SOME REMARKS ON PRAMARTS AND MILS

by ZHEN-PENG WANG† and XING-HONG XUE

(Received 10 January, 1992)

1. Notations and summary. Let F be a Banach space, $(\Omega, \mathcal{F}, P)$ a fixed probability space, D a directed set filtering to the right with the order $\leq$, and $(\mathcal{F}_t, D)$ a stochastic basis of $\mathcal{F}$, i.e. $(\mathcal{F}_t, D)$ is an increasing family of sub-σ-algebras of $\mathcal{F}: \mathcal{F}_s \subset \mathcal{F}_t$ for any $s,t \in D$ and $s \leq t$. Throughout this paper, (X_t) is an F-valued, $(\mathcal{F}_t)$-adapted sequence, i.e. X_t is $\mathcal{F}_t$-measurable, $t \in D$. We also assume that $X_t \in L^1$, i.e. $\int \|X_t\| < \infty$. We use $I(H)$ to denote the indicator function of an event H. Let ∞ be a such element: $t < \infty$, $t \in D$, $\bar{D} = D \cup \infty$, and $\mathcal{F}_\infty = \sigma\left(\bigcup_{t \in D} \mathcal{F}_t\right)$. A stopping time is a map $\tau: \Omega \to \bar{D}$ such that $(\tau \leq t) \in \mathcal{F}_t$, $t \in D$. A stopping time τ is called simple (countable) if it takes finitely (countably) many values in $D(\bar{D})$. Let T and T_c be the sets of simple and countable stopping times respectively and $T_f = \{\tau \in T_c : \tau < \infty \text{ a.s.}\}$. Clearly, $(T, \leq)$ and $(T_f, \leq)$ are directed sets filtering to the right. For $\tau \in T_c$, let

$$\mathcal{F}_\tau = \{H \in \mathcal{F} : H(\tau = t) \in \mathcal{F}_t \text{ for all } t \in D\}, \qquad X_\tau = \sum_{t \in D} X_t I(\tau = t),$$

and

$$\mathcal{B} = \left\{(X_t) : \sup_{\tau \in T} \int \|X_\tau\| < \infty\right\},$$

$$\mathcal{C} = \left\{(X_t) : \int_{(\tau<\infty)} \|X_\tau\| < \infty,\ \tau \in T_c\right\},$$

$$\bar{\mathcal{C}} = \left\{(X_t) : \text{there is } \sigma \in T_f \text{ such that } \int_{(\tau<\infty)} \|X_\tau\| < \infty,\ \sigma \leq \tau \in T_c\right\},$$

$$\mathcal{S} = \{(X_t) : (X_\tau, \tau \in T) \text{ converges stochastically (i.e. in probability) in the norm topology}\},$$

$$\mathcal{E} = \{(X_t) : (X_\tau, \tau \in T) \text{ converges essentially in the norm topology}\}.$$

Clearly, $\bar{\mathcal{C}} \supset \mathcal{C} \supset \mathcal{B}$ and $\mathcal{E} \subset \mathcal{S}$. If $(\mathcal{F}_t)$ satisfies the Vitali condition V, particularly, if $D = \mathbf{N} \equiv \{1, 2, \ldots\}$, then $(X_t) \in \mathcal{E}$ if and only if $(X_t) \in \mathcal{S}$ (cf. **[18]**, **[23]**, and **[20]**). Hence, in this case, $\mathcal{S} = \mathcal{E}$.

Mucci (**[21]**, **[22]**) and Millet and Sucheston (**[19]**, **[20]**) introduced the notations of martingales in the limit, pramarts, and subpramarts, generalizing those of martingales, amarts (Edgar and Sucheston **[7]**), uniform amarts (Bellow **[2]**), and submartingales, and provided some sufficient conditions to ensure that $(X_t) \in \mathcal{S}$ (cf. monographs **[10]** and **[15]**).

Definition 1 (**[21]**, **[20]**). A stochastic process $(X_t, \mathcal{F}_t, D)$ is called a *martingale in the limit* if

$$\operatorname{ess}\lim_{t \in D} \operatorname{ess}\sup_{t \leq s \in D} \|X_t - E(X_s \mid \mathcal{F}_t)\| = 0 \text{ a.s.}$$

† Research supported by the National Natural Science Foundation of China.

https://doi.org/10.1017/S0017089500009800 Published online by Cambridge University Press

DEFINITION 2 ([**19**], [**20**]). (i) A stochastic process $(X_t, \mathcal{F}_t, D)$ is called a *pramart* if

$$\text{s}.\lim_{\sigma\le\tau;\sigma,\tau\in T}\|X_\sigma - E(X_\tau \mid \mathcal{F}_\sigma)\| = 0,$$

i.e., for each $\epsilon > 0$, there exists $\sigma_0 \in T$ such that, for all σ, $\tau \in T$ and $\sigma_0 \le \sigma \le \tau$,

$$P\{\|X_\sigma - E(X_\tau \mid \mathcal{F}_\sigma)\| > \epsilon\} < \epsilon.$$

(ii) A stochastic process $(X_t, \mathcal{F}_t, D)$ is called a *subpramart*, if F is a Banach lattice, and if

$$\text{s}.\lim_{\sigma\le\tau;\sigma,\tau\in T}\|(X_\sigma - E(X_\tau \mid \mathcal{F}_\sigma))^+\| = 0.$$

Millet and Sucheston [**20**] proved that if the Vitali condition V holds (it is also necessary), then every pramart (X_t) is a martingale in the limit. There is a more general class of adapted processes.

DEFINITION 3 [**31**]. (i) A stochastic process $(X_t, \mathcal{F}_t, D)$ is called a *mil* if

$$\text{s}.\lim_{\sigma\le t;\sigma\in T;t\in D}\|X_\sigma - E(X_t \mid \mathcal{F}_\sigma)\| = 0,$$

i.e., for each $\epsilon > 0$, there exists $\sigma_0 \in T$ such that, for all $\sigma_0 \le \sigma \in T$ and $\sigma \le t \in D$,

$$P\{\|X_\sigma - E(X_t \mid \mathcal{F}_\sigma)\| > \epsilon\} < \epsilon.$$

The mil defined here was called mil(3) in [**31**], and when $D = \mathbf{N}$, it is equivalent to Talagrand's mil [**28**] (see [**31**]). Clearly, pramart $\Rightarrow$ mil, and martingale in the limit $\Rightarrow$ mil [**31**]. The following theorem was proved by Millet and Sucheston [**19**].

THEOREM A [**19**]. *Let $(X_n) \equiv (X_n, n \in \mathbf{N})$ be a pramart of class $\mathcal{B}$. If F has the Radon–Nikodym property, then $(X_n) \in \mathcal{E}$.*

In a real-valued case Millet and Sucheston [**20**] proved this result.

