Γ-equivariant family of horoballs in X. Assume that the Bowen–Margulis measure m_{BM} on $\Gamma\backslash\mathcal{G}X$ is finite and mixing for the geodesic flow on $\Gamma\backslash\mathcal{G}X$. Then for every $t_0 \in \mathbb{R}$, for the weak-star convergence of measures on $\mathcal{G}_{+,0}X \times \breve{\mathcal{G}}X$, we have*

$$\lim_{t\to+\infty} \|m_{\mathrm{BM}}\|\, e^{-\delta_\Gamma t} \sum_{\substack{i\in I^-/_{\sim_-},\ j\in I^+/_{\sim_+},\ \gamma\in\Gamma \\ \partial_\infty D_i \,\cap\, \partial_\infty H_{\gamma j}=\emptyset,\ \lambda_{i,\gamma j}\leq t-t_0}} \Delta_{\rho_{i,\gamma j}} \otimes \Delta_{\mathrm{g}^t \rho_{\gamma^{-1} i,j}} = \widetilde{\sigma}^+_{\mathcal{D}^-} \otimes \widetilde{\mu}^{0+}_{\mathcal{D}^+,t_0}. \tag{10}$$

Proof. Let us first give some notation that will be useful in this proof. For all $s \in \mathbb{R}$ and (i, j) in $I^- \times I^+$ such that the closures $\overline{D_i}$ and $\overline{H_{j,s}}$ of D_i and $H_{j,s}$ in $X \cup \partial_\infty X$ have empty intersection, let $\lambda_{i,j,s} = d(D_i, H_{j,s}) > 0$ be the length of the common perpendicular from D_i to $H_{j,s}$, and let $\alpha^-_{i,\,j,\,s} \in \breve{\mathcal{G}}X$ be its parametrization: it is the unique element of $\breve{\mathcal{G}}X$ such that

- $\alpha^-_{i,\,j,\,s}(t) = \alpha^-_{i,\,j,\,s}(0) \in D_i$ if $t \leq 0$,
- $\alpha^-_{i,\,j,\,s}(t) = \alpha^-_{i,\,j,\,s}(\lambda_{i,\,j,\,s}) \in H_{j,s}$ if $t \geq \lambda_{i,\,j,\,s}$, and
- $\alpha^-_{i,j,s}|_{[0,\,\lambda_{i,j,s}]} = \alpha_{i,\,j,\,s}$ is the shortest geodesic arc starting from a point of D_i and ending at a point of $H_{j,s}$.

We have $\lambda_{i,j} = 0$ if $\overline{D_i} \cap \overline{H_j} \neq \emptyset$ and $\lambda_{i,j} = \lambda_{i,j,0} > 0$ if $\overline{D_i} \cap \overline{H_j} = \emptyset$, so that $\lambda_{i,j,s} = \lambda_{i,j} + s$ when both terms $\lambda_{i,j}$ and $\lambda_{i,j,s}$ are defined and positive. Note that we have $\lambda_{i,\gamma j,s} = \lambda_{\gamma^{-1}i,j,s}$ for every $\gamma \in \Gamma$, by equivariance. When $\lambda_{i,j,s} > 0$, we define $\alpha^+_{i,j,s} = \mathrm{g}^{\lambda_{i,j,s}} \alpha^-_{i,j,s} \in \breve{\mathcal{G}}X$, which is isometric exactly on $[-\,\lambda_{i,j,s}, 0]$.

The term on the left in equation (10) is independent of the choice of the representatives of i and j. Let us fix $(i,\, j) \in I^- \times I^+$ and let us prove that for the weak-star convergence of measures on $\mathcal{G}_{+,0}X \times \breve{\mathcal{G}}X$, we have

$$\lim_{t\to+\infty} \|m_{\mathrm{BM}}\|\, e^{-\delta_\Gamma\, t} \sum_{\substack{\gamma\in\Gamma \\ \partial_\infty D_i \,\cap\, \partial_\infty H_{\gamma j} = \emptyset,\ \lambda_{i,\,\gamma j} \leq t - t_0}} \Delta_{\rho_{i,\gamma j}} \otimes \Delta_{\mathrm{g}^t \rho_{\gamma^{-1} i,j}} = \widetilde{\sigma}^+_{D_i} \otimes \mu^{0+}_{j}{}_{|W^{0+}_{t_0,j}}. \tag{11}$$

The result follows by a (locally finite) summation using equation (9).

For all $\tau \in\,]0, 1]$ and $s_0 \geq t_0$, Theorem 11.3 of [**BPP**] (in the case with potential 0) applied to the locally finite Γ-equivariant families $(D_{\alpha i})_{\alpha\in\Gamma}$ and $(H_{\beta j,s_0})_{\beta\in\Gamma}$ (see also [**PaP5**, Eq. (12)] in the manifold case) gives, for the weak-star convergence of measures on $\breve{\mathcal{G}}X \times \breve{\mathcal{G}}X$,

$$\lim_{t\to+\infty} \|m_{\mathrm{BM}}\|\, e^{-\delta_\Gamma\, t} \sum_{\substack{\gamma\in\Gamma,\ \overline{D_i}\,\cap\,\overline{H_{\gamma j,\,s_0}} = \emptyset \\ t-\tau < \lambda_{i,\,\gamma j,\,s_0} \leq t}} \Delta_{\alpha^-_{i,\gamma j,\,s_0}} \otimes \Delta_{\alpha^+_{\gamma^{-1} i,\,j,\,s_0}} = \frac{1 - e^{-\delta_\Gamma\,\tau}}{\delta_\Gamma}\, \widetilde{\sigma}^+_{D_i} \otimes \widetilde{\sigma}^-_{H_{j,\,s_0}}. \tag{12}$$

Let us consider two compact subsets Ω^- of $\partial^1_+ D_i$ and Ω^+ of W^+_j with positive measure for $\widetilde{\sigma}^+_{D_i}$ and μ^+_j respectively, whose boundaries have zero measure for $\widetilde{\sigma}^+_{D_i}$ and μ^+_j respectively. For all $s_0 \geq t_0$ and $\tau > 0$, the product $B = \Omega^- \times h_j(\Omega^+ \times [s_0, s_0 + \tau])$ is contained in $\partial^1_+ D_i \times W^{0+}_{t_0,j}$.

Step 1. Let us first relate the two right-hand sides of equations (11) and (12) evaluated on the Borel set B.

By respectively equations (7) and (8), an easy integral computation and the commutativity of the diagram (6), we have

$$\begin{aligned}
(\widetilde{\sigma}^+_{D_i} \otimes \mu^{0+}_j)(B) &= \int_{(\rho,\ell,s)\in\Omega^-\times\Omega^+\times[s_0,s_0+\tau]} d\,\widetilde{\sigma}^+_{D_i}(\rho)\, d\mu^{0+}_j(\mathrm{g}^s \ell) \\
&= \int_{(\rho,\ell,s)\in\Omega^-\times\Omega^+\times[s_0,s_0+\tau]} d\,\widetilde{\sigma}^+_{D_i}(\rho)\, e^{-\delta_\Gamma s} d\widetilde{\sigma}^-_{H_j}(\ell_{|\,]-\infty,0]})\, ds
\end{aligned}$$

https://doi.org/10.1017/etds.2023.116 Published online by Cambridge University Press

$$\begin{aligned}
&= \int_{(\rho,\ell)\in\Omega^-\times\Omega^+} d\,\widetilde{\sigma}^+_{D_i}(\rho)\left(\int_{s_0}^{s_0+\tau} e^{-\delta_\Gamma s} e^{\delta_\Gamma s_0}\,ds\right) d\,\widetilde{\sigma}^-_{H_{j,s_0}}((g^{s_0}\ell)_{|\,]-\infty,0]})\\
&= \int_{(\rho,\ell)\in\Omega^-\times\Omega^+} \frac{1-e^{-\delta_\Gamma\,\tau}}{\delta_\Gamma}\, d\widetilde{\sigma}^+_{D_i}(\rho)\, d\,\widetilde{\sigma}^-_{H_{j,s_0}}((\mathrm{g}^{s_0}\ell)_{|\,]-\infty,0]})\\
&= \int_{(\rho,\rho')\in\Omega^-\times \mathrm{g}^{s_0}{}_{|}N_j^+(\Omega^+)} \frac{1-e^{-\delta_\Gamma\,\tau}}{\delta_\Gamma}\, d\,\widetilde{\sigma}^+_{D_i}(\rho)\, d\,\widetilde{\sigma}^-_{H_{j,s_0}}(\rho'). \qquad (13)
\end{aligned}$$

Step 2. Let us now relate the two index sets of the left-hand sides of equations (11) and (12), except for the ranges of $\lambda_{i,\,\gamma j}$ and $\lambda_{i,\,\gamma j,\,s_0}$, which will be taken care of in Step 3.

For every $\gamma\in\Gamma$, if $\overline{D_i}\ \cap\ \overline{H_{\gamma j,s_0}}\ =\emptyset$ (so that $\alpha^-_{i,\,\gamma j,\,s_0}$ and $\alpha^+_{\gamma^{-1}i,\,j,\,s_0}$ are defined), then $\partial_\infty D_i\cap\partial_\infty H_{\gamma j}=\emptyset$ (so that $\rho_{i,\gamma j}$ and $\rho_{\gamma^{-1}i,j}$ are defined) and $\alpha^-_{i,\,\gamma j}(0)=\rho_{i,\,\gamma j}(0)$.

Conversely, since the set Ω^- is compact and by the local finiteness of the family $(H_j)_{j\in I_+}$, hence of $(H_{j,t_0})_{j\in I_+}$, there exists a finite subset F of Γ (depending on $i,\,j,\,\Omega^-,\,t_0$), such that for all $\gamma\in\Gamma-F$ and $s_0\geq t_0$, if $\partial_\infty D_i\cap\partial_\infty H_{\gamma j}=\emptyset$ (so that $\rho_{i,\gamma j}$ is defined) and if $\rho_{i,\gamma j}(0)\in\pi(\Omega^-)$, then $\overline{D_i}\ \cap\ \overline{H_{\gamma j,s_0}}\ =\emptyset$ (so that $\alpha^-_{i,\,\gamma j,\,s_0}$ is defined).

Step 3. Let us finally relate the two pairs of Dirac masses on the left-hand sides of equations (11) and (12), as well as the ranges of $\lambda_{i,\,\gamma j}$ and $\lambda_{i,\,\gamma j,\,s_0}$.

If $\gamma\in\Gamma-F$ furthermore satisfies $\lambda_{i,\,\gamma j}\geq T$ for some $T>0$ (which excludes only finitely many more $\gamma\in\Gamma$), then the generalized geodesics $\rho_{i,\gamma j}$ and $\alpha^-_{i,\,\gamma j,\,s_0}$ coincide on $]-\infty,T+s_0]$, hence on $]-\infty,T+t_0]$. Therefore, they are at distance at most ϵ for any given $\epsilon>0$ if T is large enough (uniformly in s_0 and γ) by equation (1).

Since X has extendible geodesics, for every $\gamma\in\Gamma$ such that $\partial_\infty D_i\cap\partial_\infty H_{\gamma j}=\emptyset$ (or equivalently $\partial_\infty D_{\gamma^{-1}i}\cap\partial_\infty H_j=\emptyset$), let $\widetilde{\rho}_{\gamma^{-1}i,j}\in\mathcal{G}X$ be any geodesic line such that we have $\widetilde{\rho}_{\gamma^{-1}i,j}|_{[0,+\infty[}=\rho_{\gamma^{-1}i,j}|_{[0,+\infty[}$. For t large enough, the generalized geodesics $\mathrm{g}^t\widetilde{\rho}_{\gamma^{-1}i,j}$ and $\mathrm{g}^t\rho_{\gamma^{-1}i,j}$, which coincide on $[-t,+\infty[$, are arbitrarily close (uniformly in γ) by equation (1). Hence, we may replace $\mathrm{g}^t\rho_{\gamma^{-1}i,j}$ by $\mathrm{g}^t\widetilde{\rho}_{\gamma^{-1}i,j}$ in the formula (11) that we want to prove.

Note that $\mathrm{g}^t\widetilde{\rho}_{\gamma^{-1}i,j}$ belongs to W^{0+}_j, and that $\mathrm{g}^{\lambda_{i,\gamma j}}\widetilde{\rho}_{\gamma^{-1}i,j}$ belongs to W^+_j. Since

$$\mathrm{g}^t\widetilde{\rho}_{\gamma^{-1}i,j}=\mathrm{g}^{t-\lambda_{i,\gamma j}}(\mathrm{g}^{\lambda_{i,\gamma j}}\widetilde{\rho}_{\gamma^{-1}i,j}),$$

it follows from Lemma 2.1(2) that $\mathrm{g}^t\widetilde{\rho}_{\gamma^{-1}i,j}$ is close to the subset $h_j(\Omega^+\times[s_0,s_0+\tau])$ if and only if $t-\lambda_{i,\gamma j}$ is close to $[s_0,s_0+\tau]$ and $\mathrm{g}^{\lambda_{i,\gamma j}}\widetilde{\rho}_{\gamma^{-1}i,j}$ is close to Ω^+. In particular, if $\mathrm{g}^t\widetilde{\rho}_{\gamma^{-1}i,j}$ is close to $h_j(\Omega^+\times[s_0,s_0+\tau])$ and t is large enough, then $\lambda_{i,\gamma j}$ is large enough, and $\lambda_{i,\gamma j,s_0}$ is close to $[t-\tau,t]$ (uniformly in γ).

Finally, the negative geodesic ray $\mathrm{g}^{s_0}{}_{|}N^+_j(\mathrm{g}^{\lambda_{i,\gamma j}}\widetilde{\rho}_{\gamma^{-1}i,j})$, which is close to the subset $\mathrm{g}^{s_0}{}_{|}N^+_j(\Omega^+)$ by Lemma 2.1(1) and (3), coincides with the generalized geodesic $\alpha^+_{\gamma^{-1}i,j,s_0}$ on the whole interval $]-\lambda_{i,\gamma j,s_0},+\infty[$. Since $\lambda_{i,\gamma j,s_0}$ is large (uniformly in γ) when t is large, and again by equation (1), this implies that the generalized geodesic lines $\mathrm{g}^{s_0}{}_{|}N^+_j(\mathrm{g}^{\lambda_{i,\gamma j}}\widetilde{\rho}_{\gamma^{-1}i,j})$ and $\alpha^+_{\gamma^{-1}i,j,s_0}$ are close (uniformly in γ).

To conclude the proof of the convergence in Theorem 3.1, we evaluate the two sides of formula (12) on the relatively compact Borel subset $\Omega^-\times\mathrm{g}^{s_0}{}_{|}N^+_j(\Omega^+)$ of $\check{\mathcal{G}}X\times\check{\mathcal{G}}X$, whose boundary has measure zero for the limit measure. By formula (13), this implies that formula (11) holds when evaluated on the relatively compact Borel subset

$B = \Omega^- \times h_j(\Omega^+ \times [s_0, s_0+\tau])$, whose boundary has measure zero for the limit measure. The result follows. $\square$

Let us now give a version of Theorem 3.1 in the discrete tree case. Referring to [**BPP**, §2.6] for background, let $\mathbb{X}$ be a locally finite simplicial tree without terminal vertices, with geometric realization $X = |\mathbb{X}|_1$ (with edge lengths equal to 1) and with boundary at infinity $\partial_\infty \mathbb{X} = \partial_\infty X$. We denote by $V\mathbb{X}$ the set of vertices of $\mathbb{X}$, identified with its image in X. Let Γ be a non-elementary discrete subgroup of the inversion-free automorphism group $\mathrm{Aut}(\mathbb{X})$ of $\mathbb{X}$, and let $\delta_\Gamma > 0$ be its critical exponent. We refer also to [**BPP**, §2.6] for the definition of the space of generalized discrete geodesic lines

$$\check{\mathcal{G}}\mathbb{X} = \{\ell \in \check{\mathcal{G}}X : \ell(0) \in V\mathbb{X},\ \ell_\pm \in V\mathbb{X} \cup \partial_\infty \mathbb{X}\}$$

of $\mathbb{X}$, and the definition of the discrete-time geodesic flow $(\mathrm{g}^n)_{n\in\mathbb{Z}}$ on $\check{\mathcal{G}}\mathbb{X}$, given by setting $\mathrm{g}^n\ell : t \mapsto \ell(t+n)$ for all $\ell \in \check{\mathcal{G}}\mathbb{X}$, $t \in \mathbb{R}$ and $n \in \mathbb{Z}$.

By taking the intersections with $\check{\mathcal{G}}\mathbb{X}$ of the previously defined objects for X, we define (see [**BPP**, §2.6])

- the closed subspaces $\mathcal{G}\mathbb{X}$, $\mathcal{G}_\pm\mathbb{X}$ and $\mathcal{G}_{\pm,0}\mathbb{X}$ of $\check{\mathcal{G}}\mathbb{X}$,
- the stable horoball $HB_+(w)$, the strong stable leaf $W^+(w)$, the stable leaf $W^{0+}(w)$ and the truncated stable leaf

$$W^{0+}_{n_0}(w) = \bigcup_{n\in\mathbb{Z}, n\geq n_0} \mathrm{g}^n W^+(w)$$

  of $w \in \mathcal{G}_+\mathbb{X}$, where $n_0 \in \mathbb{Z}$, and
- the outer/inner unit normal bundles $\partial^1_\pm \mathbb{D}$ of a non-empty proper simplicial subtree $\mathbb{D}$ of $\mathbb{X}$.

We define similarly (see [**BPP**, §2.6]) the outer/inner skinning measure $\tilde{\sigma}^\pm_{\mathbb{D}}$ on $\partial^1_\pm\mathbb{D}$ and the Bowen–Margulis measures $\tilde{m}_{\mathrm{BM}}$ on $\mathcal{G}\mathbb{X}$ and m_{BM} on $\Gamma\backslash\mathcal{G}\mathbb{X}$ associated with any choice of Patterson–Sullivan density $(\mu_x)_{x\in V\mathbb{X}}$.

