BULL. AUSTRAL. MATH. SOC. 04A20, 05A05
VOL. 7 (1972), 353-358.

Transversals of squares

N.H. Williams

Let κ and λ be cardinal numbers. Take any family $A = \{A_\nu ;\ \nu \in N\}$ where each A_ν is a product $A_\nu = B_\nu \times C_\nu$ with $|B_\nu| = |C_\nu| = \aleph_\alpha$, such that if $B \times C \subseteq A_\mu \times A_\nu$ (for $\mu \neq \nu$) then $|B|, |C| < \lambda$. We investigate under what conditions on α, κ, λ and $|N|$ there will be a set T with $1 \leq |T \cap A_\nu| < \kappa$ for each ν.

This note discusses the following question. Let κ and λ be cardinals (finite or infinite). Let $\{A_\nu ;\ \nu \in N\}$ be a family of sets where each A_ν is of the form $B_\nu \times C_\nu$ with $|B_\nu| = |C_\nu| = \aleph_\alpha$, such that if $B \times C \subseteq A_\mu \times A_\nu$ (for $\mu \neq \nu$) then $|B|, |C| < \lambda$. For which values of α, κ, λ and $|N|$ is there a set T such that $1 \leq |T \cap A_\nu| < \kappa$ for each ν in N ?

The corresponding question when one supposes that $|A_\nu| = \aleph_\alpha$ and $|A_\mu \cap A_\nu| < \lambda$ if $\mu \neq \nu$ has been extensively discussed by Erdös and Hajnal [1]. We shall use methods based on those in [1] to obtain the positive results here.

The Generalized Continuum Hypothesis will be assumed throughout.

NOTATION. Given a set A, define the *two-dimensional cardinality* $\|A\|$ of A (when it exists) by $\|A\| = \kappa$ if there are B, C with $|B| = |C| = \kappa$ for which $B \times C \subseteq A$, and the same is not true of any B', C' with $|B'| = |C'| > \kappa$. A family $A = \{A_\nu ;\ \nu \in N\}$ will be called

Received 5 June 1972.

a $\kappa\times\kappa$-*family* if each A_ν is of the form $B_\nu \times C_\nu$ where $|B_\nu| = |C_\nu| = \kappa$. Define $\mathrm{Sq}(A)$ to be the least cardinal λ such that $\|A_\mu \cap A_\nu\| < \lambda$ (for $\mu \neq \nu$ from N). A set T will be called a λ-*transversal* of A if $1 \leq |T \cap A_\nu| < \lambda$ for each A_ν in A . Here we always suppose $\lambda < |A_\nu|$, since otherwise possibly $T \cap A_\nu = A_\nu$. For a set A , put $\mathrm{dom}(A) = \{a;\ \exists b(\langle a, b\rangle \in A)\}$ and $\mathrm{codom}(A) = \{b;\ \exists a(\langle a, b\rangle \in A)\}$.

Cardinals will be identified with initial ordinals. If κ is a cardinal, $\mathrm{Cf}(\kappa)$ is defined to be the least λ such that κ can be written as a union of λ sets each of power less than κ . Thus κ is regular just when $\mathrm{Cf}(\kappa) = \kappa$, and $\mathrm{Cf}(\kappa) = 2$ when κ is finite (and $\kappa > 1$). Define κ^+ to be the least cardinal greater than κ .

We start by making the following trivial observation.

LEMMA 1. *Let* R, S *be sets of power* $\aleph_\alpha$. *Let* $A = \{A_\nu;\ \nu < \kappa\}$ *be an* $\aleph_\alpha\times\aleph_\alpha$-*family. Suppose that either*

(i) $\kappa < \aleph_\alpha$ *and for some* $\lambda < \aleph_\alpha$, $\|A_\nu \cap (R\times S)\| < \lambda$ *for each* ν , *or*

(ii) $\kappa < \mathrm{Cf}(\aleph_\alpha)$ *and* $\|A_\nu \cap (R\times S)\| < \aleph_\alpha$ *for each* ν .

Then $R \times S \nsubseteq \bigcup A$.

Proof. Suppose *(i)* to hold. For each ν , either $|R \cap \mathrm{dom}(A_\nu)| < \lambda$ or $|S \cap \mathrm{codom}(A_\nu)| < \lambda$. Since $|\bigcup\{R\cap\mathrm{dom}(A_\nu);\ |R\cap\mathrm{dom}(A_\nu)| < \lambda\}| \leq \lambda\kappa < \aleph_\alpha$, we may choose x in $R - \bigcup\{\mathrm{dom}(A_\nu);\ |R\cap\mathrm{dom}(A_\nu)| < \lambda\}$. Similarly we may choose y in $S - \bigcup\{\mathrm{codom}(A_\nu);\ |S\cap\mathrm{codom}(A_\nu)| < \lambda\}$. But then $\langle x, y\rangle \in (R\times S) - \bigcup A$, so $R \times S \nsubseteq \bigcup A$. The situation is similar if *(ii)* holds.