THEOREM B [**18**]. *Let (X_t) be a real-valued subpramart satisfying condition* (d), i.e.

$$\liminf_D \int X_t^- + \liminf_D \int X_t^+ < \infty.$$

Then $(X_t) \in \mathcal{S}$.

Later, Egghe [**9**], [**10**], Słaby [**26**], [**27**] and Frangos [**11**] worked on a problem raised by Sucheston: if F has the Radon–Nikodym property, does every L^1-bounded pramart (X_t) belong to $\mathcal{S}$? When F is a Banach lattice, Słaby and Frangos proved the following positive subpramart convergence theorem, extending Heinich's positive submartingale convergence theorem [**16**].

THEOREM C ([**11**] and [**27**]). *Let F be a Banach lattice with the Radon–Nikodym property. If (X_t) is a positive subpramart satisfying* $\liminf_t \int \|X_t\| < \infty$, *and if $(\mathcal{F}_t)$ satisfies the Vitali condition V and F is separable, then $(X_t) \in \mathcal{E}$.*

They also solved Sucheston's problem when F is a separable dual (Frangos [**11**]), or a weakly sequentially complete space (Słaby [**27**]). The following theorem completely solved the problem.

THEOREM D. (i) ([**28**], also see [**31**].) *Let* (X_t) *be a mil satisfying* $\liminf_t \int \|X_t\| < \infty$. *If F has the Radon–Nikodym property, then* $(X_t) \in \mathcal{S}$.

(ii) ([**31**].) *If* $F = R_1$ *and* (X_t) *is a mil satisfying*

$$\liminf_t \left\{ \min \int X_t^+, \int X_t^- \right\} < \infty,$$

then $(X_t) \in \mathcal{S}$.

Part (ii) of Theorem D is an improvement of Mucci's L^1-bounded, real-valued martingale in the limit convergence theorem [**22**].

Suggested by Chow's submartingale convergence theorem ([**4**], also see Remark 2), Yamasaki [**34**] proved the following theorem.

THEOREM E [**34**]. *Suppose that* (X_n) *is a real-valued martingale in the limit and* $(X_n) \in \mathscr{C}$. *Then* $(X_n) \in \mathscr{E}$.

Yamasaki [**34**] also provided an example, showing that there exists a real-valued martingale in the limit which belongs to $\mathscr{C}$, but for which $\int X_n^+ \uparrow \infty$.

In this paper we show that Theorem E can be extended to vector-valued mils, and for pramarts the condition $\liminf_t \int \|X_t\| < \infty$ in Theorem D can be weakened to $\liminf_{\tau \in T} \int \|X_\tau\| < \infty$. On the other hand, it is of interest to characterize a subclass of $\mathcal{S}$. Using Bellow's uniform amart convergence theorem, we can get: if F has the Radon–Nikodym property and the net $(X_\tau, \tau \in T)$ is uniformly integrable, then (X_n) is a uniform amart if and only if $(X_n) \in \mathscr{E}$ (Gut [**14**]). Krengel and Sucheston [**17**] also provided an example showing that there exists a real-valued $(X_n) \in \mathscr{B} \cap \mathscr{E}$ such that (X_n) is uniformly integrable, but (X_n) is not an amart. However, Talagrand [**28**] and Wang and Xue [**31**] proved that if (X_t) is uniformly integrable, then $(X_t) \in \mathcal{S}$ if and only if (X_t) is a mil, and Xue [**32**] proved that the converse of Theorem A is true: if $(X_n) \in \mathcal{S} \cap \mathscr{B}$, then (X_n) is a pramart. In this paper we prove that the converse of Theorem E is also true. More specifically, we have Theorem 1.

THEOREM 1. *Suppose that F has the Radon–Nikodym property.*

(a) *If* $(X_t) \in \bar{\mathscr{C}}$, *then*

(i) $(X_t) \in \mathcal{S} \Leftrightarrow (X_t)$ *is a pramart* $\Leftrightarrow (X_t)$ *is a mil, and if the Vitali condition V holds,*

$$(X_t) \in \mathcal{S} \Leftrightarrow (X_t) \textit{ is a martingale in the limit};$$

(ii) *if* (X_t) *is a pramart (mil), so is* $(\|X_t\|)$, *and under the Vitali condition V, if* (X_t) *is a martingale in the limit, so is* $(\|X_t\|)$.

(b) *If* (X_t) *is a pramart satisfying* $\liminf_{\tau \in T} \int \|X_\tau\| < \infty$, *then* $(X_t) \in \mathcal{S}$.

Suppose that F is a Banach lattice. Then $(X_t) \in \mathcal{S} \Rightarrow (|X_t|) \in \mathcal{S}$, where

https://doi.org/10.1017/S0017089500009800 Published online by Cambridge University Press

$|X_t| = X_t^+ + X_t^-$. Since, for $x \in F$,

$$X_t \vee x = \frac{X_t + x + |X_t - x|}{2}, \qquad X_t \wedge x = \frac{X_t + x - |X_t - x|}{2} \tag{1}$$

(cf. [**25**, Proposition 2.5]), we have the following corollary.

COROLLARY 1. *Suppose that F is a Banach lattice with the Radon–Nikodym property. Then the set of pramarts (mils) of class $\bar{\mathscr{C}}$ is a vector lattice; and, under the Vitali condition V, the set of martingales in the limit of class $\bar{\mathscr{C}}$ is also a vector lattice.*

When F is a Banach lattice, we have similar results for subpramarts. The part (i) of the following theorem is an improvement of Theorem C.

THEOREM 2. *Let F be a Banach lattice with the Radon–Nikodym property and (X_t) a positive subpramart. If one of the following holds, then $(X_t) \in \mathscr{S}$:*

(i) $\liminf_{\tau \in T} \int \|X_\tau\| < \infty$;

(ii) *$(X_t) \in \bar{\mathscr{C}}$ and $\mathscr{F}_\infty$ is nonatomic or* $\mathrm{s}\,.\liminf_{\tau \in T} \|X_\tau\| < \infty$ *a.s., where*

$$\mathrm{s}\,.\liminf_{\tau \in T} \|X_\tau\| = \operatorname{ess\,sup}\left\{\xi : \lim_{\tau \in T} P(\|X_\tau\| < \xi) = 0\right\},$$

the stochastic lower limit of $(\|X_\tau\|, \tau \in T)$.

For real-valued processes, we show that condition (d) in Theorem B can be weakened to a one-sided condition and the requirement of being positive in Theorem 2 can be dropped.