Given $w \in \mathcal{G}_+\mathbb{X}$, its stable horoball $HB_+(w)$ is a subtree of $\mathbb{X}$ and we again denote by $N^+_w : W^+(w) \to \partial^1_- HB_+(w)$ the canonical homeomorphism defined in §2. We now have a homeomorphism $h_w : W^+(w) \times \mathbb{Z} \to W^{0+}(w)$ defined by $(\ell, m) \mapsto \mathrm{g}^m\ell$. The conditional measure $\mu_{W^{0+}(w)}$ of the Bowen–Margulis measure $\tilde{m}_{\mathrm{BM}}$ (for the discrete-time geodesic flow on $\check{\mathcal{G}}\mathbb{X}$) on the stable leaf $W^{0+}(w)$ of w is now defined, for $m \in \mathbb{Z}$ and $\ell \in W^+(w)$, by

$$d\mu_{W^{0+}(w)}(\mathrm{g}^m\ell) = e^{-\delta_\Gamma m}\, d\mu_{W^+(w)}(\ell)\, dm, \tag{14}$$

where dm is the counting measure on $\mathbb{Z}$.

Let $\mathcal{D}^- = (\mathbb{D}^-_i)_{i\in I^-}$ and $\mathcal{D}^+ = (\mathbb{H}^+_j)_{j\in I^+}$ be locally finite Γ-equivariant families of non-empty proper simplicial subtrees of $\mathbb{X}$, with $\mathbb{H}^+_j$ a horoball for every $j \in I^+$. We consider the geometric realizations $D_i = |\mathbb{D}_i|_1$ of $\mathbb{D}_i$ and $H_j = |\mathbb{H}_j|_1$ of $\mathbb{H}_j$. For every $n_0 \in \mathbb{Z}$, we define the horoball H_{j,n_0} such that H_j is the n_0-neighbourhood of H_{j,n_0} if $n_0 \geq 0$ and H_{j,n_0} is the $(-n_0)$-neighbourhood of H_j if $n_0 \leq 0$. For every $n_0 \in \mathbb{Z}$, as at the end of §2, we define the measures $\tilde{\sigma}^+_{\mathcal{D}^-}$ and $\tilde{\mu}^{0+}_{\mathcal{D}^+,n_0}$ on $\check{\mathcal{G}}\mathbb{X}$, and their induced measures $\sigma^+_{\mathcal{D}^-}$ and $\mu^{0+}_{\mathcal{D}^+,n_0}$ on $\Gamma\backslash\check{\mathcal{G}}\mathbb{X}$.

https://doi.org/10.1017/etds.2023.116 Published online by Cambridge University Press

For all $m \in \mathbb{Z}$ and $(i, j) \in I^- \times I^+$, the elements $\rho_{i,j}$ and $\alpha^{\pm}_{i,j,m}$, respectively defined just before and just after the statement of Theorem 3.1, actually belong to $\check{\mathcal{G}}\mathbb{X}$.

Note that for many interesting lattices in $\mathrm{Aut}(\mathbb{X})$ (and this will turn out to be the case for the application in §4.5), the time-one geodesic flow is not mixing (it is not even ergodic), though the time-two geodesic flow is mixing on a half-subspace; see [**BPP**, end of §4.4] for explanations. This explains the usefulness of assertion (2) in the next statement.

Fix a basepoint $x^\bullet \in V\mathbb{X}$. Let $V_{\mathrm{even}}\mathbb{X}$ be the subset of $V\mathbb{X}$ of vertices of X at even distance from $x^\bullet$. Let

$$\check{\mathcal{G}}_{\mathrm{even}}\mathbb{X} = \{\ell \in \check{\mathcal{G}}\mathbb{X} : \ell(0) \in V_{\mathrm{even}}\mathbb{X}\} \quad \text{and} \quad \mathcal{G}_{\mathrm{even}}\mathbb{X} = \check{\mathcal{G}}_{\mathrm{even}}\mathbb{X} \cap \mathcal{G}\mathbb{X}.$$

THEOREM 3.2. *Let $\mathbb{X}$ be a locally finite simplicial tree without terminal vertices. Let Γ be a non-elementary discrete subgroup of* $\mathrm{Aut}(\mathbb{X})$. *Let $\mathcal{D}^- = (\mathbb{D}_i^-)_{i \in I^-}$ and $\mathcal{D}^+ = (\mathbb{H}_j^+)_{j \in I^+}$ be locally finite Γ-equivariant families of non-empty proper simplicial subtrees of X, with $\mathbb{H}_j^+$ a horoball for every $j \in I^+$.*

(1) *Assume that the Bowen–Margulis measure m_{BM} on $\Gamma\backslash\mathcal{G}\mathbb{X}$ endowed with the discrete-time geodesic flow is finite and mixing. Then for every $n_0 \in \mathbb{Z}$, for the weak-star convergence of measures on $\mathcal{G}_{+,0}\mathbb{X} \times \check{\mathcal{G}}\mathbb{X}$, we have*

$$\lim_{n\to+\infty} \|m_{\mathrm{BM}}\|\, e^{-\delta_\Gamma\, n} \sum_{\substack{i\in I^-/_{\sim_-},\ j\in I^+/_{\sim_+},\ \gamma\in\Gamma \\ \partial_\infty\mathbb{D}_i \,\cap\, \partial_\infty\mathbb{H}_{\gamma j}=\emptyset,\ \lambda_{i,\gamma j}\le n-n_0}} \Delta_{\rho_{i,\gamma j}} \otimes \Delta_{\mathtt{g}^n \rho_{\gamma^{-1} i, j}} = \widetilde{\sigma}^+_{\mathcal{D}^-} \otimes \widetilde{\mu}^{0+}_{\mathcal{D}^+, n_0}.$$

(2) *Assume that Γ preserves $V_{\mathrm{even}}\mathbb{X}$. Assume that the restriction to $\Gamma\backslash\mathcal{G}_{\mathrm{even}}\mathbb{X}$ of the Bowen–Margulis measure m_{BM} is finite and mixing for the time-two map of the discrete-time geodesic flow. Assume that the endpoints of every common perpendicular between disjoint elements of $\mathcal{D}^-$ and $\mathcal{D}^+$ belong to $V_{\mathrm{even}}\mathbb{X}$. Then for every $n_0 \in \mathbb{Z}$, for the weak-star convergence of measures on $\mathcal{G}_{+,0}\mathbb{X} \times \check{\mathcal{G}}\mathbb{X}$, we have*

$$\lim_{n\to+\infty} \frac{\|m_{\mathrm{BM}}\|}{2}\, e^{-2\,\delta_\Gamma\, n} \sum_{\substack{i\in I^-/_{\sim_-},\ j\in I^+/_{\sim_+},\ \gamma\in\Gamma \\ \partial_\infty\mathbb{D}_i \,\cap\, \partial_\infty\mathbb{H}_{\gamma j}=\emptyset,\ \lambda_{i,\gamma j}\le 2n-2n_0}} \Delta_{\rho_{i,\gamma j}} \otimes \Delta_{\mathtt{g}^{2n} \rho_{\gamma^{-1} i, j}}$$
$$= \widetilde{\sigma}^+_{\mathcal{D}^-}\,{}_{|\check{\mathcal{G}}_{\mathrm{even}}\mathbb{X}} \otimes \widetilde{\mu}^{0+}_{\mathcal{D}^+, 2n_0}\,{}_{|\check{\mathcal{G}}_{\mathrm{even}}\mathbb{X}}.$$

Proof. (1) Let us fix $i \in I^-$ and $j \in I^+$. It follows from (the case with zero potential of) [**BPP**, Theorem 11.9] in the same way as [**BPP**, Theorem 11.3] follows from [**BPP**, Theorem 11.1] that for every integer $m_0 \ge n_0$, we have

$$\lim_{n\to+\infty} \|m_{\mathrm{BM}}\|\, e^{-\delta_\Gamma\, n} \sum_{\substack{\gamma\in\Gamma \\ \mathbb{D}_i^- \cap \mathbb{H}^+_{\gamma j, m_0}=\emptyset,\ \lambda_{i,\gamma j, m_0}=n}} \Delta_{\alpha^-_{i,\gamma j}} \otimes \Delta_{\alpha^+_{\gamma^{-1} i, j, m_0}} = \widetilde{\sigma}^+_{D_i} \otimes \widetilde{\sigma}^-_{H_{j, m_0}}$$

for the weak-star convergence of measures on the locally compact space $\check{\mathcal{G}}\mathbb{X} \times \check{\mathcal{G}}\mathbb{X}$. The proof of Theorem 3.2(1) is then similar to that of Theorem 3.1 using this equation instead of equation (12).

(2) Let us fix $i \in I^-$ and $j \in I^+$. It follows from (the case with zero potential of) now [**BPP**, Theorem 11.11] (and more precisely of equation (11.28) in its proof with $t = 2n$) in

the same way as [**BPP**, Theorem 11.3] follows from [**BPP**, Theorem 11.1] that for every integer $m_0 \geq n_0$, we have

$$\lim_{n\to+\infty} \frac{\|m_{\mathrm{BM}}\|}{2}\, e^{-2\,\delta_\Gamma\, n} \sum_{\substack{\gamma\in\Gamma \\ \mathbb{D}_i^- \cap \mathbb{H}^+_{\gamma j, 2m_0} = \emptyset \\ \lambda_{i,\gamma j, 2m_0} = 2n}} \Delta_{\alpha^-_{i,\gamma j}} \otimes \Delta_{\alpha^+_{\gamma^{-1} i, j, 2m_0}} = \widetilde{\sigma}^+_{D_i}\,{}_{|\check{\mathcal{G}}_{\mathrm{even}}\mathbb{X}} \otimes \widetilde{\sigma}^-_{H_{j,2m_0}}\,{}_{|\check{\mathcal{G}}_{\mathrm{even}}\mathbb{X}}$$

for the weak-star convergence of measures on the locally compact space $\check{\mathcal{G}}_{\mathrm{even}}\mathbb{X} \times \check{\mathcal{G}}_{\mathrm{even}}\mathbb{X}$. The proof of Theorem 3.2(2) is then similar to that of Theorem 3.1 using this equation instead of equation (12). □

In order to conclude §3, let us give equidistribution statements in the quotient by Γ of the two previous results. In order to simplify them, we assume that D is a proper non-empty closed convex subset of X and that H is a (closed) horoball of X such that the Γ-equivariant families $\mathcal{D}^- = (\gamma D)_{\gamma\in\Gamma}$ and $\mathcal{D}^+ = (\gamma H)_{\gamma\in\Gamma}$ are locally finite. In the simplicial tree case as above, we assume that D and H are the geometric realizations of simplicial subtrees $\mathbb{D}$ and $\mathbb{H}$ of $\mathbb{X}$.

We denote by Γ_D and Γ_H the stabilizers of D and H in Γ, respectively. For every $\gamma \in \Gamma$ such that the point at infinity of γH does not belong to $\partial_\infty D$, we define the *multiplicity* of the common perpendicular from D to γH by

$$m_\gamma = \frac{1}{\mathrm{Card}(\Gamma_D \cap (\gamma \Gamma_H \gamma^{-1}))}$$

and we denote by ρ_γ the unique geodesic ray in $\partial^1_+ D$ converging to the point at infinity of γH. Note that for all $\alpha \in \Gamma_D$ and $\beta \in \Gamma_H$, we have

$$m_\gamma = m_{\alpha\gamma\beta} \quad \text{and} \quad \alpha\rho_\gamma = \rho_{\alpha\gamma\beta}.$$

THEOREM 3.3

(1) *For every $t_0 \in \mathbb{R}$, if (X, Γ) satisfies the assumptions of Theorem 3.1 for $\mathcal{D}^\pm$ as above, and if the measures $\sigma^+_{\mathcal{D}^-}$ and $\mu^{0+}_{\mathcal{D}^+,t_0}$ on $\Gamma\backslash\check{\mathcal{G}}X$ are finite and non-zero, then for the weak-star convergence of measures on $(\Gamma\backslash\mathcal{G}_{+,0}X) \times (\Gamma\backslash\check{\mathcal{G}}X)$ we have*

$$\lim_{t\to+\infty} \|m_{\mathrm{BM}}\|\, e^{-\delta_\Gamma\, t} \sum_{\substack{\gamma\in\Gamma_D\backslash\Gamma/\Gamma_H \\ 0<d(D,\gamma H)\leq t-t_0}} m_\gamma\, \Delta_{\Gamma\rho_\gamma} \otimes \Delta_{\mathtt{g}^t\Gamma\rho_\gamma} = \sigma^+_{\mathcal{D}^-} \otimes \mu^{0+}_{\mathcal{D}^+,t_0}. \tag{15}$$

(2) *For every $n_0 \in \mathbb{Z}$, if $(\mathbb{X}, \Gamma)$ satisfies the assumptions of Theorem 3.2(1) for $\mathcal{D}^\pm$ as above, and if the measures $\sigma^+_{\mathcal{D}^-}$ and $\mu^{0+}_{\mathcal{D}^+,n_0}$ on $\Gamma\backslash\check{\mathcal{G}}\mathbb{X}$ are finite and non-zero, then for the weak-star convergence of measures on $(\Gamma\backslash\mathcal{G}_{+,0}\mathbb{X}) \times (\Gamma\backslash\check{\mathcal{G}}\mathbb{X})$ we have*

$$\lim_{n\to+\infty} \|m_{\mathrm{BM}}\|\, e^{-\delta_\Gamma\, n} \sum_{\substack{\gamma\in\Gamma_D\backslash\Gamma/\Gamma_H \\ \partial_\infty D\,\cap\,\gamma\partial_\infty H=\emptyset,\ d(D,\gamma H)\leq n-n_0}} m_\gamma\, \Delta_{\Gamma\rho_\gamma} \otimes \Delta_{\mathtt{g}^n\Gamma\rho_\gamma} = \sigma^+_{\mathcal{D}^-} \otimes \mu^{0+}_{\mathcal{D}^+,n_0}.$$

https://doi.org/10.1017/etds.2023.116 Published online by Cambridge University Press

(3) *For every $n_0 \in \mathbb{Z}$, if $(\mathbb{X}, \Gamma)$ satisfies the assumptions of Theorem 3.2(2) for $\mathcal{D}^{\pm}$ as above, and if the measures $\sigma^+_{\mathcal{D}^-}$ and $\mu^{0+}_{\mathcal{D}^+, 2n_0}$ on $\Gamma\backslash\check{\mathcal{G}}\mathbb{X}$ are finite and non-zero, then for the weak-star convergence of measures on $(\Gamma\backslash\mathcal{G}_{+,0}\mathbb{X}) \times (\Gamma\backslash\check{\mathcal{G}}\mathbb{X})$ we have*

$$\lim_{n\to+\infty} \frac{\|m_{\mathrm{BM}}\|}{2}\, e^{-2\,\delta_\Gamma\, n} \sum_{\substack{\gamma\in\Gamma_D\backslash\Gamma/\Gamma_H \\ \partial_\infty D\,\cap\,\gamma\partial_\infty H=\emptyset,\ d(D,\gamma H)\le 2n-2n_0}} m_\gamma\ \Delta_{\Gamma\rho_\gamma} \otimes \Delta_{\mathtt{g}^{2n}\Gamma\rho_\gamma}$$
$$= \sigma^+_{\mathcal{D}^-}{}_{\mid \Gamma\backslash\check{\mathcal{G}}_{\mathrm{even}}\mathbb{X}} \otimes \mu^{0+}_{\mathcal{D}^+,2n_0}{}_{\mid \Gamma\backslash\check{\mathcal{G}}_{\mathrm{even}}\mathbb{X}}. \tag{16}$$

Proof. The first assertion follows from Theorem 3.1 in the same way as Corollary 12.3 in the manifold case and Theorem 12.8 in the tree case of [**BPP**] follows from Theorem 11.1 of [**BPP**]. The second and third assertions follow respectively from Theorem 3.2(1) and (2) in the same way as Theorems 12.9 and 12.12 of [**BPP**] follow from Theorems 11.9 and 11.11 of [**BPP**]. □

Remark. Assume first in this remark that X is a (negatively curved) symmetric space, that Γ is an arithmetic lattice and that D has smooth boundary. Note that the Bowen–Margulis measure is then the Liouville measure, and in particular is a smooth measure. For all $\ell \in \mathbb{N}$ and $f \in \mathcal{C}^\ell_c(\Gamma\backslash T^1X)$, we denote by $\|f\|_\ell$ the ℓth Sobolev norm of f. We identify $\mathcal{G}_{+,0}X$ and $\check{\mathcal{G}}X$ with T^1X by uniquely extending geodesic rays and segments to geodesic lines. Then one could prove, as in [**PaP5**, Theorem 15(2)] (see also [**BPP**, Theorem 12.7(2)]), by replacing the above equation (12) by the difference of the evaluations at $T = t$ and $T = t - \tau$ of equation (28) of [**PaP5**], that there exists $\tau' > 0$ such that we have an error term of the form $\mathrm{O}_{t_0}(e^{-\kappa' t}\|\Psi^-\|_\ell\|\Psi^+\|_\ell)$ when evaluating (before taking the limit on the left-hand side) the two sides of equation (15) on a pair of functions $\Psi^\pm \in \mathcal{C}^\ell_c(\Gamma\backslash T^1X)$.

Assume now, with the notation of §4.5, that X is the geometric realization of the Bruhat–Tits tree $\mathbb{X}_v$ of a (PGL_2, K_v) and $\Gamma = \mathrm{PGL}_2(R_v)$ is the Nagao lattice. One could prove a similar error term in equation (16) replacing a Sobolev regularity by a locally constant regularity, as in remark (ii) in [**BPP**, p. 282] using [**BPP**, Proposition 15.7(2)] in order to check the main assumption of that remark.

4. *Applications to equidistribution of Farey fractions*

In this section we give five examples of applications of the results of §3, by taking arithmetic families of points (of Farey fractions type) with a given average density in an expanding closed horosphere, and we study their equidistribution properties. As their proofs, though having similar schemes, make reference to many different papers, and require numerous different computations and checks, it has not been possible, if only for the sake of the readability of this paper, to regroup them into one statement. More corollaries of Theorem 3.3(1) may be obtained by varying a non-uniform arithmetic lattice Γ in the isometry group of a negatively curved symmetric space X. In §§4.1 and 4.2, we denote by $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ the image in $\mathrm{PSL}_2(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C})/\{\pm\,\mathrm{id}\}$ of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{C})$.

4.1. *Standard Farey fractions and Marklof's theorem.*

Let us now check that as a corollary of Theorem 3.3(1), we obtain a new and geometric proof of the case $n = 2$ of

[**Mar2**, Theorem 6]. We give extra details in the proof of Corollary 4.1, as it will serve as a model for the next four examples.