COROLLARY 2. *Let* $A = \{A_\nu;\ \nu < \kappa\}$ *be any* $\aleph_\alpha\times\aleph_\alpha$-*family such that either*

(i) $\kappa < \aleph_\alpha$ *and* $\mathrm{Sq}(A) < \aleph_\alpha$, *or*

https://doi.org/10.1017/S0004972700045196 Published online by Cambridge University Press

(ii) $\kappa < \mathrm{Cf}(\aleph_\alpha)$ *and* $\mathrm{Sq}(A) \le \aleph_\alpha$. *Then* A *has a* 2*-transversal. (Note that a* 2*-transversal is a genuine transversal, in the usual sense.)*

Proof. By the Lemma, for each ν we may choose x_ν in $A_\nu - \cup\{A_\mu;\ \mu \ne \nu\}$. Then $T = \{x_\nu;\ \nu < \kappa\}$ is a 2-transversal of A .

Examples show that if κ is increased above the limits in Corollary 2, then a 2-transversal may not always exist. For *(i)*, the following result from [*1*, §4.5] may be used. Given $\lambda < \aleph_\alpha$, there is a family $\{B_\nu;\ \nu < \aleph_\alpha\}$ of sets each of power $\aleph_\alpha$ such that $|B_\mu \cap B_\nu| < \lambda$ if $\mu \ne \nu$, for which there is no set T such that $1 \le |B_\nu \cap T| < \lambda$ for each ν . If one now takes the family $A = \{\aleph_\alpha \times B_\nu;\ \nu < \aleph_\alpha\}$, then clearly A is an $\aleph_\alpha \times \aleph_\alpha$-family with $\mathrm{Sq}(A) \le \lambda$, and yet A has no λ-transversal. Using the same method with the appropriate family constructed in [*1*, §4.4] gives an $\aleph_\alpha \times \aleph_\alpha$-family $A = \{A_\nu;\ \nu < \mathrm{Cf}(\aleph_\alpha)\}$ with $\mathrm{Sq}(A) = \aleph_\alpha$ which has no λ-transversal for any λ with $\lambda < \mathrm{Cf}(\aleph_\alpha)$.

These examples show that the results in the next two theorems are the best that can be expected.

THEOREM 3. *Let* $A = \{A_\nu;\ \nu < \mathrm{Cf}(\aleph_\alpha)\}$ *be an* $\aleph_\alpha \times \aleph_\alpha$*-family with* $\mathrm{Sq}(A) \le \aleph_\alpha$. *Then there is a* $\mathrm{Cf}(\aleph_\alpha)$*-transversal for* A .

Proof. Given the family A , we shall construct a $\mathrm{Cf}(\aleph_\alpha)$-transversal T . By Lemma 1 *(ii)*, for each μ with $\mu < \mathrm{Cf}(\aleph_\alpha)$ we have that $A_\mu \nsubseteq \cup\{A_\nu;\ \nu < \mu\}$, and so we can choose $x_\mu \in A_\mu - \cup\{A_\nu;\ \nu < \mu\}$. Put $T = \{x_\mu;\ \mu < \mathrm{Cf}(\aleph_\alpha)\}$. Then for any ν it follows that $T \cap A_\nu \subseteq \{x_\mu;\ \mu \le \nu\}$, and so $|T \cap A_\nu| < \mathrm{Cf}(\aleph_\alpha)$. Thus T is indeed a $\mathrm{Cf}(\aleph_\alpha)$-transversal for A .

We need to make use of the following result, the proof of which comes from a simple modification to the proof of an analogous result of Tarski [*2*, Théoreme 5].

LEMMA 4. *Let* S *be a set of power* $\aleph_\alpha$, *and suppose that* $\mathrm{cf}(\aleph_\alpha) \neq \mathrm{cf}(\lambda)$. *Then* $S \times S$ *cannot be decomposed into a family* A *of more than* $\aleph_\alpha$ *subsets where* $\mathrm{Sq}(A) \leq \lambda$, *with* $\|A\| \geq \lambda$ *for each* A *in* A.