THEOREM 3. *Let $F = R_1$.*

(a) *If* (i) *or* (ii) *holds, then $(X_t) \in \mathscr{S}$:*

(i) *(X_t) is a subpramart satisfying* $\liminf_{\tau \in T} \int X_\tau^+ < \infty$;

(ii) *(X_t) is a subpramart or a mil, $(X_t^+) \in \bar{\mathscr{C}}$, and $\mathscr{F}_\infty$ is nonatomic or* $\mathrm{s}\,.\liminf_{\tau \in T} \|X_\tau\| < \infty$ *a.s..*

(b) *If $(X_t) \in \mathscr{S}$ and $(X_t^-) \in \bar{\mathscr{C}}$, then (X_t) is a subpramart.*

Part of Theorem 3 was proved by Wang [**29**]. The proof here is new.

COROLLARY 2. *Suppose that $F = R_1$.*

(a) *If $(X_t^-) \in \bar{\mathscr{C}}$ and* $\liminf_{\tau \in T} \int X_\tau^+ < \infty$, *then $(X_t) \in \mathscr{S}$ if and only if (X_t) is a subpramart.*

(b) *If $(X_t) \in \bar{\mathscr{C}}$ and if* $\mathrm{s}\,.\liminf_{\tau \in T} \|X_\tau\| < \infty$ *a.s. or $\mathscr{F}_\infty$ is nonatomic, then*

$$(X_t) \in \mathscr{S} \Leftrightarrow (X_t) \text{ is a subpramart} \Leftrightarrow (X_t) \text{ is a pramart.}$$

Austin, Edgar, and Ionescu Tulcea [**1**] (also see [**7**]) proved that L^1-bounded, real-valued amarts form a vector lattice. This result was extended by Ghoussoub [**12**] to L^1-bounded, Banach lattice-valued order amarts when the Radon–Nikodym property holds, and by Schmidt [**24**] to L^1-bounded, l^1-valued uniform amarts. It is natural to ask: can we change "class $\bar{\mathscr{C}}$" to "L^1-bounded class" in Corollary 1? For martingales in the

https://doi.org/10.1017/S0017089500009800 Published online by Cambridge University Press

limit and mils, it is impossible. Indeed, Bellow and Dvoretzky **[3]** presented an example showing that there exists a uniformly integrable, real-valued martingale in the limit (X_n) such that $(|X_n|)$ is not a martingale in the limit, and Talagrand **[28]** constructed an L^1-bounded, real-valued martingale in the limit (X_n) with $X_n \to 0$ a.s. such that $(|X_n|)$ is not a mil. Hence, the set of L^1-bounded, real-valued martingales in the limit (mils) is not a vector lattice. Talagrand also presented an L^1-bounded, l^2-valued pramart (X_n) such that $(|X_n|)$ is not a pramart. However, Talagrand **[28]** and Wang **[29]** proved that the set of L^1-bounded, real-valued pramarts is a vector lattice. In fact, Talagrand and Wang proved that if (X_n) is a real-valued pramart satisfying $\liminf_n \int |X_n| < \infty$, then so is $(|X_n|)$. The following theorem is an improvement of their result.

THEOREM 4. *If (X_t) is a real-valued pramart satisfying* $\liminf_{\tau \in T} \min\{\int X_\tau^-, \int X_\tau^+\} < \infty$, *then $(|X_t|)$ is a pramart, hence, for each $\lambda \in R_1$, both $(X_t \vee \lambda)$ and $(X_t \wedge \lambda)$ are pramarts.*

In Section 2 we characterize real-valued pramarts and subpramarts via Snell's envelopes. We prove Theorems 1–4 in Section 3. In Section 4 we make some comments pointing out that: (i), for pramarts, the condition $\liminf_{\tau \in T} \int \|X_\tau\| < \infty$ is weaker than the condition $\liminf_t \int \|X_t\| < \infty$; (ii), in general, the condition $(X_t^-) \in \bar{\mathscr{C}}$ in Theorem 3 cannot be dropped; (iii) the condition $\liminf_{\tau \in T} \min\{\int X_\tau^+, \int X_\tau^-\} < \infty$ in Theorem 4 is necessary; (iv), in general, the set $\bar{\mathscr{C}}$ is larger than $\mathscr{C}$.

2. Snell's envelopes and characterizations of pramarts and subpramarts. In this section we assume that $F = R_1$. We use the following Snell's envelopes to characterize real-valued pramarts and subpramarts. For a real-valued process (X_t) we denote

$$Y_t = \operatorname{ess}\sup_{t \le \tau \in T} E(X_\tau \mid \mathscr{F}_t), \qquad P_t = \operatorname{ess}\sup_{t \leqq \tau \in T} E(X_\tau^+ \mid \mathscr{F}_t), \qquad R_t = \operatorname{ess}\inf_{t \le \tau \in T} E(X_\tau \mid \mathscr{F}_t).$$

The following lemma is well known in the martingale theory and the theory of optimal stopping (cf. **[5]**, **[13]**, and **[23]**).

LEMMA 1. *$(Y_\tau, \mathscr{F}_\tau, T)$ is a generalized supermartingale* (i.e. *Y_τ takes values in $(-\infty, \infty]$, $EY_\tau^- < \infty$ and $E(Y_\tau \mid \mathscr{F}_\sigma) \le Y_\sigma$ a.s. for all τ, $\sigma \in T$ and $\tau \ge \sigma$*), *and for any $\tau \in T$ there exist $(\tau_n) \subset T$ such that $\tau \le \tau_n$ and*

$$E(X_{\tau_n} \mid \mathscr{F}_\tau) \uparrow Y_\tau = \operatorname{ess}\sup\{E(X_\sigma \mid \mathscr{F}_\tau) : \tau \le \sigma \in T\}.$$

Therefore, if $\int |Y_t| < \infty$, $t \in D$, then $(-Y_t)$ is a subpramart. Moreover, if $\sup_t \int |Y_t| < \infty$, then (Y_t) is an amart, hence a pramart.

PROPOSITION 1. (i) *(X_t) is a subpramart if and only if $(X_\tau - R_\tau, \tau \in T)$ converges to zero in probability. In this case,* $\lim_{\tau \in T} P(R_\tau = -\infty) = 0$.

(ii) *(X_t) is a pramart if and only if $(Y_\tau - R_\tau, \tau \in T)$ converges to zero in probability. In this case,* $\lim_{\tau \in T} P(R_\tau = -\infty) = \lim_{\tau \in T} P(Y_\tau = \infty) = 0$.