Let $G = \mathrm{PSL}_2(\mathbb{R})$ and let Γ be the modular group $\mathrm{PSL}_2(\mathbb{Z})$. For all $r, t \in \mathbb{R}$, let

$$\mathfrak{n}_-(r) = \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \Phi^t = \begin{bmatrix} e^{-t/2} & 0 \\ 0 & e^{t/2} \end{bmatrix}.$$

Let

$$H = \{\mathfrak{n}_-(r) : r \in \mathbb{R}\},$$

and let

$$\Gamma_H = H \cap \Gamma = \{\mathfrak{n}_-(r) : r \in \mathbb{Z}\}.$$

We see $\Gamma_H \backslash H$ as contained in $\Gamma \backslash G$, and we endow $\Gamma_H \backslash H$ with its H-invariant probability measure $\mu_{\Gamma_H \backslash H}$. We endow $\mathbb{R}/\mathbb{Z}$ with its probability Haar measure dx, so that the map $r \mapsto \mathfrak{n}_-(r)$ induces a measure- preserving homeomorphism $\mathbb{R}/\mathbb{Z} \to \Gamma_H \backslash H$.

For every $t \in \mathbb{R}$, we consider the subset $\mathcal{F}_t$ of $\mathbb{R}/\mathbb{Z}$ consisting of the *(standard) Farey fractions of height at most $e^{t/2}$*, defined by

$$\mathcal{F}_t = \left\{ \frac{p}{q} \bmod 1 : \ p, q \in \mathbb{Z}, (p, q) = 1, 0 < q \leq e^{t/2} \right\}.$$

Note that in the definition of both Φ^t and $\mathcal{F}_t$, Marklof replaces t by $2t$, but our convention is more natural, considering the left-hand part of equation (20) below.

Let $\Theta : \Gamma \backslash G \to \Gamma \backslash G$ be the Cartan involutive homeomorphism $\Gamma g \mapsto \Gamma\, {}^t g^{-1}$, so that for every continuous function with compact support $f : \mathbb{R}/\mathbb{Z} \times \Gamma \backslash G \to \mathbb{R}$ and for every $s \in \mathbb{R}$, we have

$$\int f \, dx \otimes d\, (\Theta_* \, (\Phi^{-s})_* \, \mu_{\Gamma_H \backslash H}) = \int_{(x,y) \in (\mathbb{R}/\mathbb{Z}) \times (\Gamma_H \backslash H)} f(x, \Theta(y\Phi^{-s})) \, dx \, d\mu_{\Gamma_H \backslash H}(y).$$

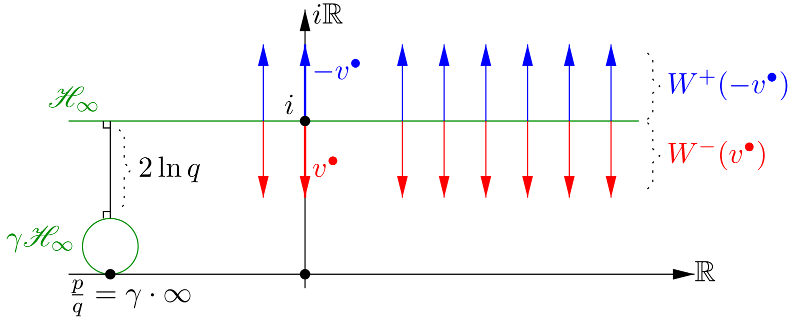
COROLLARY 4.1. (Marklof [**Mar2**, Theorem 6]) *For every $t_0 \in \mathbb{R}$, for the weak-star convergence of measures on $\mathbb{R}/\mathbb{Z} \times \Gamma \backslash G$, we have*

$$\lim_{t \to +\infty} \frac{1}{\mathrm{Card}\, \mathcal{F}_{t-t_0}} \sum_{r \in \mathcal{F}_{t-t_0}} \Delta_r \otimes \Delta_{\Gamma \mathfrak{n}_-(r) \Phi^t} = e^{t_0} \int_{s=t_0}^{+\infty} dx \otimes d\, (\Theta_* \, (\Phi^{-s})_* \, \mu_{\Gamma_H \backslash H}) \, e^{-s} \, ds. \tag{17}$$

Proof. We consider in this proof $X = \mathbb{H}^2_{\mathbb{R}}$, where $\mathbb{H}^n_{\mathbb{R}}$ is the upper half-space model of the real hyperbolic space of dimension n (with constant sectional curvature -1). We again denote by $\iota : T^1\mathbb{H}^n_{\mathbb{R}} \to T^1\mathbb{H}^n_{\mathbb{R}}$ the antipodal map $v \mapsto -v$. We normalize, as we may, the Patterson density $(\mu_x)_{x \in X}$ of the (non-uniform arithmetic) lattice Γ of the orientation-preserving isometry group G of X to consist of probability measures. The critical exponent of Γ is

$$\delta_\Gamma = 1. \tag{18}$$

We start the proof by recalling precisely a bijection between G and the unit tangent bundle of $\mathbb{H}^2_{\mathbb{R}}$. We denote by $\cdot$ the action of G by homographies on $\mathbb{H}^2_{\mathbb{R}} \cup \partial_\infty \mathbb{H}^2_{\mathbb{R}}$, as well as its derived action on $T^1\mathbb{H}^2_\mathbb{R}$. We fix $v^\bullet = (i, -i) \in T^1\mathbb{H}^2_\mathbb{R}$, which is the unit tangent vector at the base point i of $\mathbb{H}^2_\mathbb{R}$ pointing vertically down (its length is not adequate in the picture below, but this makes the picture easier to understand).



We denote by $\widetilde{\varphi} : G \to T^1\mathbb{H}^2_\mathbb{R}$ the orbital map at $v^\bullet$, defined by $g \mapsto g \cdot v^\bullet$, which is a G-equivariant (for the left actions) homeomorphism, and by $\varphi : \Gamma\backslash G \to \Gamma\backslash T^1\mathbb{H}^2_\mathbb{R}$ its quotient homeomorphism. We define $S = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$, which is an order-two element of Γ. The involution S satisfies the following remarkable properties, in the connected center-free semisimple real Lie group G, that it anticommutes with the standard Cartan subgroup $\Phi^\mathbb{R} = \{\Phi^t : t \in \mathbb{R}\}$ of G and that the conjugation by S is the standard Cartan involution $g \mapsto {}^t g^{-1}$ of G:

$$\text{for all } g \in G, \text{ we have } \quad {}^t g^{-1} = SgS^{-1} \quad \text{and} \quad \text{for all } s \in \mathbb{R}, \text{ we have } \quad S\Phi^s S^{-1} = \Phi^{-s}. \tag{19}$$

Hence, with Θ defined just before the statement of Corollary 4.1, for all $x \in \Gamma\backslash G$ and $s \in \mathbb{R}$, we have

$$\Theta(x\Phi^s) = \Theta(x)\Phi^{-s}.$$

The element S represents a generator of the standard Weyl group $N_G(\Phi^\mathbb{R})/Z_G(\Phi^\mathbb{R})$ (whose order is 2). The following properties say that the action of the geodesic flow g^t on $T^1\mathbb{H}^2_\mathbb{R}$ corresponds to multiplication on the right by Φ^t in G, and that the antipodal map on $T^1\mathbb{H}^2_\mathbb{R}$ corresponds to multiplication on the right by S in G:

$$\text{for all } t \in \mathbb{R} \text{ and } g \in G, \text{ we have } \quad \mathrm{g}^t\widetilde{\varphi}(g) = \widetilde{\varphi}(g\Phi^t) \quad \text{and} \quad \iota\,\widetilde{\varphi}(g) = \widetilde{\varphi}(gS). \tag{20}$$

By the above two centred formulas and since $S \in \Gamma$, the homeomorphism φ relates the antipodal map ι on $\Gamma\backslash T^1\mathbb{H}^2_\mathbb{R}$ to the Cartan involution Θ on $\Gamma\backslash G$ by

$$\iota \circ \varphi = \varphi \circ \Theta.$$

Let $\mathcal{H}_\infty = \{z \in \mathbb{H}^2_\mathbb{R} : \operatorname{Im} z \geq 1\}$, which is a (closed) horoball centred at ∞ in $\mathbb{H}^2_\mathbb{R}$. The subgroup Γ_H is equal to the stabilizer $\Gamma_{\mathcal{H}_\infty}$ of $\mathcal{H}_\infty$ in Γ. We define

$$\mathcal{D}^- = \mathcal{D}^+ = (\gamma \cdot \mathcal{H}_\infty)_{\gamma\in\Gamma}, \tag{21}$$

which are locally finite Γ-equivariant families of horoballs. The map from $\Gamma - \Gamma_{\mathcal{H}_\infty}$ to $\mathbb{R}$ defined by $\gamma = \left[\begin{smallmatrix} p & r \\ q & s \end{smallmatrix}\right] \mapsto \gamma \cdot \infty = p/q$ (where we assume, as we may, that $q > 0$) induces

https://doi.org/10.1017/etds.2023.116 Published online by Cambridge University Press

a bijection from $\Gamma_{\mathcal{H}_\infty}\backslash(\Gamma - \Gamma_{\mathcal{H}_\infty})/\Gamma_{\mathcal{H}_\infty}$ to the additive group $\mathbb{Q}/\mathbb{Z}$ such that we have $d(\mathcal{H}_\infty, \gamma\cdot\mathcal{H}_\infty) = 2\ln q$ (see the above picture). In particular, for all $t, t_0 \in \mathbb{R}$, we have

$$d(\mathcal{H}_\infty, \gamma\cdot\mathcal{H}_\infty) \le t - t_0 \quad \text{if and only if} \quad q \le e^{(t-t_0)/2}. \tag{22}$$

Identifying geodesic rays in $\mathcal{G}_{+,0}X$ and geodesic lines in $\mathcal{G}X$ with their unit tangent vector at time 0, we have

$$\partial^1_+\mathcal{H}_\infty = W^-(v^\bullet) = \widetilde{\varphi}(H),$$

so that, by the left equivariance of $\widetilde{\varphi}$, the orbits of the right action of H on G correspond to the strong unstable leaves for the geodesic flow on $T^1\mathbb{H}^2_{\mathbb{R}}$. Similarly, using equation (20), we have

$$\partial^1_-\mathcal{H}_\infty = W^+(-v^\bullet) = \iota W^-(v^\bullet) = \widetilde{\varphi}(H\,S) \quad \text{and} \quad W^{0+}(-v^\bullet) = \widetilde{\varphi}(H\Phi^{\mathbb{R}}\,S).$$

More precisely, using the right-hand parts of equations (19) and (20),

$$\text{for all } s, r \in \mathbb{R}, \text{ we have} \quad \widetilde{\varphi}(\mathfrak{n}_-(r)\,\Phi^{-s}\,S) = \widetilde{\varphi}(\mathfrak{n}_-(r)\,S\,\Phi^{s}) = \mathrm{g}^s\,\iota\,\widetilde{\varphi}(\mathfrak{n}_-(r)). \tag{23}$$

The endpoint map $\widetilde{\psi} : \partial^1_+\mathcal{H}_\infty \to \mathbb{R}$ defined by $\rho \mapsto \rho_+$ is a Γ_H-equivariant homeomorphism, such that we have $\widetilde{\varphi}^{-1}(\widetilde{\psi}^{-1}(r)) = \mathfrak{n}_-(r)$ for all $r \in \mathbb{R}$. We denote by $\psi : \Gamma_H\backslash\partial^1_+\mathcal{H}_\infty \to \mathbb{R}/\mathbb{Z}$ the quotient homeomorphism, and we identify $\Gamma_H\backslash\partial^1_+\mathcal{H}_\infty$ with its image in $\Gamma\backslash T^1\mathbb{H}^2_{\mathbb{R}}$. For every $\gamma = \left[\begin{smallmatrix} p & r \\ q & s\end{smallmatrix}\right] \in \Gamma - \Gamma_H$, with $\rho_\gamma \in \partial^1_+\mathcal{H}_\infty$ the geodesic ray entering perpendicularly in $\gamma\cdot\mathcal{H}_\infty$, we have

$$\widetilde{\varphi}^{-1}(\rho_\gamma) = \mathfrak{n}_-(\gamma\cdot\infty) \quad \text{and} \quad \psi_*(\Delta_{\Gamma\rho_\gamma}) = \Delta_{\gamma\cdot\infty \bmod 1} = \Delta_{p/q \bmod 1}. \tag{24}$$

Furthermore, by [**PaP4**, Theorem 9.11] or [**PaP5**, Proposition 20(2)] with $n = 2$, the skinning measure $\widetilde{\sigma}^{\pm}_{\mathcal{H}_\infty} = \iota_*\widetilde{\sigma}^{\mp}_{\mathcal{H}_\infty}$ is equal to twice the Riemannian volume of $\partial^1_{\mp}\mathcal{H}_\infty$, so that

$$\psi_*(\sigma^+_{\mathcal{D}^-}) = 2\,dx \quad \text{and} \quad (\varphi^{-1})_*(\sigma^+_{\mathcal{D}^-}) = 2\,\mu_{\Gamma_H\backslash H}. \tag{25}$$

By for instance [**PaP4**, Theorem 9.10] or [**PaP5**, Proposition 20(1)] with $n = 2$, we have

$$\|m_{\mathrm{BM}}\| = 4\pi\ \mathrm{vol}(\Gamma\backslash\mathbb{H}^2_{\mathbb{R}}) = \frac{4\pi^2}{3}.$$

Mertens's formula [**HaW**, Theorem 330] (see also [PaP3, §3] for a geometric proof) implies that, as $t \to +\infty$,

$$\mathrm{Card}\,\mathcal{F}_{t-t_0} \sim \frac{3}{\pi^2}e^{2(t-t_0)/2} = \frac{3}{\pi^2}e^{t-t_0}.$$

Since no element of Γ pointwise fixes a non-trivial geodesic segment of $\mathbb{H}^2_{\mathbb{R}}$, for every $\gamma \in \Gamma$ such that $d(\mathcal{H}_\infty, \gamma\cdot\mathcal{H}_\infty) > 0$, we have

$$m_\gamma = 1.$$

For every $t_0 \in \mathbb{R}$, let us consider the truncation $\Phi^{\ge t_0} = \{\Phi^t : t \ge t_0\}$ of the Cartan subgroup $\Phi^{\mathbb{R}}$. For all $t \in \mathbb{R}$ and $\gamma \in \Gamma - \Gamma_H$, by the two left-hand parts of equations (20) and (24), we have

$$(\varphi^{-1})_*(\Delta_{\Gamma\mathrm{g}^t\rho_\gamma}) = \Delta_{\Gamma n_-(\gamma\cdot\infty)\Phi^t}. \tag{26}$$

By equation (23), the homeomorphism φ^{-1} maps the truncated stable leaf

$$\Gamma W^{0+}_{t_0}(-v^\bullet) = \bigcup_{s \geq t_0} \Gamma \mathrm{g}^s \partial^1_- \mathcal{H}_\infty = \bigcup_{s \geq t_0} \Gamma \mathrm{g}^s W^+(-v^\bullet) = \bigcup_{s \geq t_0} \Gamma \mathrm{g}^s \iota W^-(v^\bullet)$$

to the truncated orbit $\Gamma H(\Phi^{\geq t_0})^{-1} S$ in $\Gamma \backslash G$ of the lower triangular subgroup of G. Furthermore, by the left-hand part of equation (19) and since $S \in \Gamma$ for the first equality, by equation (23) for the third equality, by equations (7) and (18) for the fourth equality, and since $\iota_* \sigma^-_{\mathcal{D}^+} = \sigma^+_{\mathcal{D}^-}$ and by the right-hand part of equation (25) for the last equality, for all $s, r \in \mathbb{R}$ with $s \geq t_0$, we have

$$\begin{aligned} d((\varphi^{-1})_*(\mu^{0+}_{\mathcal{D}^+, t_0}))(\Theta(\Gamma n_-(r)\Phi^{-s})) &= d((\varphi^{-1})_*(\mu^{0+}_{\mathcal{D}^+, t_0}))(\Gamma n_-(r)\Phi^{-s} S) \\ &= d\mu^{0+}_{\mathcal{D}^+, t_0}(\Gamma \widetilde{\varphi}(n_-(r)\Phi^{-s} S)) = d\mu^{0+}_{\mathcal{D}^+}(\Gamma \mathrm{g}^s \iota\, \widetilde{\varphi}(n_-(r))) = e^{-s} d\sigma^-_{\mathcal{D}^+}(\Gamma \iota\, \widetilde{\varphi}(n_-(r)))\, ds \\ &= e^{-s} (\varphi^{-1})_*\, \iota_*\, d\sigma^-_{\mathcal{D}^+}(\Gamma n_-(r))\, ds = 2\, d\mu_{\Gamma_H \backslash H}(\Gamma n_-(r))\, e^{-s}\, ds. \end{aligned}$$

Therefore, by the left-hand part of equation (25), for all $x \in \mathbb{R}/\mathbb{Z}$, $y \in \Gamma_H \backslash H$ and $s \geq t_0$, we have

$$d((\psi \times \varphi^{-1})_*(\sigma^+_{\mathcal{D}^-} \otimes \mu^{0+}_{\mathcal{D}^+, t_0}))(x, \Theta(y\, \Phi^{-s})) = 4\, dx\, d\mu_{\Gamma_H \backslash H}(y)\, e^{-s}\, ds. \tag{27}$$

By the linearity of the pushforward of measures and by equation (22), the left-hand part of equation (24), and equation (26), as $t \to +\infty$, we have

$$\begin{aligned} &(\psi \times \varphi^{-1})_* \Bigg(\|m_{\mathrm{BM}}\|\, e^{-\delta_\Gamma t} \sum_{\substack{\gamma \in \Gamma_{\mathcal{H}_\infty} \backslash \Gamma / \Gamma_{\mathcal{H}_\infty} \\ 0 < d(\mathcal{H}_\infty, \gamma \cdot \mathcal{H}_\infty) \leq t - t_0}} m_\gamma\, \Delta_{\Gamma \rho_\gamma} \otimes \Delta_{\mathrm{g}^t \Gamma \rho_\gamma} \Bigg) \\ &\quad = \frac{4\pi^2}{3}\, e^{-t} \sum_{r \in \mathcal{F}_{t - t_0}} \Delta_r \otimes \Delta_{\Gamma \mathrm{n}_-(r)\Phi^t} \\ &\quad \sim 4\, e^{-t_0}\, \frac{1}{\operatorname{Card} \mathcal{F}_{t - t_0}} \sum_{r \in \mathcal{F}_{t - t_0}} \Delta_r \otimes \Delta_{\Gamma \mathrm{n}_-(r)\Phi^t}. \end{aligned} \tag{28}$$

Since the product map $\psi \times \varphi^{-1}$ is a homeomorphism from $(\Gamma W^-(v^\bullet)) \times (\Gamma W^{0+}_{t_0}(-v^\bullet))$ to $(\mathbb{R}/\mathbb{Z}) \times (\Gamma H(\Phi^{\geq t_0})^{-1})$, its pushforward map on measures is continuous for the weak-star convergence. Hence, Corollary 4.1 follows from equations (27) and (28) by Theorem 3.3(1) applied to the families $\mathcal{D}^\pm$ defined in equation (21). □

Remarks

(1) Using the final Remark of §3 and an approximation by linear combinations of functions with separate variables, one could prove that there exist $\tau' > 0$ and $\ell \in \mathbb{N}$ such that for every $\Psi \in \mathcal{C}^\ell_c(\mathbb{R}/\mathbb{Z} \times \Gamma \backslash G)$, we have an error term of the form $\mathrm{O}_{t_0}(e^{-\kappa' t} \|\Psi\|_\ell)$ when evaluating (before taking the limit on the left-hand side) the two sides of equation (17) on the function Ψ. See also [**Mar3**] when $n = 2$ and [**Li**] when $n \geq 3$ for an effective version of Marklof's result.