LEMMA 5. *Let* λ *and* β *be given, and if* $\beta = \gamma + 1$ *suppose that* $\mathrm{cf}(\aleph_\gamma) \neq \mathrm{cf}(\lambda)$. *Let* S *be any set with* $|S| < \aleph_\beta$, *and suppose* $B = \{B_\nu;\ \nu \in N\}$ *is a family with* $\mathrm{Sq}(B) \leq \lambda$, *where always* $B_\nu \subseteq S \times S$ *and* $\|B_\nu\| \geq \lambda$. *Then* $|N| < \aleph_\beta$.

Proof. Since $|B| \leq |S|^+$, if $|S|^+ < \aleph_\beta$ then certainly $|N| < \aleph_\beta$. And if $|S|^+ = \aleph_\beta$, then Lemma 4 applies to give the result.

THEOREM 6. *Let* λ *and* α *be given, and if* $\alpha = \gamma + 1$ *suppose that either* $\lambda = \aleph_\gamma$ *or* $\mathrm{cf}(\lambda) \neq \mathrm{cf}(\aleph_\gamma)$. *Then every* $\aleph_\alpha \times \aleph_\alpha$*-family* A *of* $\aleph_\alpha$ *sets with* $\mathrm{Sq}(A) \leq \lambda$ *has a* λ^+*-transversal.*

Proof. The case when $\lambda^+ = \aleph_\alpha$ is covered by Theorem 3, so we may suppose $\lambda^+ < \aleph_\alpha$. Take a suitable family $A = \{A_\nu;\ \nu < \aleph_\alpha\}$.

Write $A \sim X$ for $\{\langle a, b\rangle \in A;\ a \notin \mathrm{dom}(X)$ and $b \notin \mathrm{codom}(X)\}$.

Use transfinite induction to define elements x_μ when $\mu < \aleph_\alpha$ as follows. Put $X_\mu = \{x_\nu;\ \nu < \mu\}$ and $X^*_\mu = \mathrm{dom}(X_\mu) \times \mathrm{codom}(X_\mu)$. Write

$$A'_\mu = (A_\mu \sim X_\mu) - \bigcup\{A_\rho;\ |A_\rho \cap X_\mu| \geq \lambda\}.$$

Choose x_0 in A_0. When $\mu > 0$, if $A'_\mu = \emptyset$ put $x_\mu = x_0$; otherwise choose x_μ from A'_μ.

Since $|X_\mu| < \aleph_\alpha$ when $\mu < \aleph_\alpha$, by applying Lemma 5 to $B_\mu = \{A_\rho \cap X^*_\mu;\ \|A_\rho \cap X^*_\mu\| \geq \lambda\}$, it follows that $|B_\mu| < \aleph_\alpha$. The choice of x_ν ensures that $\|A_\rho \cap X^*_\mu\| \geq \lambda$ exactly when $|A_\rho \cap X_\mu| \geq \lambda$. Now since $\|A_\mu\| = \aleph_\alpha$, also $\|A_\mu \sim X_\mu\| = \aleph_\alpha$ and so Lemma 1 *(i)* shows that

https://doi.org/10.1017/S0004972700045196 Published online by Cambridge University Press

$A_\mu \sim X_\mu \nsubseteq \bigcup\{A_\rho;\ |A_\rho \cap X_\mu| \geq \lambda\}$, unless of course $|A_\mu \cap X_\mu| \geq \lambda$. Thus either $|A_\mu \cap X_\mu| \geq \lambda$ or else $x_\mu \in A_\mu$.

Put $T = \{x_\nu;\ \nu < \aleph_\alpha\}$; so always $|T \cap A_\nu| \geq 1$. And for any A_μ , if for some ρ it happens that $|A_\mu \cap \{x_\nu;\ \nu < \rho\}| = \lambda$, then for all ν , where $\nu \geq \rho$, either $x_\nu = x_0$ or $x_\nu \notin A_\mu$. Hence always $|T \cap A_\mu| \leq \lambda$, and so T is a λ^+-transversal for A .

In the case $\alpha = \gamma + 1$, I do not know if the restriction on λ in Theorem 6 can be lifted. However, by changing B_μ to $\{A_\rho \cap X^*_\mu;\ \|A_\rho \cap X^*_\mu\| > \lambda\}$, one establishes the following result.

THEOREM 7. *For all* λ , *if* A *is any* $\aleph_\alpha \times \aleph_\alpha$*-family of* $\aleph_\alpha$ *sets with* $\mathrm{Sq}(A) \leq \lambda$, *then* A *has a* λ^{++}*-transversal.*

In the case of an $\aleph_\alpha \times \aleph_\alpha$-family of power greater than $\aleph_\alpha$, one cannot always expect to find even an $\aleph_\alpha$-transversal. This contrasts with the results of [1]. Consider the following $\aleph_\alpha \times \aleph_\alpha$-family A of $\aleph_{\alpha+1}$ sets with $\mathrm{Sq}(A) = 2$, for which any transversal must meet some member of A in a set of power $\aleph_\alpha$.