Proof. For any $\tau \in T$, by Lemma 1, we can choose $\tau \le \tau_n \in T$ such that $E(X_{\tau_n} \mid \mathcal{F}_\tau) \downarrow R_\tau$. Since $X_t \ge R_t$ a.s.,

$$\sup_{\tau \le \sigma \in T} P(X_\tau - E(X_\sigma \mid \mathcal{F}_\tau) > \epsilon) \le P\Big(\operatorname{ess}\sup_{\tau \le \sigma \in T} (X_\tau - E(X_\sigma \mid \mathcal{F}_\tau)) > \epsilon\Big)$$

$$= P(X_\tau - R_\tau > \epsilon) = P(|X_\tau - R_\tau| > \epsilon) = \int \lim_n I(X_\tau - E(X_{\tau_n} \mid \mathcal{F}_\tau) > \epsilon)$$

$$= \lim_n \int I(X_\tau - E(X_{\tau_n} \mid \mathcal{F}_\tau) > \epsilon) \le \sup_{\tau \le \sigma \in T} P(X_\tau - E(X_\sigma \mid \mathcal{F}_\tau) > \epsilon).$$

Hence, (X_t) is a subpramart if and only if $(X_\tau - R_\tau, \tau \in T)$ converges to zero in probability. In this case, $\lim_{\tau \in T} P(R_\tau = -\infty) \le \lim_{\tau \in T} P(X_\tau - R_\tau > 1) = 0$. Finally (ii) follows from (i) and the symmetric property.

Corollary 3. *Suppose that $(X_\tau, \tau \in T)$ converges to zero in probability. Then*

(i) *(X_t) is a subpramart if and only if $(R_\tau, \tau \in T)$ converges to zero in probability;*

(ii) *if, in addition, (X_t) is nonnegative, then (X_t) is a pramart if and only if $(Y_\tau, \tau \in T)$ converges to zero in probability.*

Remark 1. When (X_t) is nonpositive, (i) in Proposition 1 was proved by Millet and Sucheston [**20**]. The proof here is adopted from their paper. When $D = \mathbf{N}$, (ii) in Corollary 3 is an analogue of Theorem 11 in [**28**].

3. Proofs of Theorems 1–4. To prove Theorems 1–4, we need the following lemmas. Let

$$W_t = \operatorname{ess}\sup_{t \le \tau \in T} E(\|X_\tau\| \mid \mathcal{F}_t), \qquad A_t = (W_t = \infty).$$

Definition 4 (cf. [**10**]). Let F be a Banach lattice. A stochastic process $(X_t, \mathcal{F}_t, D)$ is called a *GBT* (*a game which becomes better with time*), if

$$\text{s.} \lim_{t \le s; t, s \in D} \|(X_t - E(X_s \mid \mathcal{F}_t))^+\| = 0.$$

Lemma 2. (i) ([**33**]) *$A_t \subset A_s$ a.s. for all $s \le t$. Hence $A^* \equiv \operatorname{ess}\lim A_t$ exists.*

(ii) *If $(X_t) \in \mathscr{C}$, then $P(A^*) = 0$ or A^* is a union of atoms of $\mathcal{F}_\infty$.*

Proof. When $(X_t) \in \mathscr{C}$, (ii) was proved in [**33**]. Now assume that $(X_t) \in \bar{\mathscr{C}}$ and for some $\sigma \in T_f$ and each $\sigma \le \tau \in T_c$, $\int_{(\tau<\infty)} \|X_\tau\| < \infty$. Choose $(t_n) \subset D$ such that $P(\sigma < t_n) \uparrow 1$. Since $(X_t I(\sigma \le t_n), \mathcal{F}_t, t_n \le t \in D) \in \mathscr{C}$, $A^* \cap (\sigma \le t_n)$ is a union of atoms of $\mathcal{F}_\infty$ or ϕ, $n \ge 1$, and so is A^*.

Lemma 3. *Suppose that (X_t) is a real-valued GBT (subpramart or mil) and $(X_t^+) \in \bar{\mathscr{C}}$. Then $(X_t)((X_\tau, \tau \in T))$ converges stochastically to a r.v. ξ such that $-\infty < \xi \le +\infty$, $(\xi = +\infty) = A^{+*}$ a.s., and A^{+*} is a union of atoms of $\mathcal{F}_\infty$ or ϕ, where $A^{+*} = \operatorname{ess}\lim A_t^+$, $A_t^+ = (P_t = \infty)$.*

https://doi.org/10.1017/S0017089500009800 Published online by Cambridge University Press

Proof. As the proof of Lemma 2 we may assume that $(X_t^+) \in \mathscr{C}$. Then the conclusions for GBTs and subpramarts follow from Theorems 9 and 10 in **[33]**. Now assume that (X_t) is a mil. Choose $(t_n) \subset D$ such that $A_{t_n}^+ \downarrow A^{+*}$. For any fixed $n \geq 1$ and $K \geq 1$, by Lemma 1, it is easy to see that $(X_t I(P_{t_n} \leq K), t \geq t_n)$ is a mil and

$$\liminf_t EX_t^+ I(P_{t_n} \leq K) \leq EP_{t_n} I(P_{t_n} \leq K) < \infty.$$

Hence $(X_t I(P_{t_n} \leq K)) \in \mathscr{S}$ (Theorem D). Since $\bigcup_{n \geq 1} \bigcup_{K \geq 1} (P_{t_n} \leq K) = \Omega \backslash A^{+*}$, $(X_t I(\Omega \backslash A^{+*})) \in \mathscr{S}$. Since every mil is a GBT, $\text{s}\,.\lim_T X_\tau I(A^{+*}) = \infty I(A^{+*})$.

REMARK 2. Suppose that (X_n) is a real-valued process and $(X_n^+) \in \mathscr{C}$. Chow **[4]** proved that if (X_n) is a submartingale, then (X_n) converges a.s. to a r.v. ξ which takes values in $(-\infty, +\infty]$. Yamasaki **[34]** obtained the same result for martingales in the limit. In Lemma 3, we extend their results to subpramarts and mils and show that $(\xi = +\infty)$ is a union of atoms of $\mathscr{F}_\infty$ or ϕ.

Let F^* be the dual space of F and F^{*+} the positive cone of F^* if F is a Banach lattice.