(2) A version of Corollary 4.1 with congruences is possible. Let $N \in \mathbb{N} - \{0\}$, and let $\Gamma_0[N]$ be the Hecke congruence subgroup of level N of Γ, preimage of the upper triangular subgroup by the morphism of reduction modulo N of the coefficients.

Up to replacing $\mathcal{F}_t$ by $\{p/q \in \mathcal{F}_t : q \equiv 0 \mod N\}$, to replacing Γ by $\Gamma_0[N]$ and to replacing Θ_* by an averaging operator over cosets of $\Gamma_0[N]$ in Γ (coming from the fact that the lattice $\Gamma_0[N]$ is no longer invariant under the Cartan involution $g \mapsto {}^t g^{-1}$), one could obtain as in [**Mar1**, Theorem 2(B)] a joint partial equidistribution of Farey fractions with a congruence assumption on their denominator and with an error term. See also [**Hee**].

4.2. *Equidistribution of complex Farey fractions at a given density.* Let K be an imaginary quadratic number field, with discriminant D_K, ring of integers $\mathcal{O}_K$, finite group of unit integers $\mathcal{O}_K^\times$ (which is equal to $\{\pm 1\}$ unless $D_K = -4, -3$), and Dedekind's zeta function ζ_K.

Let $G = \mathrm{PSL}_2(\mathbb{C})$ and let Γ be the *Bianchi group* $\mathrm{PSL}_2(\mathcal{O}_K)$. For all $r \in \mathbb{C}$ and $t \in \mathbb{R}$, we consider the elements of G defined by

$$\mathfrak{n}_-(r) = \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \Phi^t = \begin{bmatrix} e^{-t/2} & 0 \\ 0 & e^{t/2} \end{bmatrix}.$$

Let $H = \{\mathfrak{n}_-(r) : r \in \mathbb{C}\}$. We denote by

$$M = \left\{ \begin{bmatrix} e^{-i\,\theta/2} & 0 \\ 0 & e^{i\,\theta/2} \end{bmatrix} : \theta \in \mathbb{R} \right\}$$

the compact factor of the centralizer of the standard Cartan subgroup $\Phi^{\mathbb{R}} = \{\Phi^t : t \in \mathbb{R}\}$ of G, which normalizes H. Note that both Γ and M are invariant under the standard Cartan involution $g \mapsto {}^t g^{-1}$. Let

$$\begin{aligned} \Gamma_H &= N_G(H) \cap \Gamma = (HM) \cap \Gamma \\ &= \left\{ \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \begin{bmatrix} 1 & a^{-1}b \\ 0 & 1 \end{bmatrix} : a \in \mathcal{O}_K^\times,\ b \in \mathcal{O}_K \right\}, \end{aligned}$$

which is a semidirect product $(M \cap \Gamma) \ltimes (H \cap \Gamma)$. The discrete group Γ_H admits a properly discontinuously action $\star$ on the left on H so that $H \cap \Gamma$ acts firstly by translations and $M \cap \Gamma$ secondly by conjugation: for all $a \in \mathcal{O}_K^\times$, $b \in \mathcal{O}_K$ and $r \in \mathbb{C}$, we have

$$\begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \star \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \left(\begin{bmatrix} 1 & a^{-1}b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}^{-1} = \begin{bmatrix} 1 & a^2 r + ab \\ 0 & 1 \end{bmatrix}. \tag{29}$$

We see, as we may, $\Gamma_H \backslash H$ contained in $\Gamma \backslash G / M$ (as the image $(\Gamma \cap H) \backslash H / (M \cap \Gamma)$ of H in the set of double cosets). We endow $\Gamma_H \backslash H$ with the induced measure $\mu_{\Gamma_H \backslash H}$ of a Haar measure on H by the branched cover $H \to \Gamma_H \backslash H$, normalized to be a probability measure, which we also see as a probability measure on $\Gamma \backslash G / M$ (with support $\Gamma_H \backslash H$). We denote by $\mathcal{O}'_K$ the semidirect product $\mathcal{O}_K^\times \ltimes \mathcal{O}_K$, which acts on the left, with kernel of order two, on $\mathbb{C}$ by $((a, b), r) \mapsto a^2 r + ab$. Note that for every $t \in \mathbb{R}$, by equation (29) and since Φ^t centralizes M, the double class $\Gamma \mathfrak{n}_-(r) \Phi^t M$ is well defined for every equivalence class $r \in \mathcal{O}'_K \backslash \mathbb{C}$. We endow the quotient space $\mathcal{O}'_K \backslash \mathbb{C}$ with the induced measure dx of the Lebesgue measure on $\mathbb{C}$ by the branched cover $\mathbb{C} \to \mathcal{O}'_K \backslash \mathbb{C}$, normalized to be a probability measure.

https://doi.org/10.1017/etds.2023.116 Published online by Cambridge University Press

For every $t \in \mathbb{R}$, we consider the subset $\mathcal{F}_t$ of $\mathcal{O}'_K\backslash\mathbb{C}$ consisting of the *complex Farey fractions of height at most* $e^{t/2}$, defined by

$$\mathcal{F}_t = \mathcal{O}'_K\backslash\left\{\frac{p}{q} :\ p, q \in \mathcal{O}_K,\, p\mathcal{O}_K + q\mathcal{O}_K = \mathcal{O}_K,\, 0 < |q| \leq e^{t/2}\right\}.$$

Note that the above set of fractions p/q is indeed invariant under $\mathcal{O}'_K$.

Let $\Theta : \Gamma\backslash G/M \to \Gamma\backslash G/M$ be the Cartan involutive homeomorphism defined by $\Gamma g M \mapsto \Gamma\, {}^t g^{-1} M$.

COROLLARY 4.2. *For every $t_0 \in \mathbb{R}$, for the weak-star convergence of probability measures on $(\mathcal{O}'_K\backslash\mathbb{C}) \times (\Gamma\backslash G/M)$, we have*

$$\lim_{t\to+\infty} \frac{1}{\operatorname{Card} \mathcal{F}_{t-t_0}} \sum_{r\in\mathcal{F}_{t-t_0}} \Delta_r \otimes \Delta_{\Gamma \mathfrak{n}_-(r)\Phi^t M}$$
$$= 2\, e^{2t_0} \int_{s=t_0}^{+\infty} (dx) \otimes (\Theta_* \,(\Phi^{-s})_*\, \mu_{\Gamma_H\backslash H})\, e^{-2s}\, ds.$$

This statement implies Corollary 1.2 when $D_K \neq -4, -3$, since then we have $\mathcal{O}'_K\backslash\mathbb{C} = \mathcal{O}_K\backslash\mathbb{C} = \mathbb{C}/\mathcal{O}_K$ and $\Gamma_H = H \cap \Gamma$. As a remark similar to the remarks at the end of §4.1, one could obtain an error term under an additional regularity assumption, and a joint partial equidistribution result for complex Farey fractions with their denominator congruent to 0 modulo any fixed element N in $\mathbb{Z}_K - \{0\}$.

Proof. We mostly indicate the differences with the proof of Corollary 4.1. We now consider $X = \mathbb{H}^3_{\mathbb{R}}$ with coordinates $(z, u) \in \mathbb{C}\times\,]0, +\infty[$. The critical exponent of the (non-uniform arithmetic) lattice Γ of the orientation-preserving isometry group G of X is now

$$\delta_\Gamma = 2.$$

We denote by $\cdot$ the action of G by homographies on $\partial_\infty\mathbb{H}^3_{\mathbb{R}} = \mathbb{C} \cup \{\infty\}$, by isometries on $\mathbb{H}^3_{\mathbb{R}}$ through the Poincaré extension, and by the derived action on $T^1\mathbb{H}^3_{\mathbb{R}}$. We now fix the unit tangent vector $v^\bullet = ((0, 1), (0, -1)) \in T^1\mathbb{H}^3_{\mathbb{R}}$. The stabilizer of $v^\bullet$ in G is equal to M and is hence centralized by $\Phi^{\mathbb{R}}$. The orbital map $\widetilde{\varphi} : g \mapsto g \cdot v^\bullet$ now defines a homeomorphism $\varphi : \Gamma\backslash G/M \to \Gamma\backslash T^1\mathbb{H}^3_{\mathbb{R}}$. The order-two element $S = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ still belongs to Γ. It normalizes M and $\Phi^{\mathbb{R}}$, and formulae (19) and (20) are still satisfied.

Now let $\mathcal{H}_\infty = \{(z, u) \in \mathbb{H}^3_{\mathbb{R}} : u \geq 1\}$. With $\Phi^{\geq t_0} = \{\Phi^t : t \geq t_0\}$, we again have

$$\partial^1_+\mathcal{H}_\infty = W^-(v^\bullet) = \widetilde{\varphi}(H) \quad \text{and} \quad W^{0+}_{t_0}(-v^\bullet) = \bigcup_{s\geq t_0} g^s \partial^1_-\mathcal{H}_\infty = \widetilde{\varphi}(H(\Phi^{\geq t_0})^{-1} S). \tag{30}$$

The subgroup Γ_H is again equal to the stabilizer $\Gamma_{\mathcal{H}_\infty}$ of the horoball $\mathcal{H}_\infty$ in Γ. We again consider the locally finite Γ-equivariant families of horoballs

$$\mathcal{D}^+ = \mathcal{D}^- = (\gamma \cdot \mathcal{H}_\infty)_{\gamma\in\Gamma}.$$

https://doi.org/10.1017/etds.2023.116 Published online by Cambridge University Press

The map $\gamma = \left[\begin{smallmatrix} p & r \\ q & s \end{smallmatrix}\right] \mapsto \gamma \cdot \infty = p/q$ now induces, for every $t \in \mathbb{R}$, a bijection from the set $\{[\gamma] \in \Gamma_{\mathcal{H}_\infty} \backslash (\Gamma - \Gamma_{\mathcal{H}_\infty}) / \Gamma_{\mathcal{H}_\infty} : d(\mathcal{H}_\infty, \gamma \cdot \mathcal{H}_\infty) \leq t\}$ to $\mathcal{F}_t$. With ρ_γ the element of $\partial_+^1 \mathcal{H}_\infty$ whose point at infinity is $\gamma \cdot \infty$, the endpoint map $\widetilde{\psi} : \partial_+^1 \mathcal{H}_\infty \to \mathbb{C}$ now induces a homeomorphism $\psi : \Gamma_H \backslash \partial_+^1 \mathcal{H}_\infty \to \mathcal{O}'_K \backslash \mathbb{C}$, such that

$$\psi_*(\Delta_{\Gamma \rho_\gamma}) = \Delta_{\mathcal{O}'_K \gamma \cdot \infty}.$$

Let us compute the total mass of the induced Lebesgue measure $d\,\mathrm{Leb}_{\mathcal{O}'_K \backslash \mathbb{C}}$ on $\mathcal{O}'_K \backslash \mathbb{C}$, yielding dx after renormalization to a probability measure. Since the branched cover $\mathcal{O}_K \backslash \mathbb{C} \to \mathcal{O}'_K \backslash \mathbb{C}$ is $(|\mathcal{O}_K^\times|/2)$-sheeted outside the singular part and since $\mathcal{O}_K$ is generated as a $\mathbb{Z}$-lattice of $\mathbb{C}$ by 1 and $(D_K + i\sqrt{|D_K|})/2$, we have

$$\|d\,\mathrm{Leb}_{\mathcal{O}'_K \backslash \mathbb{C}}\| = \frac{2}{|\mathcal{O}_K^\times|}\, \|d\,\mathrm{Leb}_{\mathcal{O}_K \backslash \mathbb{C}}\| = \frac{\sqrt{|D_K|}}{|\mathcal{O}_K^\times|}.$$

Again by [**PaP4**, Theorem 9.11] or [**PaP5**, Proposition 20(2)], now with $n = 3$, we have

$$\psi_*(\sigma^+_{\mathcal{D}^-}) = 4\, d\,\mathrm{Leb}_{\mathcal{O}'_K \backslash \mathbb{C}} = \frac{4\sqrt{|D_K|}}{|\mathcal{O}_K^\times|}\, dx \quad \text{and} \quad (\varphi^{-1})_*(\sigma^+_{\mathcal{D}^-}) = \frac{4\sqrt{|D_K|}}{|\mathcal{O}_K^\times|}\, d\mu_{\Gamma_H \backslash H}.$$

Again by [**PaP4**, Theorem 9.10] or [**PaP5**, Proposition 20(1)], now with $n = 3$ and with Humbert's volume formula (see, for instance, [**EGM**, §§8.8 and 9.6]), we have

$$\|m_{\mathrm{BM}}\| = 4\,\mathrm{Vol}(\mathbb{S}^2)\,\mathrm{Vol}(\Gamma \backslash \mathbb{H}^3_{\mathbb{R}}) = \frac{4}{\pi}\, |D_K|^{3/2}\, \zeta_K(2).$$

Mertens's formula for the quadratic imaginary number fields (see also [**PaP3**, Theorem 3.1]) gives, using the action of $k \in \mathcal{O}_K$ on $(p, q) \in \mathcal{O}_K \times \mathcal{O}_K$ by horizontal shears $k \cdot (p, q) = (p + kq)$, as $t \to +\infty$,

$$\begin{aligned}
&\mathrm{Card}\,\mathcal{F}_{t-t_0} \\
&\quad \sim \frac{2}{|\mathcal{O}_K^\times|}\mathrm{Card}\Big(\mathcal{O}_K \backslash \Big\{\frac{p}{q} :\ p, q \in \mathcal{O}_K,\, p\mathcal{O}_K + q\mathcal{O}_K = \mathcal{O}_K,\, 0 < |q| \leq e^{(t-t_0)/2}\Big\}\Big) \\
&\quad = \frac{2}{|\mathcal{O}_K^\times|^2}\mathrm{Card}(\mathcal{O}_K \backslash \{(p, q) \in \mathcal{O}_K \times \mathcal{O}_K : p\mathcal{O}_K + q\mathcal{O}_K = \mathcal{O}_K,\, 0 < |q|^2 \leq e^{t-t_0}\}) \\
&\quad \sim \frac{2\pi}{|\mathcal{O}_K^\times|^2\, \zeta_K(2)\sqrt{|D_K|}}\, e^{2t-2t_0}.
\end{aligned}$$

Since $\mathcal{O}_K$ has finite index in $\mathcal{O}'_K$, there are only finitely many elliptic elements in Γ up to conjugation by $\Gamma \cap H$ whose fixed point set contains ∞ as a point at infinity. There are only finitely many $\Gamma_{\mathcal{H}_\infty}$-orbits of images of $\mathcal{H}_\infty$ by Γ meeting $\mathcal{H}_\infty$. Hence, there exists a finite subset F of the set of double cosets $\Gamma_{\mathcal{H}_\infty} \backslash \Gamma / \Gamma_{\mathcal{H}_\infty}$ such that for every element $[\gamma] \in \Gamma_{\mathcal{H}_\infty} \backslash \Gamma / \Gamma_{\mathcal{H}_\infty} - F$, we have

$$d(\mathcal{H}_\infty, \gamma \cdot \mathcal{H}_\infty) > 0 \quad \text{and} \quad m_\gamma = 1.$$

We have, similarly to equation (26), for all $\gamma \in \Gamma - \Gamma_{\mathcal{H}_\infty}$ and $t \in \mathbb{R}$,

$$(\varphi^{-1})_*(\Delta_{\Gamma \mathtt{g}^t \rho_\gamma}) = \Delta_{\Gamma n_{-(\gamma \cdot \infty)} \Phi^t M}$$

https://doi.org/10.1017/etds.2023.116 Published online by Cambridge University Press

and, for all $y \in \Gamma_H\backslash H$ and $s \in \mathbb{R}$ with $s \geq t_0$,

$$d((\varphi^{-1})_*(\mu^{0+}_{\mathcal{D}^+,t_0}))(\Theta(y\,\Phi^{-s})) = \|\sigma^-_{\mathcal{D}^+}\|\,d\mu_{\Gamma_H\backslash H}(y)\,e^{-2s}\,ds$$
$$= \frac{4\sqrt{|D_K|}}{|\mathcal{O}_K^\times|}\,d\mu_{\Gamma_H\backslash H}(y)\,e^{-2s}\,ds.$$

The end of the proof of Corollary 4.2 now proceeds like that of Corollary 4.1. □

4.3. *Equidistribution of Heisenberg Farey fractions at a given density.* Let K, D_K, $\mathcal{O}_K$, $\mathcal{O}_K^\times$, ζ_K be as at the beginning of §4.2. Let $\mathtt{tr}$ and $\mathtt{n}$ be the (absolute) trace and norm of K. We denote by $\langle a, \alpha, c\rangle$ the ideal of $\mathcal{O}_K$ generated by $a, \alpha, c \in \mathcal{O}_K$.

