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COMPACTNESS OF EMBEDDINGS OF FUNCTION SPACES ON QUASI-BOUNDED DOMAINS AND THE DISTRIBUTION OF EIGENVALUES OF RELATED ELLIPTIC OPERATORS

HANS-GERD LEOPOLD[1] AND LESZEK SKRZYPCZAK[2]

[1] *Mathematisches Institut, Friedrich-Schiller Universität, Ernst Abbe Platz 1–2, 07740 Jena, Germany* (hans-gerd.leopold@uni-jena.de)
[2] *Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Umultowska 87, 61-614 Poznań, Poland* (lskrzyp@amu.edu.pl)

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Abstract We prove sufficient and necessary conditions for compactness of the Sobolev embeddings of Besov and Triebel–Lizorkin spaces defined on bounded and unbounded uniformly E-porous domains. The asymptotic behaviour of the corresponding entropy numbers is calculated. Some applications to the spectral properties of elliptic operators are described.

Keywords: compact embeddings; Besov and Triebel–Lizorkin spaces; quasi-bounded domains; elliptic operators; distribution of eigenvalues

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1. Introduction

Let Ω be a domain in $\mathbb{R}^n$ and let $\tilde{W}^k_p(\Omega)$ be a function space obtained by completing $C_0^\infty(\Omega)$ in the usual Sobolev norm. It is well known that Sobolev embeddings of the function spaces $\tilde{W}^k_p(\Omega)$ into $L_q(\Omega)$ can be compact if the domain Ω is bounded. It was noted by Clark in 1965 that the Sobolev embeddings can also be compact if the domain is unbounded but sufficiently narrow at infinity and satisfies a certain regularity condition; see [**3**]. This is related to the discreteness of the spectrum of the Dirichlet Laplacian on Ω [**4**]. We refer the reader to [**1**] for further discussion and references.

Operator properties of the Sobolev embeddings were studied later by König; see [**7**,**8**]. He dealt with Sobolev spaces on the quasi-bounded domains satisfying an additional regularity condition, the so-called C^ℓ_k condition, where $\ell \in \mathbb{R}_+$ and $k = 1, \dots, n$. Roughly speaking, this means that the boundary consists of sufficiently smooth manifolds of dimension at least $(n-k)$. He found some asymptotic estimates of approximation, the Gelfand and Kolmogorov numbers of embeddings $\tilde{W}^k_p(\Omega) \hookrightarrow L_q(\Omega)$ and applied them to estimates of eigenvalues of elliptic operators. In contrast to these works, we concentrate on the entropy numbers of the embeddings. We also work with the domains with boundaries that satisfy a different regularity condition, which seems to be weaker and allows some fractality of the boundary. Moreover, for any quasi-bounded domain we introduce a new constant, called the box-packing constant, that is very helpful both to formulate sufficient and necessary conditions for compactness of the embeddings, as well as to describe the asymptotic behaviour of entropy numbers. In particular, König's results do not cover Example 5.6.

More precisely, we consider the Besov and Triebel–Lizorkin spaces defined on a wide range of unbounded domains, so-called uniformly E-porous domains. Recently, Triebel proved the wavelet characterization of the spaces; see [**14**]. Based on the characterization, we obtain sufficient and necessary conditions for compactness of the Sobolev embeddings. Moreover, we study the degree of the compactness of the embeddings in terms of entropy numbers. The exact asymptotic behaviour of the entropy numbers is calculated. In particular, for any quasi-bounded uniformly E-porous domain $\Omega \subset \mathbb{R}^n$ we define the box-packing constant $b(\Omega)$, $n \leqslant b(\Omega) \leqslant \infty$, and prove that the embedding

$$\tilde{W}^{k_1}_{p_1}(\Omega) \hookrightarrow \tilde{W}^{k_2}_{p_2}(\Omega)$$

is compact if

$$k_1 - k_2 - n\left(\frac{1}{p_1} - \frac{1}{p_2}\right) > b(\Omega)\left(\frac{1}{p_2} - \frac{1}{p_1}\right)_+.$$