LEMMA 4.(i) *Suppose that F is a Banach lattice. If (X_t) is a positive GBT, then so are $(\|X_t\|)$ and $(f(X_t)), f \in F^{*+}$.*

(ii) *If F is a Banach lattice and (X_t) is a positive subpramart, then so are $(\|X_t\|)$ and $(f(X_t)), f \in F^{*+}$.*

(iii) *If (X_t) is a mil, then $(\|X_t\|)$ is a GBT and $(f(X_t))$ is a mil, $f \in F^*$.*

Proof. If F is a Banach lattice and (X_t) is positive, then for $t,s \in D$ and $\sigma \in T$,

$$\|(X_t - E(X_s \mid \mathscr{F}_t))^+\| \geq (\|X_t\| - \|E(X_s \mid \mathscr{F}_t)\|)^+ \geq (\|X_t\| - E(\|X_s\| \mid \mathscr{F}_t))^+,$$

and

$$\|(X_t - E(X_s \mid \mathscr{F}_t))^+\| \geq \frac{f(X_t - E(X_s \mid \mathscr{F}_t))^+}{\|f\|} \geq \frac{(f(X_t) - E(f(X_s) \mid \mathscr{F}_t))^+}{\|f\|}, f \in F^{*+} \backslash 0,$$

(i) holds. Similarly, we get (ii). Since

$$\|X_t - E(X_s \mid \mathscr{F}_t)\| \geq \|X_t\| - E(\|X_s\| \| \mathscr{F}_t),$$

and

$$\|X_\sigma - E(X_s \mid \mathscr{F}_\sigma)\| \geq \frac{|f(X_\sigma - E(X_s \mid \mathscr{F}_\sigma))|}{\|f\|} = \frac{|f(X_\sigma) - E(f(X_s) \mid \mathscr{F}_\sigma)|}{\|f\|}, f \in F^* \backslash 0,$$

(iii) holds.

LEMMA 5. *Suppose that (X_t) is a mil and $(X_t) \in \bar{\mathscr{C}}$. Then $P(A^*) = 0$.*

Proof. Assume that $P(A^*) > 0$. By Lemma 2, we may assume that A^* is an atom of $\mathscr{F}_\infty$ and $X_t = x_t$ on A^*. Since (X_t) is a mil, by Lemmas 3 and 4,

$$\lim_t f(x_t) \text{ exists and is finite for each } f \in F^*,$$

and

$$\|x_t\| \to \infty,$$

which contradicts the Banach–Steinhaus theorem.

When $D = N$, the following lemma was proved in [**32**]. The proof here is new.

LEMMA 6. *Assume that* $(X_t) \in \mathscr{B} \cap \mathscr{S}$. *Then* (X_t) *is a pramart.*

Proof. Assume that $(X_\tau, \tau \in T)$ converges stochastically to X. By Fatou's lemma, $X \in L^1$. Let

$$Z_t = X_t - E(X \mid \mathscr{F}_t).$$

Then $(Z_t) \in \mathscr{B}$. Clearly, we need only to show that (Z_t) is a pramart. Since $(E(X \mid \mathscr{F}_\tau), \tau \in T)$ converges stochastically to X, $(\|Z_\tau\|, \tau \in T)$ converges stochastically to zero. Hence, we need only to show that $(S_\tau, \tau \in T)$ converges stochastically to zero, where

$$S_t = \operatorname{ess} \sup_{t \le \sigma \in T} E(\|Z_\sigma\| \mid \mathscr{F}_t).$$

First we show that $(S_t) \in \mathscr{S}$. By Lemma 1, (S_t) is a pramart satisfying

$$\sup_{t \in D} \int |S_t| = \sup_{\tau \in T} \int \|Z_\tau\| \le \sup_{\tau \in T} \int \|X_\tau\| + \int \|X\| < \infty.$$

Hence, by Theorem B, $(S_t) \in \mathscr{S}$. Assume that $(S_\tau, \tau \in T)$ does not converge stochastically to zero, then $(\|Z_\tau\| - S_\tau, \tau \in T)$ converges stochastically to a nonpositive r.v. ξ such that $P(\xi < 0) > 0$, since $S_t \ge \|Z_t\|$ and (S_t), $(\|Z_t\|) \in \mathscr{S}$. Then, by Fatou's lemma,

$$\liminf_{\tau \in T} \int (S_\tau - \|Z_\tau\|) \ge \int \lim_{\tau \in T} (S_\tau - \|Z_\tau\|) > 0. \tag{2}$$

On the other hand, by Lemma 1,

$$\lim_{\tau \in T} \int S_\tau = \lim_{\tau \in T} \left(\sup_{\tau \le \tau' \in T} \int \|Z_{\tau'}\| \right) = \limsup_{\tau \in T} \int \|Z_\tau\| < \infty,$$

which contradicts (2).

LEMMA 7. *If* $(X_t) \in \mathscr{S} \cap \bar{\mathscr{C}}$, *then* $P(A^*) = 0$.

Proof. Assume that $(X_t) \in \mathscr{S} \cap \bar{\mathscr{C}}$ and $P(A^*) > 0$. Then, by Lemma 2, we may assume that A^* is an atom of $\mathscr{F}_\infty$, and A_t is an atom of $\mathscr{F}_t$ such that $A_t \supset A^*$, $\|X_t\| = a_t$ on A_t, and $\lim a_t = a \in R_1$. We may also assume that $a_t < a + 1$, $t \in D$. For any $\sigma \in T_f$, there is $t_0 \in D$, $P(A^* \cap (\sigma = t_0)) > 0$. Since $(\sigma = t_0) \in \mathscr{F}_{t_0}$, $A_{t_0} \subset (\sigma = t_0)$. Since $W_{t_0} = \infty$ on A_{t_0}, there is $t_0 \le \tau_1 \in T$ such that $E(\|X_{\tau_1}\| \mid \mathscr{F}_{t_0}) > (a+2)/P(A_{t_0})$ on A_{t_0}, and there is $t_1 \in D$ such that $P(A^* \cap (\tau_1 = t_1)) > 0$ (then $A_{t_1} \subset (\tau_1 = t_1)$). Assume that we have chosen $t_{n-1} \le \tau_n \in T$ such that $E(\|X_{\tau_n}\| \mid \mathscr{F}_{t_{n-1}}) \ge (a+2)/P(A_{t_{n-1}})$ on $A_{t_{n-1}}$. Then there is $t_n \in D$ such that $P(A^* \cap (\tau_n = t_n)) > 0$, (hence $A_{t_n} \subset (\tau_n = t_n)$). And we can choose $t_n \le \tau_{n+1} \in T$ such that $E(\|X_{\tau_{n+1}}\| \mid \mathscr{F}_{t_n}) \ge (a+2)/P(A_{t_n})$ on A_{t_n}. Let $\tau = \sum_{n \ge 1} \tau_n I(A_{t_{n-1}} \setminus A_{t_n}) + \sigma I(\Omega \setminus A_{t_0}) + \infty I(A^*)$.