Let q be the non-degenerate Hermitian form $-z_0\overline{z_2} - z_2\overline{z_0} + |z_1|^2$ of signature $(1, 2)$ on $\mathbb{C}^3$ with coordinates (z_0, z_1, z_2). Let $G = \mathrm{PSU}_q = \mathrm{SU}_q/(\mathbb{U}_3\ \mathrm{id})$ be the projective special unitary group of q, where $\mathrm{SU}_q = \{g \in \mathrm{GL}_3(\mathbb{C}) : q \circ g = q,\ \det g = 1\}$ and $\mathbb{U}_3$ is the group of cube roots of unity. Let Γ be the image of $\mathrm{SU}_q \cap \mathrm{SL}_3(\mathcal{O}_K)$ in G, which is a (non-uniform) arithmetic lattice in G, called the (projective special) *Picard modular group* of K.

Denoting by $\begin{bmatrix} a & \overline{\gamma} & b \\ \alpha & A & \beta \\ c & \overline{\delta} & d \end{bmatrix}$ the image in G of $\begin{pmatrix} a & \overline{\gamma} & b \\ \alpha & A & \beta \\ c & \overline{\delta} & d \end{pmatrix} \in \mathrm{SU}_q$, let

$$H = \left\{ \mathfrak{n}_-(w_0, w) = \begin{bmatrix} 1 & \overline{w} & w_0 \\ 0 & 1 & w \\ 0 & 0 & 1 \end{bmatrix} : w_0, w \in \mathbb{C},\ 2\,\mathrm{Re}\,w_0 = |w|^2 \right\},$$

$$\Phi^{\mathbb{R}} = \left\{ \Phi^t = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^t \end{bmatrix} : t \in \mathbb{R} \right\} \text{ and } M = \left\{ \begin{bmatrix} \zeta & 0 & 0 \\ 0 & \overline{\zeta}^2 & 0 \\ 0 & 0 & \zeta \end{bmatrix} : \zeta \in \mathbb{C},\ |\zeta| = 1 \right\}.$$

Note that H, $\Phi^{\mathbb{R}}$ and M are Lie subgroups of G, that M is the compact factor of the centralizer in G of the standard Cartan subgroup $\Phi^{\mathbb{R}}$ of G, and that the subgroup $M\Phi^{\mathbb{R}}$ normalizes the *Heisenberg group* H. The groups Γ and M are invariant under the standard Cartan involution

$$g \mapsto {}^*g^{-1},$$

where ${}^*g$ is the image in G of the transpose-conjugate matrix of any matrix in SU_q representing g.

Let

$$\Gamma_H = N_G(H) \cap \Gamma = (MH) \cap \Gamma = \left\{ \begin{bmatrix} u & u\overline{v} & uv_0 \\ 0 & \overline{u}^2 & \overline{u}^2 v \\ 0 & 0 & u \end{bmatrix} : \begin{array}{c} u \in \mathcal{O}_K^\times,\ v, v_0 \in \mathcal{O}_K \\ \mathtt{tr}(v_0) = \mathtt{n}(v) \end{array} \right\},$$

which admits a properly discontinuously action $\star$ on the left on H by

$$\begin{bmatrix} u & u\overline{v} & uv_0 \\ 0 & \overline{u}^2 & \overline{u}^2\,v \\ 0 & 0 & u \end{bmatrix} \star \begin{bmatrix} 1 & \overline{w} & w_0 \\ 0 & 1 & w \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & u^3(\overline{w} + \overline{v}) & w_0 + v_0 + w\overline{v} \\ 0 & 1 & \overline{u}^3(w + v) \\ 0 & 0 & 1 \end{bmatrix}, \tag{31}$$

where $H \cap \Gamma$ acts firstly by left translations and $M \cap \Gamma$ secondly by conjugations on H. The inclusion map $H \to G$ induces an identification between the quotient $\Gamma_H \backslash H$ and the image of H in $\Gamma \backslash G/M$. We endow $\Gamma_H \backslash H$ with the induced measure $\mu_{\Gamma_H \backslash H}$ of a Haar measure on H, by the branched cover $H \to \Gamma_H \backslash H$, normalized to be a probability measure, which we also see as a probability measure on $\Gamma \backslash G/M$ (with support $\Gamma_H \backslash H$).

For every $t \in \mathbb{R}$, we consider the subset $\mathcal{F}_t$ of $\Gamma_H \backslash H$ consisting of the *Heisenberg Farey fractions of height at most* e^t, defined by

$$\mathcal{F}_t = \Gamma_H \backslash \left\{ \mathfrak{n}_-\left(\frac{a}{c}, \frac{\alpha}{c}\right) : \begin{array}{c} a, \alpha, c \in \mathcal{O}_K,\ \langle a, \alpha, c\rangle = \mathcal{O}_K, \\ \mathtt{tr}(a\,\overline{c}) = \mathtt{n}(\alpha), \end{array} \ 0 < \mathtt{n}(c) \leq e^{2\,t} \right\}.$$

Note that the above set of elements $\mathfrak{n}_-(a/c, \alpha/c)$ is indeed invariant under Γ_H, by equation (31). Let $\Theta : \Gamma \backslash G/M \to \Gamma \backslash G/M$ be the Cartan involutive homeomorphism defined by $\Gamma g M \mapsto \Gamma\, {}^*g^{-1} M$.

COROLLARY 4.3. *For every $t_0 \in \mathbb{R}$, for the weak-star convergence of probability measures on $(\Gamma_H \backslash H) \times (\Gamma \backslash G/M)$, we have*

$$\lim_{t \to +\infty} \frac{1}{\mathrm{Card}\ \mathcal{F}_{t-t_0}} \sum_{r \in \mathcal{F}_{t-t_0}} \Delta_r \otimes \Delta_{\Gamma\, r\, \Phi^t M}$$
$$= 4\, e^{4\, t_0} \int_{s=t_0}^{+\infty} (\mu_{\Gamma_H \backslash H}) \otimes (\Theta_*\, (\Phi^{-s})_*\, \mu_{\Gamma_H \backslash H})\, e^{-4s}\, ds.$$

As a remark similar to the remarks at the end of §4.1, one could obtain an error term under an additional regularity assumption, and a joint partial equidistribution result for Heisenberg Farey points $\mathfrak{n}_-(a/c, \alpha/c)$ modulo Γ_H with their denominators c congruent to 0 modulo any fixed element N in $\mathcal{O}_K - \{0\}$.

Proof. We mostly indicate the differences with the proof of Corollary 4.1. We refer to [**Gol**] as well as [**PaP1**, §6.1], [**PaP6**, §3] for background on complex hyperbolic geometry. We follow the conventions of the latter reference concerning the normalization of the sectional curvature and the choice of the Hermitian form with signature (1, 2).

We now consider $X = \mathbb{H}^2_{\mathbb{C}}$ the Siegel domain model of the complex hyperbolic plane, that is, the complex manifold

$$\{(w_0, w) \in \mathbb{C}^2 : 2\ \mathrm{Re}\ w_0 - |w|^2 > 0\},$$

endowed with the Riemannian metric

$$ds^2_{\mathbb{H}^2_{\mathbb{C}}} = \frac{1}{(2\ \mathrm{Re}\ w_0 - |w|^2)^2}((dw_0 - dw\ \overline{w})(\overline{dw_0} - w\ \overline{dw}) + (2\ \mathrm{Re}\ w_0 - |w|^2)\ dw\ \overline{dw}). \tag{32}$$

This metric is normalized so that its sectional curvatures are in $[-4, -1]$. The boundary at infinity of $\mathbb{H}^2_{\mathbb{C}}$ is

$$\partial_\infty \mathbb{H}^2_{\mathbb{C}} = \{(w_0, w) \in \mathbb{C}^2 : 2\ \mathrm{Re}\ w_0 - |w|^2 = 0\} \cup \{\infty\}.$$

Using homogeneous coordinates, we identify $\mathbb{H}^2_{\mathbb{C}} \cup \partial_\infty \mathbb{H}^2_{\mathbb{C}}$ with its image in $\mathbb{P}^2(\mathbb{C})$ by the map $(w_0, w) \mapsto [w_0 : w : 1]$ and $\infty \mapsto [1 : 0 : 0]$. We denote by $\cdot$ the projective action

https://doi.org/10.1017/etds.2023.116 Published online by Cambridge University Press

of G on $\mathbb{H}^2_{\mathbb{C}} \cup \partial_\infty \mathbb{H}^2_{\mathbb{C}}$, as well as its derived action on $T^1\mathbb{H}^2_{\mathbb{C}}$. The holomorphic isometry group of $\mathbb{H}^2_{\mathbb{C}}$ is G (acting projectively on $\mathbb{P}^2(\mathbb{C})$).

The critical exponent of the (non-uniform arithmetic) lattice Γ of G is now (see, for instance, [**CI**, §6])

$$\delta_\Gamma = 4.$$

We now fix $v^\bullet = ((1, 0), (-2, 0)) \in T^1\mathbb{H}^2_{\mathbb{C}}$, which is indeed a unit tangent vector with footpoint $x^\bullet = (1, 0)$ by equation (32). The stabilizer of $v^\bullet$ in G is equal to M and hence is centralized by $\Phi^{\mathbb{R}}$. The orbital map $\widetilde{\varphi} : g \mapsto g \cdot v^\bullet$ now defines a homeomorphism $\varphi : \Gamma\backslash G/M \to \Gamma\backslash T^1\mathbb{H}^2_{\mathbb{C}}$.

For every $t \in \mathbb{R}$, the element Φ^t acts on $\mathbb{H}^2_{\mathbb{C}}$ by the map $(w_0, w) \mapsto (e^{-2t} w_0, e^{-t} w)$. The geodesic line ℓ in $\mathbb{H}^2_{\mathbb{C}}$ such that $\ell(0) = x^\bullet$ and $\ell'(0) = v^\bullet$ is $t \mapsto (e^{-2\,t}, 0)$. Hence, $\mathrm{g}^t v^\bullet = \ell'(t) = (-2\,e^{-2\,t}, 0) = d_{x^\bullet}\Phi^t(v^\bullet) = \Phi^t \cdot v^\bullet$. Therefore, by equivariance,

$$\text{for all } t \in \mathbb{R} \text{ and } g \in G, \text{ we have} \quad \mathrm{g}^t \widetilde{\varphi}(g) = \widetilde{\varphi}(g\Phi^t).$$

The order-two element $S = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \in \Gamma$ acts by the map $(w_0, w) \mapsto (1/w_0, -w/w_0)$ on $\mathbb{H}^2_{\mathbb{C}}$. It thus fixes the point $x^\bullet = (1, 0)$ and acts by $-\operatorname{id}$ on $T_{x^\bullet}\mathbb{H}^2_{\mathbb{C}}$. In particular, it maps $v^\bullet$ to $-v^\bullet$. By equivariance,

$$\text{for all } g \in G, \text{ we have} \quad \iota\, \widetilde{\varphi}(g) = \widetilde{\varphi}(gS).$$

The element S centralizes M and normalizes $\Phi^{\mathbb{R}}$; more precisely,

$$\text{for all } t \in \mathbb{R}, \text{ we have} \quad S\Phi^t S^{-1} = \Phi^{-t}.$$

Since S is the projective image of the matrix of the Hermitian form $q = -z_0\overline{z_2} - z_2\overline{z_0} + |z_1|^2$, we have ${}^*g\ S\ g = S$ for every $g \in G$, hence

$$\text{for all } g \in G, \text{ we have} \quad {}^*g^{-1} = S\ g\ S^{-1}.$$

For all $x \in \Gamma\backslash G$ and $s \in \mathbb{R}$, we again have $\Theta(x\Phi^s) = \Theta(x)\Phi^{-s}$ and $\iota \circ \varphi = \varphi \circ \Theta$.

The (closed) horoball in $\mathbb{H}^2_{\mathbb{C}}$ centred at ∞ whose boundary $\partial\mathcal{H}_\infty$ contains $x^\bullet$ is

$$\mathcal{H}_\infty = \{(w_0, w) \in \mathbb{H}^2_{\mathbb{C}} : 2\ \mathrm{Re}\ w_0 - |w|^2 \geq 2\}.$$

The Heisenberg group H acts simply transitively on $\partial\mathcal{H}_\infty$ and on $\partial^1_\pm\mathcal{H}_\infty$, which contains $\pm v^\bullet$. Thus again with $\Phi^{\geq t_0} = \{\Phi^t : t \geq t_0\}$, equation (30) is still satisfied. By for instance [**PaP6**, p. 90], the stabilizer $\Gamma_{\mathcal{H}_\infty}$ in Γ of the horoball $\mathcal{H}_\infty$, as well as that of $\partial^1_\pm\mathcal{H}_\infty$, is equal to Γ_H. The Γ-equivariant families of horoballs

$$\mathcal{D}^- = \mathcal{D}^+ = (\gamma \cdot \mathcal{H}_\infty)_{\gamma\in\Gamma}$$

are again locally finite, since ∞ is a bounded parabolic fixed point of Γ.

For every $\gamma \in \Gamma$ having a representative (whose choice does not change the following claims) in SU_q with first column $\begin{pmatrix} a \\ \alpha \\ c \end{pmatrix} \in \mathcal{M}_{3,1}(\mathcal{O}_K)$, we have $\gamma \notin \Gamma_{\mathcal{H}_\infty}$ if and only if $c \neq 0$ (see, for instance, [**PaP1**, Eq. (42)]) and then,

https://doi.org/10.1017/etds.2023.116 Published online by Cambridge University Press

(i) since $\infty = [1:0:0]$, the point at infinity $\gamma \cdot \infty$ is equal to $(a/c, \alpha/c)$;
(ii) since H acts simply transitively on $\partial_\infty \mathbb{H}^2_{\mathbb{C}} - \{\infty\}$, there exists a unique $r_\gamma \in H$ such that $r_\gamma \cdot 0 = \gamma \cdot \infty$, and we have $r_\gamma = \mathfrak{n}_-(a/c, \alpha/c)$;
(iii) by [**PaP1**, Lemma 6.3], we have $d(\mathcal{H}_\infty, \gamma \cdot \mathcal{H}_\infty) = \ln |c| = \frac{1}{2} \ln(\mathfrak{n}(c))$.

Therefore, by [**PaP1**, Proposition 6.5(2)] with $\mathcal{I} = \mathcal{O}_K$, the map $\gamma \mapsto r_\gamma$ induces, for all $t, t_0 \in \mathbb{R}$, a bijection from $\{[\gamma] \in \Gamma_{\mathcal{H}_\infty} \backslash (\Gamma - \Gamma_{\mathcal{H}_\infty}) / \Gamma_{\mathcal{H}_\infty} : d(\mathcal{H}_\infty, \gamma \cdot \mathcal{H}_\infty) \le t - t_0\}$ to $\mathcal{F}_{t-t_0}$.

Again using the simple transitivity of the action of H on $\partial^1_\pm \mathcal{H}_\infty$, we have a Γ_H-equivariant homeomorphism $\widetilde{\psi} : \partial^1_+ \mathcal{H}_\infty \to H$ which associates to $v \in \partial^1_+ \mathcal{H}_\infty$ the unique element $\widetilde{\psi}(v) \in H$ such that $\widetilde{\psi}(v) \cdot (v^\bullet) = v$.

For every $\gamma \in \Gamma - \Gamma_{\mathcal{H}_\infty}$, with ρ_γ the element of $\partial^1_+ \mathcal{H}_\infty$ whose point at infinity is $\gamma \cdot \infty$, the map $\widetilde{\psi}$ induces a homeomorphism $\psi : \Gamma_H \backslash \partial^1_+ \mathcal{H}_\infty \to \Gamma_H \backslash H$ such that

$$\psi_*(\Delta_{\Gamma \rho_\gamma}) = \Delta_{\Gamma_H r_\gamma}.$$

In the remainder of the proof of Corollary 4.3, we use the same normalization of the Patterson–Sullivan measures $(\mu_x)_{x \in \mathbb{H}^2_{\mathbb{C}}}$ as in [**PaP6**, §4]. We denote by

$$\delta_{x,y} = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

the Kronecker symbol.