Let S_ν for $\nu < \aleph_{\alpha+1}$ be pairwise disjoint sets each of power $\aleph_\alpha$. Now put

$$R = S_0 \times \left\{R;\ R \subseteq \bigcup\{S_\nu;\ \nu < \aleph_{\alpha+1}\} \text{ and } |R| = \aleph_\alpha\right\} ,$$

so $|R| = \aleph_{\alpha+1}$. Thus there is an enumeration $\{\langle y_\nu, R_\nu\rangle;\ \nu < \aleph_{\alpha+1}\}$ of R . Put

$$A = \{S_0 \times S_\nu;\ \nu < \aleph_{\alpha+1}\} \cup \left\{\left(\{y_\nu\} \cup S_{\nu+1}\right) \times R_\nu;\ \nu < \aleph_{\alpha+1}\right\} .$$

Then A is indeed an $\aleph_\alpha \times \aleph_\alpha$-family with $|A| = \aleph_{\alpha+1}$ and $\mathrm{Sq}(A) = 2$. However, let T be any transversal of A . In particular, T meets each of the sets $S_0 \times S_\nu$; choose $\langle x_\nu, y_\nu\rangle \in T \cap \left(S_0 \times S_\nu\right)$. Since $|S_0| = \aleph_\alpha$, there are x in S_0 and $H \subseteq \aleph_{\alpha+1}$ with $|H| = \aleph_{\alpha+1}$ such

https://doi.org/10.1017/S0004972700045196 Published online by Cambridge University Press

that $x_\nu = x$ for all ν in H . Choose R with $|R| = \aleph_\alpha$ such that $R \subseteq \{y_\nu;\ \nu \in H\}$. Then there is μ with $\mu < \aleph_{\alpha+1}$ for which $\langle x, R\rangle = \langle x_\mu, R_\mu\rangle$. But then $|T\cap((\{y_\nu\}\cup S_{\mu+1}) \times R_\mu)| = \aleph_\alpha$.

In view of this negative result, one may wish to modify the definition of a transversal. Let us call T a $\lambda\times\lambda$-transversal of A if $1 \leq \|T\cap A\| < \lambda$ for each A in A . It is however trivial that if $A = \{A_\nu;\ \nu < \aleph_\alpha\}$ is any family of $\aleph_\alpha$ sets with $\|A_\nu\| = \aleph_\alpha$, then A has a 2×2-transversal. One chooses inductively x_ν, y_ν for $\nu < \aleph_\alpha$ so that $\langle x_\nu, y_\nu\rangle \in A_\nu$, $x_\nu \in \mathrm{dom}(A_\nu) - \{x_\mu;\ \mu < \nu\}$ and $y_\nu \in \mathrm{codom}(A_\nu) - \{y_\mu;\ \mu < \nu\}$. Then $T = \{\langle x_\nu, y_\nu\rangle;\ \nu < \aleph_\alpha\}$ is a 2×2-transversal of A .

For $\aleph_\alpha\times\aleph_\alpha$-families of more than $\aleph_\alpha$ sets one can now establish the following theorem by modifying the construction in §5 of [1].

THEOREM 8. *Let* α *and* β *be given. For all* γ *with* $\gamma \leq \alpha + \mathrm{cf}(\aleph_\beta)$, *if* $A = \{A_\nu;\ \nu < \aleph_\gamma\}$ *is any* $\aleph_\alpha\times\aleph_\alpha$*-family with* $\mathrm{Sq}(A) \leq \aleph_\beta$ *then there is an* $\aleph_{\beta+1}\times\aleph_{\beta+1}$*-transversal for* A .

In fact Theorem 8 is true under the weaker assumption that $|A_\nu| = \|A_\nu\| = \aleph_\alpha$ (for each ν).

References

[1] P. Erdös and A. Hajnal, "On a property of families of sets", *Acta Math. Acad. Sci. Hungar.* 12 (1961), 87-123.

[2] Alfred Tarski, "Sur la décomposition des ensembles en sous-ensembles presque disjoints", *Fund. Math.* 14 (1929), 205-215.

Department of Mathematics,
Monash University,
Clayton,
Victoria.

https://doi.org/10.1017/S0004972700045196 Published online by Cambridge University Press