Then $\sigma \le \tau \in T_c$ and

$$E\,\|X_\tau\|\,I(\tau<\infty) \ge \sum_n \left(E\,\|X_{\tau_n}\|\,I(A_{t_{n-1}}) - E\,\|X_{t_n}\|\,I(A_{t_n})\right)$$

$$\ge \sum_n \left(E\,\|X_{\tau_n}\|\,I(A_{t_{n-1}}) - (a+1)\right) \ge \sum_n \left(E(E(\|X_{\tau_n}\| \mid \mathcal{F}_{t_{n-1}})I(A_{t_{n-1}}) - (a+1)\right) = \infty.$$

Hence, for any $\sigma \in T_f$ there is $\sigma \le \tau \in T_c$ such that $E\,\|X_\tau\|I(\tau<\infty)=\infty$, i.e. $(X_t) \notin \bar{\mathscr{C}}$, a contradiction.

Proof of Theorem 1. (a) Since $(X_t) \in \bar{\mathscr{C}}$, by Lemmas 5 and 7, either $(X_t) \in \mathscr{S}$ or being a mil implies $P(A^*)=0$, and we can choose $(t_k) \subset D$ such that $t_1 < t_2 < \ldots$ and $P(A_{t_k}) \downarrow 0$. For $m,\ k \in \mathbf{N}$ define

$$X_t^k = X_t I(W_{t_k} < m), \qquad t_k \le t \in D.$$

Since $E(\|X_\tau\| \mid \mathcal{F}_t) \le W_t$, a.s., $t \le \tau \in T$, by Lemma 1,

$$\sup_{t_k \le \tau \in T} \int \|X_\tau^k\| \le \int W_{t_k}(W_{t_k} < m) < \infty.$$

Hence, $(X_t^k,\ t_k \le t \in D)$ is of class $\mathscr{B}$. If (X_t) is a mil, then $(X_t^k,\ t_k \le t \in D)$ is a mil of class $\mathscr{B}$, and by Theorem D, $(X_t^k,\ t_k \le t \in D) \in \mathscr{S}$; if $(X_t) \in \mathscr{S}$, then, $(X_t^k,\ t_k \le t \in D) \in \mathscr{B} \cap \mathscr{S}$, and, by Lemma 6, $(X_t^k, t_k \le t \in D)$ is a pramart. Since $\bigcup_{k,m \in \mathbf{N}} (W_{t_k} < m) = \Omega \setminus A^* = \Omega$ a.s. and every pramart is a mil (a martingale in the limit, if the Vitali condition holds),we get (a).

Now we prove (b). Assume that (X_t) is a pramart satisfying $\liminf_{\tau \in T} \int \|X_\tau\| < \infty$. It is easy to show that $(X_\tau, \mathcal{F}_\tau, \tau \in T)$ is a mil, and applying Theorem D, $(X_t) \in \mathscr{S}$.

Proof of Theorem 2. *Proof of* (i). Suppose that F has the Radon–Nikodym property and (X_t) is a positive subpramart satisfying $\liminf_{\tau \in T} \int \|X_\tau\| = M < \infty$. Choose $(t_n) \subset D$ such that $t_1 \le t_2 \le \ldots$ and

$$\sup_{t_n \le \tau \le \sigma;\, \tau, \sigma \in T} P(\|(X_\tau - E(X_\sigma \mid \mathcal{F}_\tau))^+\| > 1/n) < 2^{-n}. \tag{3}$$

We claim that the following fact holds:

for all $(\tau_n, \sigma_n) \subset T$ such that $t_n \le \tau_n \le \sigma_n \le \tau_{n+1}$, and $\int \|X_{\sigma_n}\| < M+1$,

$(X_{\tau_1}, X_{\sigma_1}, \ldots, X_{\tau_n}, X_{\sigma_n}, \ldots)$ converges almost surely to a finite r.v.

Proof of the claim. For $n \ge 1$, let

$$\bar{X}_{2n-1} = X_{\tau_n}, \qquad \bar{\mathcal{F}}_{2n-1} = \mathcal{F}_{\tau_n};$$
$$\bar{X}_{2n} = X_{\sigma_n}, \qquad \bar{\mathcal{F}}_{2n} = \mathcal{F}_{\sigma_n}.$$

Then, $(\bar{X}_n, \bar{\mathcal{F}}_n, n \in \mathbf{N})$ is a positive subpramart satisfying

$$\liminf_n \int \|\bar{X}_n\| \le \liminf_n \int \|X_{\sigma_n}\| < \infty.$$

https://doi.org/10.1017/S0017089500009800 Published online by Cambridge University Press

Since $\bar{X}_n$ being Bochner integrable is separably valued, so is $(\bar{X}_n, n \in \mathbf{N})$. By Theorem C, $(\bar{X}_n, n \in \mathbf{N}) \in \mathscr{E}$, and the claim has been proved.

The above claim implies that $(X_t) \in \mathscr{S}$. In fact, if $(X_t) \notin \mathscr{S}$, then there is a $c > 0$ such that for any $t \in D$ there exist $t \leq \tau$, $\rho \in T$,

$$P(\|X_\tau - X_\rho\| > c) > c. \tag{4}$$

Choose $t_1 \leq \tau_1$, $\rho_1 \in T$ such that (4) holds. Pick $\sigma_1 \in T$ such that $\sigma_1 \geq \tau_1$, $\sigma_1 \geq \rho_1$, $\sigma_1 \geq t_2$, and $\int \|X_{\sigma_1}\| < M + 1$. Assume that τ_n,ρ_n and σ_n have been chosen, choose τ_{n+1}, ρ_{n+1} and $\sigma_{n+1} \in T$ such that $\tau_{n+1} \geq \sigma_n$, $\rho_{n+1} \geq \sigma_n$, and $\sigma_{n+1} \geq \tau_{n+1}$, $\sigma_{n+1} \geq \rho_{n+1}$, $\sigma_{n+1} \geq t_{n+2}$, $\int \|X_{\sigma_{n+1}}\| < M + 1$, and

$$P(\|X_{\tau_{n+1}} - X_{\rho_{n+1}}\| > c) > c. \tag{5}$$

Then, by the claim,

$$(X_{\tau_1}, X_{\sigma_1}, \ldots, X_{\tau_n}, X_{\sigma_n}, \ldots)$$

and

$$(X_{\rho_1}, X_{\sigma_1}, \ldots, X_{\rho_n}, X_{\sigma_n}, \ldots)$$

converge almost surely to finite r.v.s. Hence,

$$(X_{\tau_1}, X_{\rho_1}, \ldots, X_{\tau_n}, X_{\rho_n}, \ldots)$$

converges almost surely to a finite r.v., which contradicts (5).

Proof of (ii). Under the assumptions of (ii), by Lemmas 2 and 3, $P(A^*) = 0$. Define X_t^k as that in the proof of Theorem 1. Then $(X_t^k, t_k \leq t \in D)$ is a positive, L^1-bounded subpramart, therefore, by part (i) of Theorem 2, $(X_t^k, t_k \leq t \in D) \in \mathscr{S}$, which implies $(X_t) \in \mathscr{S}$.