In particular, $k_1 > k_2$, since $b(\Omega) \geqslant n$. If Ω is a bounded domain, then $b(\Omega) = n$. Moreover, we prove that if the embedding is compact and $b(\Omega) < \infty$, then

$$e_k(\tilde{W}^{k_1}_{p_1}(\Omega) \hookrightarrow \tilde{W}^{k_2}_{p_2}(\Omega)) \sim k^{-\gamma}, \tag{1.1}$$

with

$$\gamma = \frac{k_1 - k_2}{b(\Omega)} + \frac{b(\Omega) - n}{b(\Omega)}\left(\frac{1}{p_1} - \frac{1}{p_2}\right).$$

Here e_k denotes the kth entropy number of the embedding (see Definition 4.1) and $a_+ := \max(a, 0)$ for any real number a. Moreover, if

$$\left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ < \gamma \leqslant \frac{k_1 - k_2}{n},$$

then one can find a quasi-bounded domain Ω such that (1.1) holds.

We write $a \sim b$ if there exists a constant $c > 0$ (independent of the context-dependent relevant parameters) such that

$$c^{-1}a \leqslant b \leqslant ca.$$

All unimportant constants will be denoted by c, sometimes with additional indices.

The paper has the following structure. In § 2, we recall the definition of the domains, the function spaces and their wavelet characterization. In § 3, we prove criteria for the compactness of the embeddings. Section 4 is devoted to the behaviour of entropy numbers. In § 5, we give some applications to the spectral theory of elliptic operators.

2. Preliminary

2.1. Function spaces on arbitrary domains

Let Ω be an open set in $\mathbb{R}^n$ such that $\Omega \neq \mathbb{R}^n$. Such a set will be called an arbitrary domain. We assume that the reader is familiar with definitions and basic facts concerning Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ and Triebel–Lizorkin spaces $F^s_{p,q}(\mathbb{R}^n)$ defined on $\mathbb{R}^n$, as well as the Besov and Triebel–Lizorkin spaces, $B^s_{p,q}(\Omega)$ and $F^s_{p,q}(\Omega)$, defined on Ω by restrictions. All we need can be found in [**14**, Chapter 1]. We will use the common notation. In particular, we set

$$\sigma_p = n\left(\frac{1}{p} - 1\right)_+ \quad \text{and} \quad \sigma_{p,q} = n\left(\frac{1}{\min(p,q)} - 1\right)_+, \quad 0 < p,\ q \leqslant \infty,$$

and $A^s_{p,q}(\mathbb{R}^n)$, $A^s_{p,q}(\Omega)$ with $A = B$ or $A = F$.

Definition 2.1. Let Ω be an arbitrary domain in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$ and let

$$0 < p \leqslant \infty, \qquad 0 < q \leqslant \infty, \qquad s \in \mathbb{R},$$

with $p < \infty$ for the F-spaces.

(i) Let

$$\begin{aligned} \tilde{A}^s_{p,q}(\bar{\Omega}) &= \{f \in A^s_{p,q}(\mathbb{R}^n)\colon \operatorname{supp} f \subset \bar{\Omega}\}, \\ \tilde{A}^s_{p,q}(\Omega) &= \{f \in D'(\Omega)\colon f = g|_\Omega \text{ for some } g \in \tilde{A}^s_{p,q}(\bar{\Omega})\}, \\ \|f \mid \tilde{A}^s_{p,q}(\Omega)\| &= \inf \|g \mid A^s_{p,q}(\mathbb{R}^n)\|, \end{aligned}$$

where the infimum is taken over all $g \in \tilde{A}^s_{p,q}(\bar{\Omega})$ with $f = g|_\Omega$.