LEMMA 4.4. *We have*

$$\|\sigma^{\mp}_{\mathcal{D}^\pm}\| = \frac{(1 + 2\,\delta_{D_K,-3})\,|D_K|}{4\,|\mathcal{O}_K^\times|}.$$

Proof. By [**PaP6**, Lemma 12(iv)] with $n = 2$, we have $\|\sigma^{\mp}_{\mathcal{D}^\pm}\| = 8\,\mathrm{Vol}(\Gamma_{\mathcal{H}_\infty} \backslash \mathcal{H}_\infty)$, where Vol is the Riemannian volume. Denoting as in [**PaP6**, §3], for every $s \in \mathbb{R}$,

$$\mathcal{H}_s = \{(w_0, w) \in \mathbb{H}^2_{\mathbb{C}} : 2\,\mathrm{Re}\, w_0 - |w|^2 \ge s\},$$

we have $\mathcal{H}_\infty = \mathcal{H}_2$ and the horoballs $\mathcal{H}_s$ all have the same stabilizer $\Gamma_{\mathcal{H}_s} = \Gamma_{\mathcal{H}_\infty}$ for $s \in \mathbb{R}$. By the comment following [**PaP6**, Eq. (11)], we have $\mathrm{Vol}(\Gamma_{\mathcal{H}_\infty} \backslash \mathcal{H}_\infty) = \frac{1}{4}\ \mathrm{Vol}(\Gamma_{\mathcal{H}_1} \backslash \mathcal{H}_1)$. The result then follows from [**PaP6**, Lemma 16] which says that

$$\mathrm{Vol}(\Gamma_{\mathcal{H}_1} \backslash \mathcal{H}_1) = \frac{(1 + 2\delta_{D_K,-3})\,|D_K|}{8\,|\mathcal{O}_K^\times|}. \qquad \square$$

Since we normalized $\mu_{\Gamma_H \backslash H}$ to be a probability measure, it follows from Lemma 4.4 that for $x \in \Gamma_H \backslash H$,

$$\psi_*(\sigma^+_{\mathcal{D}^-}) = (\varphi^{-1})_*(\sigma^+_{\mathcal{D}^-}) = \frac{(1 + 2\,\delta_{D_K,-3})\,|D_K|}{4\,|\mathcal{O}_K^\times|}\ \mu_{\Gamma_H \backslash H}.$$

By [**PaP6**, Lemma 12(iii)] with $n = 2$ and by the Holzapfel–Stover volume formula (see [**PaP6**, Lemma 17] for the appropriate normalization of the volume form), we have

$$\|m_{\mathrm{BM}}\| = \frac{\pi^2}{2}\ \mathrm{Vol}(M) = \frac{\pi\ (1 + 2\,\delta_{D_K,-3})\,|D_K|^{5/2}\,\zeta_K(3)}{96\,\zeta(3)}.$$

https://doi.org/10.1017/etds.2023.116 Published online by Cambridge University Press

By [PaP6, Eq. (21)] and the comment following it, the index of $H \cap \Gamma$ in Γ_H is equal to $|\mathcal{O}_K^\times|/(1 + 2\,\delta_{D_K,-3})$. The map from

$$\left\{(a, \alpha, c) \in \mathcal{O}_K \times \mathcal{O}_K \times \mathcal{O}_K : \begin{array}{c}\langle a, \alpha, c\rangle = \mathcal{O}_K \\ \mathtt{tr}(a\,\overline{c}) = \mathtt{n}(\alpha),\, c \neq 0\end{array}\right\}$$

to H defined by $(a, \alpha, c) \mapsto \mathfrak{n}_-(a/c, \alpha/c)$ is $|\mathcal{O}_K^\times|$-to-one onto its image. Hence, using the (lifted linear) action of $\mathfrak{n}_-(w_0, w) \in H \cap \Gamma$ on $(a, \alpha, c) \in \mathcal{O}_K \times \mathcal{O}_K \times \mathcal{O}_K$ defined by

$$\mathfrak{n}_-(w_0, w) \cdot (a, \alpha, c) = (a + \overline{w}\,\alpha + w_0\,c, \alpha + \omega\,c, c),$$

by [PaP6, Theorem 4], for every $t_0 \in \mathbb{R}$, we have, as $t \to +\infty$,

$$\begin{aligned}\mathrm{Card}\ \mathcal{F}_{t-t_0} = {} & \frac{1 + 2\,\delta_{D_K,-3}}{|\mathcal{O}_K^\times|^2} \\ & \times \mathrm{Card}\left((H\cap\Gamma)\backslash\left\{(a, \alpha, c) \in \mathcal{O}_K \times \mathcal{O}_K \times \mathcal{O}_K : \begin{array}{c}\langle a, \alpha, c\rangle = \mathcal{O}_K \\ \mathtt{tr}(a\,\overline{c}) = \mathtt{n}(\alpha) \\ 0 < \mathtt{n}(c) \leq e^{2\,t-2\,t_0}\end{array}\right\}\right) \\ \sim {} & \frac{3\,(1 + 2\,\delta_{D_K,-3})\,\zeta(3)}{2\,\pi\,|\mathcal{O}_K^\times|^2\,\sqrt{|D_K|}\,\zeta_K(3)}\,e^{4\,t-4\,t_0}.\end{aligned}$$

Since $H \cap \Gamma$ has finite index in $\Gamma_H = \Gamma_{\mathcal{H}_\infty}$ and acts freely on $\partial\mathcal{H}_\infty$, there are only finitely many elliptic elements in Γ up to conjugation by $\Gamma \cap H$ whose fixed point set contains $\infty = [1 : 0 : 0]$ as a point at infinity. There are only finitely many $\Gamma_{\mathcal{H}_\infty}$-orbits of images of $\mathcal{H}_\infty$ by Γ meeting $\mathcal{H}_\infty$. Hence, there again exists a finite subset F of the set of double cosets $\Gamma_{\mathcal{H}_\infty}\backslash\Gamma/\Gamma_{\mathcal{H}_\infty}$ such that for every $[\gamma] \in \Gamma_{\mathcal{H}_\infty}\backslash\Gamma/\Gamma_{\mathcal{H}_\infty} - F$, we have

$$d(\mathcal{H}_\infty, \gamma \cdot \mathcal{H}_\infty) > 0 \quad \text{and} \quad m_\gamma = 1.$$

We have similarly, for all $\gamma \in \Gamma - \Gamma_{\mathcal{H}_\infty}$ and $t \in \mathbb{R}$,

$$(\varphi^{-1})_*(\Delta_{\Gamma \mathtt{g}^t \rho_\gamma}) = \Delta_{\Gamma r_\gamma \Phi^t M}$$

and by Lemma 4.4, for all $y \in \Gamma_H\backslash H$ and $s \in \mathbb{R}$ with $s \geq t_0$,

$$\begin{aligned}d((\varphi^{-1})_*(\mu^{0+}_{\mathcal{D}^+,t_0}))(\Theta(y\ \Phi^{-s})) &= \|\sigma^-_{\mathcal{D}^+}\|\ d\mu_{\Gamma_H\backslash H}(y)\,e^{-4\,s}\,ds \\ &= \frac{(1 + 2\,\delta_{D_K,-3})\,|D_K|}{4\,|\mathcal{O}_K^\times|}\,d\mu_{\Gamma_H\backslash H}(y)\,e^{-4\,s}\,ds.\end{aligned}$$

The end of the proof of Corollary 4.3 now proceeds like that of Corollary 4.1. □

4.4. *Equidistribution of quaternionic Heisenberg Farey fractions at a given density.* In this section we write $\mathbb{H}$ for Hamilton's quaternion algebra over $\mathbb{R}$, with $x \mapsto \overline{x}$ its conjugation, $\mathtt{n} : x \mapsto x\overline{x}$ its reduced norm, $\mathtt{tr} : x \mapsto x + \overline{x}$ its reduced trace. Let A be a definite ($A \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{H}$) quaternion algebra over $\mathbb{Q}$, with discriminant D_A. Let $\mathcal{O}$ be a maximal order in A, with $\mathcal{O}^\times$ its finite group of invertible elements. We denote by ${}_{\mathcal{O}}\langle a, \alpha, c\rangle$ the left ideal of $\mathcal{O}$ generated by $a, \alpha, c \in \mathcal{O}$. See [Vig] for definitions.

https://doi.org/10.1017/etds.2023.116 Published online by Cambridge University Press

Let q be the non-degenerate quaternionic Hermitian form of Witt signature $(1, 2)$ on the right vector space $\mathbb{H}^3$ over $\mathbb{H}$ with coordinates (z_0, z_1, z_2) defined by

$$q = -\mathtt{tr}(\overline{z_0}\, z_2) + \mathtt{n}(z_1).$$

With $\mathrm{U}_q = \{g \in \mathrm{GL}_3(\mathbb{H}) : q \circ g = q\}$, let $G = \mathrm{PU}_q = \mathrm{U}_q / \{\pm\, \mathrm{id}\}$ be the projective unitary group of q. Let Γ be the image of $\mathrm{U}_q \cap \mathrm{GL}_3(\mathcal{O})$ in G, which is a (non-uniform) arithmetic lattice in G.

Denoting by $\begin{bmatrix} a & \overline{\gamma} & b \\ \alpha & A & \beta \\ c & \overline{\delta} & d \end{bmatrix}$ the image in G of $\begin{pmatrix} a & \overline{\gamma} & b \\ \alpha & A & \beta \\ c & \overline{\delta} & d \end{pmatrix} \in \mathrm{U}_q$, let

$$H = \left\{ \mathfrak{n}_-(w_0, w) = \begin{bmatrix} 1 & \overline{w} & w_0 \\ 0 & 1 & w \\ 0 & 0 & 1 \end{bmatrix} : w_0, w \in \mathbb{H},\ \mathtt{tr}(w_0) = \mathtt{n}(w) \right\},$$

$$\Phi^{\mathbb{R}} = \left\{ \Phi^t = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^t \end{bmatrix} : t \in \mathbb{R} \right\},$$

$$M = \left\{ m(u, U) = \begin{bmatrix} u & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & u \end{bmatrix} : u, U \in \mathbb{H},\ \mathtt{n}(u) = \mathtt{n}(U) = 1 \right\}.$$

Since $\mathbb{R}$ is central in $\mathbb{H}$, the subgroup M is the compact factor of the centralizer in G of the standard Cartan subgroup $\Phi^{\mathbb{R}}$ of G, and the subgroup $M\Phi^{\mathbb{R}}$ normalizes the *quaternionic Heisenberg group* H, since

$$m(u, U)\, \mathfrak{n}_-(w_0, w)\, m(u, U)^{-1} = \mathfrak{n}_-(u\, w_0\, \overline{u}, U\, w\, \overline{u}).$$

Since $\mathcal{O}$ is invariant under conjugation in $\mathbb{H}$, the groups Γ and M are invariant under the standard Cartan involution

$$g \mapsto {}^*g^{-1},$$

where ${}^*g$ is the image in G of the transpose-conjugate matrix of any matrix in U_q representing g.

Let

$$\Gamma_H = N_G(H) \cap \Gamma = (MH) \cap \Gamma = \left\{ \begin{bmatrix} u & u\overline{v} & uv_0 \\ 0 & U & Uv \\ 0 & 0 & u \end{bmatrix} : \begin{array}{c} u, U \in \mathcal{O}^\times,\ \ v, v_0 \in \mathcal{O} \\ \mathtt{tr}(v_0) = \mathtt{n}(v) \end{array} \right\},$$

which admits a properly discontinuously action $\star$ on the left on H by (noting the lack of commutativity)

$$\begin{bmatrix} u & u\overline{v} & uv_0 \\ 0 & U & Uv \\ 0 & 0 & u \end{bmatrix} \star \begin{bmatrix} 1 & \overline{w} & w_0 \\ 0 & 1 & w \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & u(\overline{w} + \overline{v})\overline{U} & u(v_0 + w_0 + \overline{v}w)\overline{u} \\ 0 & 1 & U(w + v)\overline{u} \\ 0 & 0 & 1 \end{bmatrix}. \tag{33}$$

The inclusion map $H \to G$ again induces an identification between the quotient $\Gamma_H \backslash H$ and the image of H in $\Gamma\backslash G/M$. We again endow $\Gamma_H \backslash H$ with the induced measure $\mu_{\Gamma_H \backslash H}$

https://doi.org/10.1017/etds.2023.116 Published online by Cambridge University Press

of a Haar measure on H, normalized to be a probability measure, that we also see as a probability measure on $\Gamma\backslash G/M$ (with support $\Gamma_H\backslash H$).

For every $t\in\mathbb{R}$, we consider the subset $\mathcal{F}_t$ of $\Gamma_H\backslash H$ consisting of the *quaternionic Heisenberg Farey fractions of height at most* e^t, defined by

$$\mathcal{F}_t=\Gamma_H\backslash\left\{\mathfrak{n}_-(a\,c^{-1},\alpha\,c^{-1}):\begin{array}{c}a,\alpha,c\in\mathcal{O},\quad {}_{\mathcal{O}}\langle a,\alpha,c\rangle=\mathcal{O},\\ \mathtt{tr}(\overline{a}\,c\,)=\mathtt{n}(\alpha),\end{array}\quad 0<\mathtt{n}(c)\le e^{2\,t}\right\}.$$

Note that the above set of elements $\mathfrak{n}_-(a\,c^{-1},\alpha\,c^{-1})$ is indeed invariant under Γ_H, by equation (33). Let $\Theta:\Gamma\backslash G/M\to\Gamma\backslash G/M$ be the Cartan involutive homeomorphism defined by $\Gamma gM\mapsto\Gamma\,{}^*g^{-1}M$.

COROLLARY 4.5. *For every $t_0\in\mathbb{R}$, for the weak-star convergence of probability measures on $(\Gamma_H\backslash H)\times(\Gamma\backslash G/M)$, we have*

$$\lim_{t\to+\infty}\frac{1}{\mathrm{Card}\,\mathcal{F}_{t-t_0}}\sum_{r\in\mathcal{F}_{t-t_0}}\Delta_r\otimes\Delta_{\Gamma\,r\,\Phi^tM}$$
$$=10\,e^{10\,t_0}\int_{s=t_0}^{+\infty}(\mu_{\Gamma_H\backslash H})\otimes(\Theta_*\,(\Phi^{-s})_*\,\mu_{\Gamma_H\backslash H})\,e^{-10\,s}\,ds.$$

As a remark similar to the remarks at the end of §4.1, one could obtain an error term under an additional smoothness assumption, and a joint partial equidistribution result for quaternionic Heisenberg Farey points $\mathfrak{n}_-(a\,c^{-1},\alpha\,c^{-1})$ modulo Γ_H with their denominators c congruent to 0 modulo any fixed element N in $\mathbb{Z}-\{0\}$.

Proof. We mostly indicate the differences with the proof of Corollary 4.3. We refer to [**KiP**, **Mos**, **Phi**] as well as [**PaP8**, §3] for background on quaternionic hyperbolic geometry. We follow the conventions of the latter reference concerning the normalization of the sectional curvature and the choice of the quaternionic Hermitian form with Witt signature (1, 2).

We now consider $X=\mathbf{H}^2_{\mathbb{H}}$ the Siegel domain model of the quaternionic hyperbolic plane, that is, the quaternionic manifold

$$\{(w_0,w)\in\mathbb{H}^2:\mathtt{tr}(w_0)-\mathtt{n}(w)>0\},$$

endowed with the Riemannian metric

$$ds^2_{\mathbf{H}^2_{\mathbb{H}}}=\frac{1}{(\mathtt{tr}w_0-\mathtt{n}(w))^2}(\mathtt{n}(dw_0-\overline{dw}\;w)+(\mathtt{tr}(w_0)-\mathtt{n}(w))\,\mathtt{n}(dw)).\qquad(34)$$

This metric is again normalized so that its sectional curvatures are in $[-4,-1]$. The boundary at infinity of $\mathbf{H}^2_{\mathbb{H}}$ is

$$\partial_\infty\mathbf{H}^2_{\mathbb{H}}=\{(w_0,w)\in\mathbb{H}^2\;:\mathtt{tr}(w_0)-\mathtt{n}(w)=0\}\cup\{\infty\}.$$

Using right-homogeneous coordinates, we identify $\mathbf{H}^2_{\mathbb{H}}\cup\partial_\infty\mathbf{H}^2_{\mathbb{H}}$ with its image in the right projective plane $\mathbb{P}^2_r(\mathbb{H})$ over $\mathbb{H}$ by the map $(w_0,w)\mapsto[w_0:w:1]$ and $\infty\mapsto[1:0:0]$. We denote by $\cdot$ the left projective action of G on $\mathbf{H}^2_{\mathbb{H}}\cup\partial_\infty\mathbf{H}^2_{\mathbb{H}}$, as well as its derived action on $T^1\mathbf{H}^2_{\mathbb{H}}$.

https://doi.org/10.1017/etds.2023.116 Published online by Cambridge University Press

The critical exponent of the (non-uniform arithmetic) lattice Γ of G is now (see, for instance, [**CI**, Theorem 4.4(i)])

$$\delta_\Gamma = 10.$$

We again fix $v^\bullet = ((1,0),(-2,0)) \in T^1\mathbf{H}^2_{\mathbb{H}}$, which is indeed a unit tangent vector with footpoint $x^\bullet = (1,0)$ by equation (34). The stabilizer of $v^\bullet$ in G is again equal to M and is hence centralized by $\Phi^{\mathbb{R}}$. The G-equivariant orbital map $\widetilde{\varphi} : g \mapsto g \cdot v^\bullet$ now defines a homeomorphism $\varphi : \Gamma\backslash G/M \to \Gamma\backslash T^1\mathbf{H}^2_{\mathbb{H}}$.

For every $t \in \mathbb{R}$, the element Φ^t acts on $\mathbf{H}^2_{\mathbb{H}}$ by the map $(w_0, w) \mapsto (e^{-2t} w_0, e^{-t} w)$. The geodesic line ℓ in $\mathbf{H}^2_{\mathbb{H}}$ such that $\ell(0) = x^\bullet$ and $\ell'(0) = v^\bullet$ is $t \mapsto (e^{-2\,t}, 0)$. Hence, as in the complex case (see the proof of Corollary 4.3),

$$\text{for all } t \in \mathbb{R}, \text{ and } g \in G, \text{ we have} \quad \mathrm{g}^t \widetilde{\varphi}(g) = \widetilde{\varphi}(g\Phi^t).$$

The order-two element $S = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$ still belongs to Γ, it centralizes M and normalizes $\Phi^{\mathbb{R}}$, and it acts by the map $(w_0, w) \mapsto (w_0^{-1}, -w\, w_0^{-1})$ on $\mathbf{H}^2_{\mathbb{H}}$. Since S is the projective image of the matrix of the quaternionic Hermitian form $q = -\mathtt{tr}(\overline{z_0}\, z_2) + \mathtt{n}(z_1)$,

$$\text{for all } g \in G, \text{ we have} \quad {}^*g^{-1} = S\, g\, S^{-1}.$$

As in the complex case, for all $g \in G$, $t \in \mathbb{R}$ and $x \in \Gamma\backslash G/M$, we have

$$\iota\, \widetilde{\varphi}(g) = \widetilde{\varphi}(gS), \quad S\Phi^t S^{-1} = \Phi^{-t}, \quad \iota \circ \varphi = \varphi \circ \Theta \quad \text{and} \quad \Theta(x\Phi^t) = \Theta(x)\Phi^{-t}.$$

The (closed) horoball in $\mathbf{H}^2_{\mathbb{H}}$ centred at ∞ whose boundary $\partial\mathcal{H}_\infty$ contains $x^\bullet$ is

$$\mathcal{H}_\infty = \{(w_0, w) \in \mathbf{H}^2_{\mathbb{H}} : \mathtt{tr}(w_0) - \mathtt{n}(w) \geq 2\}.$$

The quaternionic Heisenberg group H again acts simply transitively on $\partial\mathcal{H}_\infty$, and on $\partial^1_\pm\mathcal{H}_\infty$ which contains $\pm v^\bullet$. Thus again with $\Phi^{\geq t_0} = \{\Phi^t : t \geq t_0\}$, equation (30) is still satisfied. By for instance the end of §3 in [**PaP8**], the stabilizer $\Gamma_{\mathcal{H}_\infty}$ in Γ of the horoball $\mathcal{H}_\infty$, as well as that of $\partial^1_\pm\mathcal{H}_\infty$, is equal to Γ_H. The Γ-equivariant families of horoballs

$$\mathcal{D}^+ = \mathcal{D}^- = (\gamma \cdot \mathcal{H}_\infty)_{\gamma\in\Gamma}$$

are again locally finite, since ∞ is again a bounded parabolic fixed point of Γ.