Proof of Theorem 3. (a) Assume that (X_t) is a real-valued subpramart and $\liminf_{\tau \in T} \int X_\tau^+ < \infty$. By Proposition 1, we can choose $(t_n) \subset D$ such that $t_1 \leq t_2 \leq \ldots$ and $P(R_{t_n} = -\infty) \to 0$. For $M > 0$, and $t_n \leq t \in D$, by Lemma 1,

$$\begin{aligned} \int |R_t|\, I(R_{t_n} > -M) &= \int -R_t I(R_{t_n} > -M) + 2 \int R_t^+ I(R_{t_n} > -M) \\ &\leq \int -R_{t_n} I(R_{t_n} > -M) + 2 \sup_{s \in D} \int R_s^+ \\ &\leq M + 2 \sup_{s \in D} \left(\inf_{s \leq \tau \in T} \int X_\tau^+ \right) = M + 2 \liminf_{\tau \in T} \int X_\tau^+ < \infty. \end{aligned}$$

Hence, $(R_t I(R_{t_n} > -M),\ t_n \leq t \in D)$ is an L^1-bounded pramart, and $(R_t I(R_{t_n} > -M)) \in \mathscr{S}$ (Theorem B). By Proposition 1, $(X_t I(R_{t_n} > -M)) \in \mathscr{S}$. Since $\bigcup_n (R_{t_n} > -\infty) = \Omega$ a.s., $(X_t) \in \mathscr{S}$. The proof of (ii) is similar to the proof in Theorem 2 and, therefore, is omitted.

(b) Now we assume that $(X_t^-) \in \mathscr{E} \cap \mathscr{S}$. For any $\epsilon > 0$, by Theorem 1, we can choose $M > 0$ and $\rho \in T$ such that

$$\sup_{\rho \leq \tau \in T} P(X_\tau > M) < \epsilon$$

and

$$\sup_{\rho\le\tau\le\sigma;\tau,\sigma\in T} P(X_\tau \wedge M - E(X_\sigma \wedge M \mid \mathcal{F}_\tau) > \epsilon) < \epsilon,$$

since $(X_t \wedge M) \in \bar{\mathscr{C}} \cap \mathscr{S}$ and it is a pramart (Theorem 1). Hence, for any $\tau,\sigma \in T$ and $\rho \le \tau \le \sigma$,

$$P(X_\tau - E(X_\sigma \mid \mathcal{F}_\tau) > \epsilon) \le P\{(X_\tau - E(X_\sigma \wedge M \mid \mathcal{F}_\tau))I(X_\tau \le M) > \epsilon\} + P(X_\tau > M)$$
$$\le P(X_\tau \wedge M - E(X_\sigma \wedge M \mid \mathcal{F}_\tau) > \epsilon) + \epsilon < 2\epsilon,$$

i.e. (X_t) is a subpramart.

Proof of Theorem 4. Without loss of generality we may and do assume that $\liminf_{\tau\in T} \int X_\tau^- < \infty$. By Theorem 3, $(X_t) \in \mathscr{S}$. The condition $\liminf_{\tau\in T} \int X_\tau^- < \infty$ implies $(P_t = \infty) = (Y_t = \infty)$ a.s. (see [**33**, Lemma 3]), hence, by Proposition 1, we can choose $(t_k) \subset D$ such that

$$P(P_{t_k} = \infty) = P(Y_{t_k} = \infty) \to 0. \tag{6}$$

Then, it is easy to show that for each $k \ge 1$ and $M > 0$, $(X_t^+ I(P_{t_k} < M),\ t \ge t_k) \in \bar{\mathscr{C}} \cap \mathscr{S}$. Applying Theorem 1, $(X_t^+ I(P_{t_k} < M), t \ge t_k)$ is a pramart, and, by (6), (X_t^+) is also a pramart. Therefore, $(|X_t|) = (2X_t^+ - X_t)$ is a pramart. For each $\lambda \in R_1$, by (1), both $(X_t \wedge \lambda)$ and $(X_t \vee \lambda)$ are pramarts.

4. Some comments. 1. For a martingale (X_n), clearly,

$$\liminf_{\tau\in T} \int \|X_\tau^+\| < \infty \Leftrightarrow \liminf_n \int \|X_n^+\| < \infty.$$

However, for subpramarts, condition $\liminf_{\tau\in T} \int \|X_\tau^+\| < \infty$ is weaker than condition $\liminf_n \int \|X_n^+\| < \infty$: let $\Omega = (0, 1]$, $\mathcal{F} = \mathcal{F}_n$ be the class of Borel sets of $(0, 1]$, and P the Lebesgue measure. Define $X_n = n^2 I(0, 1/n]$. Then (X_n) is a positive pramart, $\int X_n \uparrow \infty$, and $\liminf_{\tau\in T} \int X_\tau = 0$.

2. The following example shows that, in general, the condition $(X_t^-) \in \bar{\mathscr{C}}$ in Theorem 3 can not be dropped.

Let (X_n) be independent r.v.s such that $X_n \le 0$, $\int X_n \downarrow -\infty$ and $X_n \to 0$ a.s. and X_n is non-degenerate. Let $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$. Clearly, (X_n) is not a subpramart, and we can show that $(X_n^-) \notin \bar{\mathscr{C}}$. In fact, for each $\sigma \in T_f$, choose $n \in \mathbf{N}$ so large that $P(\sigma \le n) > 0$. Then we can find $(A_k, k \ge n)$ such that $A_k \subset (\sigma \le n)$, $A_k \in \mathcal{F}_k$, $P(A_k) > 0$, $P(A_k, A_j) = 0$, $j \ne k$. Choose $n_k \ge k$, $E|X_{n_k}| P(A_k) > 1$. Let $\tau = \sum_{k\ge n} n_k I(A_k) + \infty I(\Omega \setminus (\bigcup_k A_k)$. Then $\sigma \le \tau \in T_c$ and $\int_{(\tau<\infty)} |X_\tau| = \infty$, $(X_n^-) \notin \bar{\mathscr{C}}$.