(ii) We define

$$\bar{F}^s_{p,q}(\Omega) = \begin{cases} \tilde{F}^s_{p,q}(\Omega) & \text{if } 0 < p < \infty,\ 0 < q \leqslant \infty,\ s > \sigma_{p,q}, \\ F^0_{p,q}(\Omega) & \text{if } 1 < p < \infty,\ 1 \leqslant q \leqslant \infty,\ s = 0, \\ F^s_{p,q}(\Omega) & \text{if } 0 < p < \infty,\ 0 < q \leqslant \infty,\ s < 0, \end{cases}$$

and

$$\bar{B}^s_{p,q}(\Omega) = \begin{cases} \tilde{B}^s_{p,q}(\Omega) & \text{if } 0 < p \leqslant \infty,\ 0 < q \leqslant \infty,\ s > \sigma_p, \\ B^0_{p,q}(\Omega) & \text{if } 1 < p < \infty,\ 0 < q \leqslant \infty,\ s = 0, \\ B^s_{p,q}(\Omega) & \text{if } 0 < p < \infty,\ 0 < q \leqslant \infty,\ s < 0. \end{cases}$$

Following Triebel, we introduce E-thick (exterior thick) and E-porous domains; see [**14**, Chapter 3]. We start with the definition of porosity.

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Definition 2.2.

(i) A closed set $\Gamma \subset \mathbb{R}^n$ is said to be porous if there exists a number $0 < \eta < 1$ such that one finds for any ball $B(x, r) \subset \mathbb{R}^n$ centred at x and of radius r, with $0 < r < 1$, a ball $B(y, \eta r)$ with

$$B(y, \eta r) \subset B(x, r) \quad \text{and} \quad B(y, \eta r) \cap \Gamma = \emptyset.$$

(ii) A closed set $\Gamma \subset \mathbb{R}^n$ is said to be uniformly porous if it is porous and there is a locally finite positive Radon measure μ on $\mathbb{R}^n$ such that $\Gamma = \operatorname{supp} \mu$ and

$$\mu(B(\gamma, r)) \sim h(r), \quad \text{with } \gamma \in \Gamma,\ 0 < r < 1,$$

where $h\colon [0, 1] \to \mathbb{R}$ is a continuous strictly increasing function with $h(0) = 0$ and $h(1) = 1$ (the equivalence constants are independent of γ and r).

Remark 2.3. The closed set Γ is called a d-set if there is a locally finite positive Radon measure μ on $\mathbb{R}^n$ such that $\Gamma = \operatorname{supp} \mu$ and

$$\mu(B(\gamma, r)) \sim r^d, \quad \text{with } \gamma \in \Gamma,\ 0 < r < 1.$$

Naturally, $0 \leqslant d \leqslant n$. Any d-set with $d < n$ is uniformly porous.

Definition 2.4. Let Ω be an open set in $\mathbb{R}^n$ such that $\Omega \neq \mathbb{R}^n$ and $\Gamma = \partial\Omega$.

(i) The domain Ω is said to be E-thick if one can find for any interior cube $Q^{\mathrm{i}} \subset \Omega$, with

$$\ell(Q^{\mathrm{i}}) \sim 2^{-j} \quad \text{and} \quad \operatorname{dist}(Q^{\mathrm{i}}, \Gamma) \sim 2^{-j},\ j \geqslant j_0 \in \mathbb{N},$$

a complementing exterior cube $Q^{\mathrm{e}} \subset \mathbb{R}^n \setminus \Omega$, with

$$\ell(Q^{\mathrm{e}}) \sim 2^{-j} \quad \text{and} \quad \operatorname{dist}(Q^{\mathrm{e}}, \Gamma) \sim \operatorname{dist}(Q^{\mathrm{e}}, Q^{\mathrm{i}}) \sim 2^{-j},\ j \geqslant j_0 \in \mathbb{N}.$$

Q^{i} and Q^{e} denote cubes in $\mathbb{R}^n$ with sides parallel to the axes of coordinates and side lengths $\ell(Q^{\mathrm{i}})$ and $\ell(Q^{\mathrm{e}})$, respectively.

(ii) The domain Ω is said to be E-porous if there is a number η, with $0 < \eta < 1$, such that one finds for any ball $B(\gamma, r) \subset \mathbb{R}^n$ centred at $\gamma \in \Gamma$ and of radius r, with $0 < r < 1$, a ball $B(y, \eta r)$ with

$$B(y, \eta r) \subset B(\gamma, r) \quad \text{and} \quad B(y, \eta r) \cap \bar{\Omega} = \emptyset.$$

(iii) The domain Ω is called uniformly E-porous if it is E-porous and Γ is uniformly porous.