For every $\gamma \in \Gamma$ having a representative in U_q with first column $\begin{pmatrix} a \\ \alpha \\ c \end{pmatrix} \in \mathcal{M}_{3,1}(\mathcal{O})$, we have $\gamma \notin \Gamma_{\mathcal{H}_\infty}$ if and only if $c \neq 0$ (see, for instance, [**KiP**], [**PaP8**, Eq. (3.3)]) and then

(i) since $\infty = [1:0:0]$, the point at infinity $\gamma \cdot \infty$ is equal to $(a\, c^{-1}, \alpha\, c^{-1})$;

(ii) since H acts simply transitively on $\partial_\infty\mathbb{H}^2_{\mathbb{C}} - \{\infty\}$, there exists a unique $r_\gamma \in H$ such that $r_\gamma \cdot 0 = \gamma \cdot \infty$, and we have $r_\gamma = \mathtt{n}_-(a\, c^{-1}, \alpha\, c^{-1})$;

(iii) with $\mathcal{H}_s = \{(w_0, w) \in \mathbf{H}^2_{\mathbb{H}} : \mathtt{tr}(w_0) - \mathtt{n}(w) = s\}$ for $s > 0$, by [**PaP8**, Lemma 6.5] where we take $s = 2$ so that $\mathcal{H}_2 = \mathcal{H}_\infty$, we have $d(\mathcal{H}_\infty, \gamma \cdot \mathcal{H}_\infty) = \frac{1}{2}\ln(\mathtt{n}(c))$.

Therefore, by [**PaP8**, Proposition 4.2(ii)] with $\mathfrak{m} = \mathcal{O}$, the map $\gamma \mapsto r_\gamma$ induces, for all $t, t_0 \in \mathbb{R}$, a bijection from $\{[\gamma] \in \Gamma_{\mathcal{H}_\infty}\backslash(\Gamma - \Gamma_{\mathcal{H}_\infty})/\Gamma_{\mathcal{H}_\infty} : d(\mathcal{H}_\infty, \gamma \cdot \mathcal{H}_\infty) \leq t - t_0\}$ to $\mathcal{F}_{t-t_0}$.

As in the complex case, we have homeomorphisms $\psi : \Gamma_H\backslash\partial^1_+\mathcal{H}_\infty \to \Gamma_H\backslash H$ such that

$$\psi_*(\Delta_{\Gamma\rho_\gamma}) = \Delta_{\Gamma_H r_\gamma}.$$

In the remainder of the proof of Corollary 4.5, we use the same normalization of the Patterson–Sullivan measures $(\mu_x)_{x\in\mathbf{H}^2_{\mathbb{H}}}$ as in [PaP8, §7].

LEMMA 4.6. *We have*

$$\|\sigma^{\mp}_{\mathcal{D}^\pm}\| = \frac{D_A^2}{64\,|\mathcal{O}^\times|^2}.$$

Proof. By [PaP8, Lemma 7.2(iv)] with $n = 2$, we have $\|\sigma^{\mp}_{\mathcal{D}^\pm}\| = 80\,\mathrm{Vol}(\Gamma_{\mathcal{H}_\infty}\backslash\mathcal{H}_\infty)$, where Vol is the Riemannian volume. By [PaP8, Lemma 7.1] and the arguments in its proofs, and by [PaP8, Eq. (8.4)] for the last equality, we have

$$\begin{aligned}\mathrm{Vol}(\Gamma_{\mathcal{H}_\infty}\backslash\mathcal{H}_\infty) &= \frac{1}{10}\,\mathrm{Vol}(\Gamma_{\mathcal{H}_\infty}\backslash\partial\mathcal{H}_\infty) = \frac{1}{10}\,\frac{1}{2^5}\,\mathrm{Vol}(\Gamma_{\mathcal{H}_1}\backslash\partial\mathcal{H}_1)) = \frac{1}{2^5}\,\mathrm{Vol}(\Gamma_{\mathcal{H}_1}\backslash\mathcal{H}_1)\\ &= \frac{1}{2^5}\,\frac{D_A^2}{160\,|\mathcal{O}^\times|^2}.\end{aligned}$$

The result follows. □

Since we normalized $\mu_{\Gamma_H\backslash H}$ to be a probability measure, it follows from Lemma 4.6 that for $x \in \Gamma_H\backslash H$,

$$\psi_*(\sigma^{+}_{\mathcal{D}^-}) = (\varphi^{-1})_*(\sigma^{+}_{\mathcal{D}^-}) = \frac{D_A^2}{64\,|\mathcal{O}^\times|^2}\,\mu_{\Gamma_H\backslash H}.$$

Let $m_A = 24$ if D_A is even, and $m_A = 1$ otherwise. By respectively Lemma 7.2(iii) with $n = 2$ and Theorem 1.4 in [PaP8], we have, with p ranging over primes,

$$\|m_{\mathrm{BM}}\| = \frac{\pi^4}{48}\,\mathrm{Vol}(M) = \frac{\pi^8\,m_A}{2^{18}\cdot 3^6\cdot 5^2\cdot 7}\,\prod_{p\,|\,D_A}(p-1)(p^2+1)(p^3-1).$$

By the definition of Γ_H, the index of $H\cap\Gamma$ in Γ_H is now equal to $|\mathcal{O}^\times|^2/2$. The map from

$$\left\{(a,\alpha,c)\in\mathcal{O}\times\mathcal{O}\times\mathcal{O} : \begin{array}{c}{}_{\mathcal{O}}\langle a,\alpha,c\rangle = \mathcal{O}\\ \mathtt{tr}(\overline{a}\,c) = \mathtt{n}(\alpha),\ c\neq 0\end{array}\right\}$$

to H given by $(a,\alpha,c)\mapsto \mathfrak{n}_-(a\,c^{-1},\alpha\,c^{-1})$ is $|\mathcal{O}^\times|$-to-one onto its image. Hence, using the (lifted linear) action of $\mathfrak{n}_-(w_0,w)\in H\cap\Gamma$ on $(a,\alpha,c)\in\mathcal{O}\times\mathcal{O}\times\mathcal{O}$ defined by

$$\mathfrak{n}_-(w_0,w)\cdot(a,\alpha,c) = (a+\overline{w}\,\alpha+w_0\,c,\ \alpha+\omega\,c,\ c),$$

by [PaP8, Theorem 1.1], for every $t_0\in\mathbb{R}$, we have, as $t\to+\infty$,

$$\begin{aligned}\mathrm{Card}\,\mathcal{F}_{t-t_0} &= \frac{2}{|\mathcal{O}^\times|^3}\,\mathrm{Card}\left((H\cap\Gamma)\backslash\left\{(a,\alpha,c)\in\mathcal{O}\times\mathcal{O}\times\mathcal{O} : \begin{array}{c}{}_{\mathcal{O}}\langle a,\alpha,c\rangle = \mathcal{O}\\ \mathtt{tr}(\overline{a}\,c) = \mathtt{n}(\alpha)\\ 0<\mathtt{n}(c)\leq e^{2t-2t_0}\end{array}\right\}\right)\\ &\sim \frac{2^4\cdot 3^6\cdot 5\cdot 7\,D_A^4}{\pi^8\,m_A\,|\mathcal{O}^\times|^4\,\prod_{p\,|\,D_A}(p-1)(p^2+1)(p^3-1)}\,e^{10t-10t_0}.\end{aligned}$$

As in the complex case, there exists a finite subset F of $\Gamma_{\mathcal{H}_\infty}\backslash\Gamma/\Gamma_{\mathcal{H}_\infty}$ such that for every $[\gamma]\in\Gamma_{\mathcal{H}_\infty}\backslash\Gamma/\Gamma_{\mathcal{H}_\infty}-F$, we have

$$d(\mathcal{H}_\infty,\gamma\cdot\mathcal{H}_\infty)>0,\, m_\gamma=1,\, (\varphi^{-1})_*(\Delta_{\Gamma \mathfrak{g}^t\rho_\gamma})=\Delta_{\Gamma r_\gamma \Phi^t M}$$

and by Lemma 4.6, for all $y\in\Gamma_H\backslash H$ and $s\in\mathbb{R}$ with $s\geq t_0$,

$$\begin{aligned} d((\varphi^{-1})_*(\mu^{0+}_{\mathcal{D}^+,t_0}))(\Theta(y\,\Phi^{-s})) &= \|\sigma^-_{\mathcal{D}^+}\|\,d\mu_{\Gamma_H\backslash H}(y)\,e^{-10\,s}\,ds\\ &= \frac{D_A^2}{64\,|\mathcal{O}^\times|^2}\,d\mu_{\Gamma_H\backslash H}(y)\,e^{-10\,s}\,ds. \end{aligned}$$

The end of the proof of Corollary 4.5 now proceeds like that of Corollary 4.3. □

4.5. *Equidistribution of non-archimedean Farey fractions at a given density.* In this section we give an arithmetic application of Theorem 3.3(3), proving a joint partial equidistribution result for non-archimedean arithmetic points with given density on an expanding horosphere in the quotient of a regular tree by a non-uniform arithmetic lattice.

We refer to [**Gos**, **Ros**] for the notions and complements below, as well as to [**BPP**, §14.2] whose notation we will follow. Let K be a (global) function field of genus $\mathfrak{g}$ over a finite field $\mathbb{F}_q$ of order a positive prime power q, let v be a (normalized discrete) valuation of K, let K_v be the associated completion of K, let $\mathcal{O}_v=\{x\in K_v : v(x)\geq 0\}$ be its valuation ring, let $\pi_v\in K$ with $v(\pi_v)=1$ be a uniformizer of v, let q_v be the order of the residual field $\mathcal{O}_v/\pi_v\mathcal{O}_v$, let $|\cdot|_v=q_v^{-v(\,\cdot\,)}$ be the absolute value associated with v, and let R_v be the affine function ring associated with v. The simplest example, used in Corollary 1.3, is given by the field $K=\mathbb{F}_q(Y)$ of rational fractions over $\mathbb{F}_q$ with one indeterminate Y, $\mathfrak{g}=0$, $v=v_\infty : \frac{P}{Q}\mapsto \deg Q-\deg P$ for every $P,Q\in\mathbb{F}_q[Y]$ with $Q\neq 0$ the valuation at infinity, $K_v=\mathbb{F}_q((Y^{-1}))$, the local ring $\mathcal{O}_v=\mathbb{F}_q[[Y^{-1}]]$ of formal power series in Y^{-1}, $\pi_v=Y^{-1}$, $q_v=q$, and $R_v=\mathbb{F}_q[Y]$.

Let G be the locally compact group $\mathrm{PGL}_2(K_v)=\mathrm{GL}_2(K_v)/(K_v^\times\,\mathrm{id})$. We denote by $\left[\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right]$ the image in G of $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\mathrm{GL}_2(K_v)$. Let $\Gamma=\mathrm{PGL}_2(R_v)$ be the *Nagao lattice* in G (see, for instance, [**Wei**]). We consider the subgroups of G defined by

$$H=\left\{\mathfrak{n}_-(r)=\begin{bmatrix}1 & r\\ 0 & 1\end{bmatrix} : r\in K_v\right\},\ \Phi^{\mathbb{Z}}=\left\{\Phi^n=\begin{bmatrix}1 & 0\\ 0 & \pi_v^{-n}\end{bmatrix} : n\in\mathbb{Z}\right\},$$

and $M=\left\{\left[\begin{smallmatrix} 1 & 0\\ 0 & u\end{smallmatrix}\right] : u\in K_v,\, |u|_v=1\right\}$. Note that M centralizes the standard Cartan subgroup $\Phi^{\mathbb{Z}}$, that the diagonal subgroup $M\Phi^{\mathbb{Z}}$ normalizes H, and that both Γ and M are invariant under the standard Cartan involution $g\mapsto {}^t g^{-1}$.

Let

$$\Gamma_H=N_G(H)\cap\Gamma=(HM)\cap\Gamma=\left\{\begin{bmatrix}1 & b\\ 0 & d\end{bmatrix} : d\in R_v^\times,\, b\in R_v\right\},$$

which admits a properly discontinuously action $\star$ on the left on H by

$$\begin{bmatrix}1 & b\\ 0 & d\end{bmatrix}\star\begin{bmatrix}1 & r\\ 0 & 1\end{bmatrix}=\begin{bmatrix}1 & \dfrac{r+b}{d}\\ 0 & 1\end{bmatrix}.$$

https://doi.org/10.1017/etds.2023.116 Published online by Cambridge University Press

The inclusion map $H \to G$ again induces an identification between the quotient $\Gamma_H\backslash H$ and the image of H in $\Gamma\backslash G/M$. We again endow $\Gamma_H\backslash H$ with the induced measure $\mu_{\Gamma_H\backslash H}$ of a Haar measure on H, normalized to be a probability measure, which we also see as a probability measure on $\Gamma\backslash G/M$ (with support $\Gamma_H\backslash H$).

For every $n \in \mathbb{Z}$, we consider the subset $\mathcal{F}_n$ of $\Gamma_H\backslash H$ consisting of the *Farey fractions of height at most q_v^n with respect to v*, defined by

$$\mathcal{F}_n = \Gamma_H\backslash\left\{\mathfrak{n}_-\left(\frac{a}{c}\right) : \begin{array}{c} a, c \in R_v, \quad aR_v + cR_v = R_v \\ c \neq 0, v(c) \geq -n \end{array}\right\}.$$

Let $\Theta : \Gamma\backslash G/M \to \Gamma\backslash G/M$ be the Cartan involutive homeomorphism defined by $\Gamma g M \mapsto \Gamma\, {}^t g^{-1} M$.

COROLLARY 4.7. *For every $n_0 \in \mathbb{Z}$, for the weak-star convergence of probability measures on $(\Gamma_H\backslash H) \times (\Gamma\backslash G/M)$, we have*

$$\lim_{n\to+\infty} \frac{1}{\operatorname{Card} \mathcal{F}_{n-n_0}} \sum_{r\in\mathcal{F}_{n-n_0}} \Delta_r \otimes \Delta_{\Gamma\, r\, \Phi^{2n} M}$$
$$= (1 - q_v^{-2})\, q_v^{2n_0} \sum_{m=n_0}^{+\infty} (\mu_{\Gamma_H\backslash H}) \otimes (\Theta_*\, (\Phi^{-2m})_*\, \mu_{\Gamma_H\backslash H})\, q_v^{-2m}.$$

Corollary 1.3 follows by considering the particular valued function field $(\mathbb{F}_q(Y), v_\infty)$ indicated above. As a remark similar to the remarks at the end of §4.1, one could obtain an error term under an additional locally constant regularity assumption, and a joint partial equidistribution result for non-archimedean Farey points $\mathfrak{n}_-(a/c)$ modulo Γ_H with their denominators c congruent to 0 modulo any fixed element N in $R_v - \{0\}$.

Proof. We mostly indicate the differences with the proof of Corollary 4.3. We refer to [**Tit**, **Ser**] for background on Bruhat–Tits trees, as well as to [**BPP**, §§15.1 and 15.2] whose notation we will follow.

We now consider $\mathbb{X} = \mathbb{X}_v$ the Bruhat–Tits tree of (PGL_2, K_v), which is a regular tree of degree $q_v + 1$ endowed with a vertex transitive action of G. Note that Γ acts without inversion on $\mathbb{X}_v$ by [**Ser**, II.1.3]. The set of vertices of $\mathbb{X}_v$ is the set of homothety classes $[\Lambda]$ under $K_v^\times$ of $\mathcal{O}_v$-lattices Λ in the plane $K_v \times K_v$, and $g[\Lambda] = [g\Lambda]$ for every $g \in G$. We identify the boundary at infinity $\partial_\infty\mathbb{X}_v$ of (the geometric realization of) $\mathbb{X}_v$ and the projective line $\mathbb{P}_1(K_v) = K_v \cup \{\infty\}$ by the unique homeomorphism such that the (continuous) extension to $\partial_\infty\mathbb{X}_v$ of the isometric action of G on $\mathbb{X}_v$ is the projective action of G on $\mathbb{P}_1(K_v)$, that is, the action of G by homographies on $K_v \cup \{\infty\}$. We denote by $\cdot$ the action of G by homographies on $K_v \cup \{\infty\}$, as well as the action of G on the space $\mathcal{G}\mathbb{X}_v$ of (discrete) geodesic lines in $\mathbb{X}_v$.

The critical exponent of the (non-uniform arithmetic) lattice Γ of G is now (see, for instance, [**BPP**, Eq. (15.8)])

$$\delta_\Gamma = \ln q_v. \tag{35}$$

The standard basepoint $x^\bullet$ of $\mathbb{X}_v$ is the homothety class $[\mathcal{O}_v \times \mathcal{O}_v]$ of the standard $\mathcal{O}_v$-lattice $\mathcal{O}_v \times \mathcal{O}_v$ in $K_v \times K_v$. We consider the geodesic line $v^\bullet \in \mathcal{G}\mathbb{X}_v$ with

$v^\bullet(0) = x^\bullet$, $v^\bullet(-\infty) = \infty \in \mathbb{P}_1(K_v)$ and $v^\bullet(+\infty) = 0 \in \mathbb{P}_1(K_v)$. The stabilizer of $v^\bullet$ in G is again equal to M. The G-equivariant orbital map $\widetilde{\varphi} : g \mapsto g \cdot v^\bullet$ now defines a homeomorphism $\varphi : \Gamma\backslash G/M \to \Gamma\backslash \mathcal{G}\mathbb{X}_v$.

Since $v^\bullet(n) = [\mathcal{O}_v \times \pi_v^{-n}\mathcal{O}_v]$ for every $n \in \mathbb{Z}$ (see, for instance, [**BPP**, top of p. 310]) and by equivariance (see also [**BPP**, Eq. (15.4)]),

$$\text{for all } n \in \mathbb{Z}, \text{for all } g \in G, \text{ we have} \quad \mathtt{g}^n \widetilde{\varphi}(g) = \widetilde{\varphi}(g\,\Phi^n).$$

The order-two element $S = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ still belongs to Γ, and it normalizes M and $\Phi^{\mathbb{R}}$, more precisely $S\,\Phi^n S^{-1} = \Phi^{-n}$ for every $n \in \mathbb{Z}$. By equivariance, the antipodal map ι satisfies $\iota\,\widetilde{\varphi}(g) = \widetilde{\varphi}(gS)$ for every $g \in G$. Since the computation is independent of the ground field, we have ${}^t g^{-1} = S\,gS^{-1}$ for every $g \in G$. Hence, $\iota \circ \varphi = \varphi \circ \Theta$ and $\Theta(x\Phi^n) = \Theta(x)\Phi^{-n}$ for all $x \in \Gamma\backslash G/M$ and $n \in \mathbb{Z}$.