3. The condition $\liminf_{\tau\in T} \min\{\int X_\tau^+, \int X_\tau^-\} < \infty$ in Theorem 4 is necessary. Let (ξ_n) be i.i.d. r.v.s, $P(\xi_n = 1) = P(\xi_n = -1) = 1/2$, $X_n = \sum_1^n \xi_i$, $\mathcal{F}_n = \sigma(\xi_1, \ldots, \xi_n)$. Then, (X_n) is a martingale, hence a pramart. It is well known that $\limsup_n X_n = \infty$ a.s. Hence, $(|X_n|) \notin \mathscr{C}$

and, by Theorem 3, $(|X_n|)$ is not a pramart. It is easy to see that, in this example,

$$\liminf_{\tau\in T} \min\left\{\int X_\tau^+, \int X_\tau^-\right\} = \frac{1}{2}\lim_n \int |X_n| = \infty.$$

4. The set $\bar{\mathscr{C}}$ is larger than $\mathscr{C}$. Let (Y_n) be i.i.d r.v.s, $P(Y_n = 2) = P(Y_n = -1) = P(Y_n = 0) = 1/3$. Let $X_n = 3^n \prod_{1\le i\le n} Y_i$, $\mathscr{F}_n = \sigma(Y_1, \ldots, Y_n)$. Then (X_n) is a real-valued martingale. Let $\sigma = \inf\{n \ge 1, \ Y_n = 0\}$. Clearly, $\sigma \in T_f$ and for any $\sigma \le \tau \in T_c$, $\int_{(\tau<\infty)} |X_\tau| = 0$. Hence, $(X_n) \in \bar{\mathscr{C}}$. Now let $\tau = \inf\{n \ge 1, Y_n = -1\}$. Then $\tau \in T_f$, $\int |X_\tau| = \sum_n 6^n (P(Y_1 = 2))^n/2 = \infty$, and $(X_n) \notin \mathscr{C}$.

REFERENCES

1. D. G. Austin, G. A. Edgar, and A. Ionescu Tulcea, Pointwise convergence in terms of expectations, *Z. Wahrsch. Verw. Gebiete* **30** (1974), 17–26.

2. A. Bellow, Uniform amarts: a class of asymptotic martingales for which strong almost sure convergence obtains. *Z. Wahrsch. Verw. Gebiete* **41** (1978), 177–191.

3. A. Bellow and A. Dvoretzky, On martingales in the limit, *Ann. Probab.* **8** (1980), 602–606.

4. Y. S. Chow, Convergence theorems of martingales, *Z. Wahrsch. Verw. Gebiete* **1** (1963), 340–346.

5. Y. S. Chow, H. Robbins, and D. Siegmund, *Great expectations: The theory of optimal stopping* (Houghton Mifflin, 1971).

6. Y. S. Chow and H. Teicher, *Probability theory. Independence, interchangeability, martingales* (Springer, 1978).

7. G. A. Edgar and L. Sucheston, Amarts: a class of asymptotic martingales. A. Discrete parameter, *J. Multivariate Anal.* **6** (1976), 193–221.

8. L. Egghe, Strong convergence of pramarts in Banach spaces, *Canad. J. Math.* **33** (1981), 357–361.

9. L. Egghe, On sub- and superpramarts with values in a Banach lattice, *Measure theory, Oberwolfach* 1981, Lecture Notes in Mathematics 945 (1981), 353–365.

10. L. Egghe, *Stopping time techniques for analysts and probabilists*, London Mathematical Society Lecture Note Series 100 (Cambridge University Press, 1984).

11. N. E. Frangos, On convergence of vector valued pramarts and subpramarts, *Canad. J. Math.* **37** (1985), 260–270.

12. N. Ghoussoub, Orderamarts: A class of asymptotic martingales, *J. Multivariate Anal.* **9** (1979), 165–172.

13. N. Ghoussoub and L. Sucheston, A refinement of the Riesz decomposition for amarts and semiamarts, *J. Multivariate Analysis* **8** (1978), 146–150.

14. A. Gut, A contribution to the theory of asymptotic martingales, *Glasgow Math. J.* **23** (1982), 177–186.

15. A. Gut and K. D. Schmidt, *Amarts and set function processes*, Lecture Notes in Mathematics 1042. (Springer, 1983).

16. H. Heinich, Convergence des sous-martingales positives dans un Banach réticulé, *C.R. Acad. Sci. Paris Sér. A–B* **286** (1978), A279–280.

17. U. Krengel and L. Sucheston, On semiamarts, amarts, and processes with finite value, *Probability on Banach Spaces, Advances in Probability and Related Topics* 4 (1978), 197–266.

18. K. Krickeberg, Convergence of martingales with a directed index set, *Trans. Amer. Math. Soc.* **83** (1956), 313–337.

19. A. Millet and L. Sucheston, Characterizations of Vitali conditions with overlap in terms of convergence of amarts, *Canad. J. Math.* **31** (1979), 1033–1046.

20. A. Millet and L. Sucheston, Convergence of classes of amarts indexed by directed sets, *Canad. J. Math.* **32** (1980), 86–125.

21. A. G. Mucci, Limits for martingale-like sequences, *Pacific J. Math.* **48** (1973), 197–202.
22. A. G. Mucci, Another Martingale convergence theorem, *Pacific J. Math.* **64** (1976), 539–541.
23. J. Neveu, *Discrete-parameter martingales* (North-Holland, 1975).
24. K. D. Schmidt, The lattice property of uniform amarts, *Ann. Probab.* **17** (1989), 372–378.
25. H.-U. Schwarz, *Banach lattices and operators*, Teubner-Texte zur Mathematik 71 (B. G. Teubner, 1984).
26. M. Słaby, Convergence of positive subpramarts and pramarts in Banach spaces with unconditional bases, *Bull. Polish Acad. Sci. Math.* **31** (1983), 75–80.
27. M. Słaby, Strong convergence of vector-valued pramarts and subpramarts, *Probab. Math. Statist.* **5** (1985), 187–190.
28. M. Talagrand, Some structure results for martingales in the limit and pramarts, *Ann. Probab.* **13** (1985), 1192–1203.
29. Z. P. Wang, The lattice properties of martingale-like sequences, *Acta Math. Sinica* **30** (1987), 355–360.
30. Z. P. Wang, Local convergence of martingale-like sequences, *Chinese Ann. Math. Ser. A* **9** (1988), 203–207.
31. Z. P. Wang and X. H. Xue, On convergence of vector-valued mils indexed by a directed set, *Almost everywhere convergence* (*Columbus*, *Ohio*, 1988) ed. G. Edgar, and L. Sucheston, (Academic Press, 1989), 401–416.
32. X. H. Xue, On convergence of pramarts in Banach spaces, *Bulletin of Chinese Science* **29** (1984), 1280.
33. X. H. Xue, On convergence of subpramarts and games which become better with time, *J. Theoret. Probab.* **4** (1991), 605–623.
34. Y. Yamasaki, Another convergence theorem of martingales in the limit, *Tôhoku Math. J.* **33** (1981), 555–559.

Department of Statistics
East China Normal University
Shanghai 200062
China

Department of Statistics
Columbia University
New York
NY 10027
U.S.A.

https://doi.org/10.1017/S0017089500009800 Published online by Cambridge University Press