The group H fixes the point at infinity ∞, preserves the horoball $\mathcal{H}_\infty$ in $\mathbb{X}_v$ centred at ∞ whose boundary contains $x^\bullet$, and acts simply transitively on $\partial_\infty\mathbb{X}_v - \{\infty\} = K_v$, hence on $\partial^1_\pm\mathcal{H}_\infty$. Note that $\partial^1_+\mathcal{H}_\infty$ contains the geodesic ray $v^\bullet{}_{|\,[0,+\infty[}$ and that $\partial^1_-\mathcal{H}_\infty$ contains $(\iota v^\bullet)_{|\,]-\infty,0]}$. In particular, we have $\partial^1_+\mathcal{H}_\infty = \{\ell_{|[0,+\infty[} : \ell \in W^-(v^\bullet)\}$.

Note that defining $V_{\rm even}\mathbb{X}_v$, $\check{\mathcal{G}}_{\rm even}\mathbb{X}_v$ and $\mathcal{G}_{\rm even}\mathbb{X}_v$ for the above basepoint $x^\bullet$ as just before the statement of Theorem 3.2, we have $\partial^1_\pm\mathcal{H}_\infty \subset \check{\mathcal{G}}_{\rm even}\mathbb{X}_v$, since any two points of the horosphere $\partial\mathcal{H}_\infty$ are at even distance from one another. Furthermore, Γ preserves $V_{\rm even}\mathbb{X}_v$. Indeed, note that in a simplicial tree, if two of the distances between three points are even, then so is the third. The result then follows from [**Ser**, Corollary II.1.2], which proves that the distance $d(x^\bullet, \gamma x^\bullet)$ is even for every $\gamma \in \mathrm{GL}_2(R_v)$, since $v(\det\gamma) = 0$.

Each geodesic ray $w \in \partial^1_-\mathcal{H}_\infty$ can be extended to a unique element $\widehat{w} \in \mathcal{G}\mathbb{X}_v$ such that $\widehat{w}(+\infty)$ is the point at infinity of $\mathcal{H}_\infty$. This element belongs to $\mathcal{G}_{\rm even}\mathbb{X}_v$, is equal to $(N^+_{\iota v^\bullet})^{-1}(w)$ with the notation $N^+_{\cdot}$ of §2, and we define $\widehat{\partial^1_-\mathcal{H}_\infty} = \{\widehat{w} : w \in \partial^1_-\mathcal{H}_\infty\}$. With $\Phi^{\geq n_0} = \{\Phi^n : n \geq n_0\}$, we have

$$W^{0+}_{n_0}(\iota\, v^\bullet) = \bigcup_{n \geq n_0} \mathtt{g}^n\ \widehat{\partial^1_-\mathcal{H}_\infty} = \bigcup_{n \geq n_0} \mathtt{g}^n H\ \iota\ v^\bullet = \widetilde{\varphi}(H(\Phi^{\geq n_0})^{-1}S).$$

The subgroup Γ_H is again equal to the stabilizer $\Gamma_{\mathcal{H}_\infty}$ of the horoball $\mathcal{H}_\infty$ in Γ, and ∞ is again a bounded parabolic fixed point of Γ. We again consider the locally finite Γ-equivariant families of horoballs

$$\mathcal{D}^+ = \mathcal{D}^- = (\gamma \cdot \mathcal{H}_\infty)_{\gamma \in \Gamma}.$$

Note that the support of the skinning measure $\sigma^+_{\mathcal{D}^-}$ is contained in $\Gamma\backslash\check{\mathcal{G}}_{\rm even}\mathbb{X}_v$, hence ${\sigma^+_{\mathcal{D}^-}}_{|\,\Gamma\backslash\check{\mathcal{G}}_{\rm even}\mathbb{X}_v} = \sigma^+_{\mathcal{D}^-}$.

By [**Pau**, Proposition 6.1] when $K = \mathbb{F}_q(Y)$ and $v = v_\infty$, and by [**BPP**, Lemma 15.1] in general, for every $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma$ with $c \neq 0$, we have

$$d(\mathcal{H}_\infty, \gamma \cdot \mathcal{H}_\infty) = -2\,v(c) = 2\ \ln_{q_v}|c|_v.$$

In particular, the distances $d(\mathcal{H}_\infty, \gamma \cdot \mathcal{H}_\infty)$ for $\gamma \in \Gamma$ are even and the endpoints of the common perpendiculars between elements of $\mathcal{D}^-$ and $\mathcal{D}^+$ belong to $V_{\rm even}\mathbb{X}_v$. The map $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \mapsto \mathfrak{n}_-(a/c)$ now induces, for every $n \in \mathbb{Z}$, a bijection from

https://doi.org/10.1017/etds.2023.116 Published online by Cambridge University Press

$$\{[\gamma] \in \Gamma_{\mathcal{H}_\infty} \backslash (\Gamma - \Gamma_{\mathcal{H}_\infty}) / \Gamma_{\mathcal{H}_\infty} : d(\mathcal{H}_\infty, \gamma \cdot \mathcal{H}_\infty) \leq 2n\}$$

to $\mathcal{F}_n$. Denoting by ρ_γ the element of $\partial^1_+ \mathcal{H}_\infty$ whose point at infinity is $\gamma \cdot \infty = a/c$, the map $\widetilde{\psi} : \partial^1_+ \mathcal{H}_\infty \to H$ defined by $w \mapsto \mathfrak{n}_-(w(+\infty))$ now induces a homeomorphism $\psi : \Gamma_H \backslash \partial^1_+ \mathcal{H}_\infty \to \Gamma_H \backslash H$, such that

$$\psi_*(\Delta_{\Gamma_H \rho_\gamma}) = \Delta_{\Gamma_H \mathfrak{n}_-(\gamma \cdot \infty)}.$$

In the remainder of the proof of Corollary 4.7, we use the same normalization of the Patterson–Sullivan measures $(\mu_x)_{x \in V\mathbb{X}_v}$ as in [**BPP**, §15.3]. Since we normalized $\mu_{\Gamma_H \backslash H}$ to be a probability measure, it follows from [**BPP**, Proposition 15.3(2)] that, for $x \in \Gamma_H \backslash H$,

$$\psi_*(\sigma^+_{\mathcal{D}^-}) = (\varphi^{-1})_*(\sigma^+_{\mathcal{D}^-}) = \frac{q^{\mathfrak{g}-1}}{q-1}\, \mu_{\Gamma_H \backslash H}. \tag{36}$$

With ζ_K the Dedekind zeta function of K (see, for instance, [**Gos**, §7.8] or [**Ros**, §5]), by [**BPP**, Proposition 15.3(1)], we have

$$\|m_{\rm BM}\| = 2\, \zeta_K(-1)\, \frac{q_v + 1}{q_v}.$$

By [**BPP**, Eq. (14.3)], the subgroup $H \cap \Gamma = \mathfrak{n}_-(R_v)$ has index $|R_v^\times| = q - 1$ in Γ_H. The map from the set $\{(x, y) \in R_v \times R_v : xR_v + yR_v = R_v,\ y \neq 0\}$ to H given by $(x, y) \mapsto \mathfrak{n}_-(x/y)$ is $|R_v^\times|$-to-one onto its image. Hence, using the action by shears of R_v on $R_v \times R_v$ defined by $z \cdot (x, y) = (x + zy, y)$, by [**BPP**, Corollary 16.2] with $G = \mathrm{GL}_2(R_v)$ and $(x_0, y_0) = (1, 0)$ so that $m_{v,x_0,y_0} = q - 1$ by [**BPP**, Eq. (16.1)] with the notation of that book, for every $n_0 \in \mathbb{Z}$, as $n \to +\infty$, we have

$$\begin{aligned}\mathrm{Card}\, \mathcal{F}_{n-n_0} &= \frac{1}{|R_v^\times|^2}\, \mathrm{Card}\Big(R_v \backslash \Big\{(x, y) \in R_v \times R_v : \begin{array}{l} xR_v + yR_v = R_v \\ 0 < |y|_v \leq q_v^{n-n_0} \end{array} \Big\}\Big) \\ &\sim \frac{q^{2\mathfrak{g}-2}\, q_v^3}{(q-1)^2\, (q_v^2 - 1)\, (q_v + 1)\, \zeta_K(-1)}\, q_v^{2n-2n_0}.\end{aligned}$$

For all $n \in \mathbb{Z}$ and $[\gamma] \in \Gamma_{\mathcal{H}_\infty} \backslash \Gamma / \Gamma_{\mathcal{H}_\infty}$ outside a finite subset, we have

$$d(\mathcal{H}_\infty, \gamma \cdot \mathcal{H}_\infty) > 0, \quad m_\gamma = 1 \quad \text{and} \quad (\varphi^{-1})_*(\Delta_{\Gamma \mathfrak{g}^{2n} \rho_\gamma}) = \Delta_{\Gamma_{r_\gamma} \Phi^{2n} M}.$$

By equations (14), (35) and (36), with dm the counting measure on $\mathbb{Z}$, for every $n_0 \in \mathbb{Z}$, for $y \in \Gamma_H \backslash H$ and $m \geq n_0$, we have

$$\begin{aligned} d((\varphi^{-1})_*(\mu^{0+}_{\mathcal{D}^+, 2n_0\ |\ \Gamma \backslash \breve{\mathcal{G}}_{\rm even} \mathbb{X}_v}))(\Theta(y\, \Phi^{-2m})) &= \|\sigma^-_{\mathcal{D}^+}\|\, d\mu_{\Gamma_H \backslash H}(y)\, e^{-(\ln q_v)\, 2m}\, dm \\ &= \frac{q^{\mathfrak{g}-1}}{q-1}\, d\mu_{\Gamma_H \backslash H}(y)\, q_v^{-2m}\, dm. \end{aligned}$$

The end of the proof of Corollary 4.7 now proceeds like that of Corollary 4.1, replacing Theorem 3.3(1) by Theorem 3.3(3). □

Acknowledgement. The authors thank for its support the French-Finnish CNRS IEA PaCap.

References

[AC] J. Athreya and Y. Cheung. A Poincaré section for the horocycle flow on the space of lattices. *Int. Math. Res. Not. IMRN* **2014** (2014), 2643–2690.

[BH] M. R. Bridson and A. Haefliger. *Metric Spaces of Non-positive Curvature (Grundlehren der mathematischen Wissenschaften, 319)*. Springer Verlag, Berlin, 1999.

[BPP] A. Broise-Alamichel, J. Parkkonen and F. Paulin. *Equidistribution and Counting under Equilibrium States in Negative Curvature and Trees. Applications to non-Archimedean Diophantine Approximation (Progress in Mathematics, 329)*. Birkhäuser, Cham, 2019; with an Appendix by J. Buzzi.

[CI] K. Corlette and A. Iozzi. Limit sets of discrete groups of isometries of exotic hyperbolic spaces. *Trans. Amer. Math. Soc.* **351** (1999), 1507–1530.

[EGM] J. Elstrodt, F. Grunewald and J. Mennicke. *Groups acting on Hyperbolic Space: Harmonic Analysis and Number Theory (Springer Monographs in Mathematics)*. Springer Verlag, Berlin, 1998.

[EM] A. Eskin and C. McMullen. Mixing, counting, and equidistribution in Lie groups. *Duke Math. J.* **71** (1993), 181–209.

[Gol] W. M. Goldman. *Complex Hyperbolic Geometry*. Oxford University Press, Oxford, 1999.

[Gos] D. Goss. *Basic Structures of Function Field Arithmetic (Ergebnisse der Mathematik und ihrer Grenzgebiete, 35)*. Springer Verlag, Berlin, 1996.

[HaW] G. H. Hardy and E. M. Wright. *An Introduction to the Theory of Numbers*, 6th edn. Oxford University Press, Oxford, 2008.

[Hee] B. Heersink. Equidistribution of Farey sequences on horospheres in covers of $\mathrm{SL}(n+1, \mathbb{Z}) \backslash \mathrm{SL}(n+1, \mathbb{R})$ and applications. *Ergod. Th. & Dynam. Sys.* **41** (2021), 471–493.

[HeP] S. Hersonsky and F. Paulin. Diophantine approximation on negatively curved manifolds and in the Heisenberg group. *Rigidity in Dynamics and Geometry (Cambridge, 2000)*. Eds. M. Burger and A. Iozzi. Springer Verlag, Berlin, 2002, pp. 203–226.

[KiP] I. Kim and J. Parker. Geometry of quaternionic hyperbolic manifolds. *Math. Proc. Cambridge Philos. Soc.* **135** (2003), 291–320.

[KlM] D. Kleinbock and G. Margulis. Bounded orbits of nonquasiunipotent flows on homogeneous spaces. *Sinai's Moscow Seminar on Dynamical Systems (American Mathematical Society Translations – Series 2, 171)*. Eds. L. A. Bunimovich, B. M. Gurevich and Y. B. Pesin. American Mathematical Society, Providence, RI, 1996, pp. 141–172.

[Li] H. Li. Effective limit distribution of the Frobenius numbers. *Compos. Math.* **151** (2015), 898–916.

[Lut] C. Lutsko. Farey sequences for thin groups. *Int. Math. Res. Not. IMRN* **2022** (2022), 11642–11689.

[Mar1] J. Marklof. Horospheres and Farey fractions. *Dynamical Numbers—Interplay Between Dynamical Systems and Number Theory (Contemporary Mathematics, 532)*. Eds. S. Kolyada, Y. Manin, M. Möller, P. Moree and T. Ward. American Mathematical Society, Providence, RI, 2010, pp. 97–106.

[Mar2] J. Marklof. The asymptotic distribution of Frobenius numbers. *Invent. Math.* **181** (2010), 179–207.

[Mar3] J. Marklof. Fine-scale statistics for the multidimensional Farey sequence. *Limit Theorems in Probability, Statistics and Number Theory (Springer Proceedings in Mathematics & Statistics, 42)*. Eds. P. Eichelsbacher, G. Elsner, H. Kösters, M. Löwe, F. Merkl and S. Rolles. Springer Verlag, Berlin, 2013, pp. 49—57.

[Mos] G. D. Mostow. *Strong Rigidity of Locally Symmetric Spaces (Annals of Mathematics Studies, 78)*. Princeton University Press, Princeton, NJ, 1973.

[OS1] H. Oh and N. Shah. The asymptotic distribution of circles in the orbits of Kleinian groups. *Invent. Math.* **187** (2012), 1–35.

[OS2] H. Oh and N. Shah. Equidistribution and counting for orbits of geometrically finite hyperbolic groups. *J. Amer. Math. Soc.* **26** (2013), 511–562.

[PaP1] J. Parkkonen and F. Paulin. Prescribing the behaviour of geodesics in negative curvature. *Geom. Topol.* **14** (2010), 277–392.

[PaP2] J. Parkkonen and F. Paulin. Skinning measure in negative curvature and equidistribution of equidistant submanifolds. *Ergod. Th. & Dynam. Sys.* **34** (2014), 1310–1342.

[PaP3] J. Parkkonen and F. Paulin. On the arithmetic of cross-ratios and generalised Mertens' formulas. *Numéro Spécial: Aux croisements de la géométrie hyperbolique et de l'arithmétique (Mathématiques, 23)*. Eds. F. Dal'Bo and C. Lecuire. Annales de la Faculté des Sciences de Toulouse, Toulouse, 2014, pp. 967–1022.

[PaP4] J. Parkkonen and F. Paulin. Counting arcs in negative curvature. *Geometry, Topology and Dynamics in Negative Curvature (ICM 2010 Satellite Conference, Bangalore) (London Mathematical Society Lecture Note Series, 425)*. Eds. C. S. Aravinda, T. Farrell and J.-F. Lafont. Cambridge University Press, Cambridge, 2016.

[PaP5] J. Parkkonen and F. Paulin. Counting common perpendicular arcs in negative curvature. *Ergod. Th. & Dynam. Sys.* **37** (2017), 900–938.

[PaP6] J. Parkkonen and F. Paulin. Counting and equidistribution in Heisenberg groups. *Math. Ann.* **367** (2017), 81–119.

[PaP7] J. Parkkonen and F. Paulin. A survey of some arithmetic applications of ergodic theory in negative curvature. *Ergodic Theory and Negative Curvature: CIRM Jean Morley Chair Subseries (Lecture Notes in Mathematics, 2164)*. Ed. B. Hasselblatt. Springer Verlag, Cham, 2017, pp. 293–326.

[PaP8] J. Parkkonen and F. Paulin. Counting and equidistribution in quaternionic Heisenberg groups. *Math. Proc. Cambridge Philos. Soc.* **173** (2022), 67–104.

[PaPS] F. Paulin, M. Pollicott and B. Schapira. *Equilibrium States in Negative Curvature (Astérisque, 373)*. Société Mathématique de France, Paris 2015.

[Pau] F. Paulin. Groupe modulaire, fractions continues et approximation diophantienne en caractéristique p. *Geom. Dedicata* **95** (2002), 65–85.

[Phi] Z. Philippe. Invariants globaux des variétés hyperboliques quaternioniques. *Doctoral thesis*, Université de Bordeaux, Dec. 2016. https://tel.archives-ouvertes.fr/tel-01661448.

[Rob] T. Roblin. Ergodicité et équidistribution en courbure négative. *Mém. Soc. Math. Fr. (N.S.)* **95** (2003), vi+96 pp.

[Ros] M. Rosen. *Number Theory in Function Fields (Graduate Texts in Mathematics, 210)*. Springer Verlag, New York, 2002.

[Sar] P. Sarnak. Asymptotic behavior of periodic orbits of the horocycle flow and Eisenstein series. *Comm. Pure Appl. Math.* **34** (1981), 719–739.

[Ser] J.-P. Serre. *Arbres, amalgames, SL_2 (Astérisque, 46)*, 3ème éd. corr. Société Mathématique de France, Paris, 1983.

[Tit] J. Tits. Reductive groups over local fields. *Automorphic Forms, Representations and L-functions (Corvallis, 1977), Part 1 (Proceedings of Symposia in Pure Mathematics, XXXIII)*. Eds. A. Borel and W. Casselman. American Mathematical Society, Providence, RI, 1979, pp. 29–69.

[Vig] M. F. Vignéras. *Arithmétique des algèbres de quaternions (Lecture Notes in Mathematics, 800)*. Springer Verlag, Berlin, 1980.

[Wei] A. Weil. On the analogue of the modular group in characteristic p. *Functional Analysis and Related Fields (Chicago, 1968)*. Ed. F. E. Browder. Springer Verlag, Berlin, 1970, pp. 211–223.