Remark 2.5. If Ω is E-porous, then Ω is E-thick and $|\Gamma| = 0$. On the other hand, if Ω is E-thick and Γ is a d-set, then Ω is uniformly E-porous and $n - 1 \leqslant d < n$.

2.2. Whitney decomposition and related wavelet systems

Let Ω be an arbitrary domain in $\mathbb{R}^n$, with $\Omega \neq \mathbb{R}^n$. We recall the construction of a wavelet basis on Ω that will be needed later on; see [**14**].

Let $\psi_F \in C^u(\mathbb{R})$ and $\psi_M \in C^u(\mathbb{R})$, $u \in \mathbb{N}$, be real compactly supported Daubechies wavelets. We also assume that ψ_M satisfies the vanishing moment conditions for all $v \in \mathbb{N}_0$, $v < u$. For $j \in \mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$ and $G \in \{F, M\}^n$, let

$$\Psi^{j,L}_{G,m}(x) = 2^{(j+L)n/2} \prod_{a=1}^{n} \psi_{G_a}(2^{j+L}x_a - m_a), \quad m = (m_1, \dots, m_n) \in \mathbb{Z}^n, \tag{2.1}$$

where $L \in \mathbb{N}_0$ is fixed such that

$$\operatorname{supp} \psi^L_F \subset (-\varepsilon, \varepsilon), \qquad \operatorname{supp} \psi^L_M \subset (-\varepsilon, \varepsilon)$$

for some sufficiently small $\varepsilon > 0$ (as specified later on). Here, $\psi^L_F = \psi_F(2^L\cdot)$ and $\psi^L_M = \psi_M(2^L\cdot)$. We also set $\{F, M\}^{n*} = \{F, M\}^n \setminus \{\bar{F}\}$, where $\bar{F} = (F, \dots, F)$.

For some positive numbers c_1, c_2, c_3 we choose

$$\mathbb{Z}_\Omega = \{x^j_r \in \Omega\colon j \in \mathbb{N}_0;\ r = 1, \dots, N_j\}, \tag{2.2}$$

where $N_j \in \bar{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ exists such that

$$|x^j_r - x^j_{r'}| \geqslant c_1 2^{-j}, \quad j \in \mathbb{N}_0,\ r \neq r',$$

and

$$\operatorname{dist}\left(\bigcup_{r=1}^{N_j} B(x^j_r, c_2 2^{-j}), \Gamma\right) \geqslant c_3 2^{-j}, \quad j \in \mathbb{N}_0,\ \Gamma = \partial\Omega. \tag{2.3}$$

Let $K \in \mathbb{N}$, $D > 0$ and $c_4 > 0$. Then, the system of functions

$$\{\Phi^j_r\colon j \in \mathbb{N}_0;\ r = 1, \dots, N_j\} \quad \text{with } \operatorname{supp} \Phi^j_r \subset B(x^j_r, c_2 2^{-j}),\ j \in \mathbb{N}_0, \tag{2.4}$$

is called a u-wavelet system (with respect to Ω) if it consists of the following three types of functions:

- $\Phi^0_r = \Psi^{0,L}_{G,m}$ for some $G \in \{F, M\}^n$, $m \in \mathbb{Z}^n$ (basic wavelets);
- $\Phi^j_r = \Psi^{j,L}_{G,m}$, $j \in \mathbb{N}$, $\operatorname{dist}(x^j_r, \Gamma) \geqslant c_4 2^{-j}$ for some $G \in \{F, M\}^{n*}$, $m \in \mathbb{Z}^n$ (interior wavelets);
- $\Phi^j_r = \sum_{|m-m'|\leqslant K} d^j_{m,m'} \Psi^{j,L}_{\bar{F},m'}$, $j \in \mathbb{N}$, $\operatorname{dist}(x^j_r, \Gamma) < c_4 2^{-j}$ for some $m = m(j, r) \in \mathbb{Z}^n$, $d^j_{m,m'} \in \mathbb{R}$, with $\sum_{|m-m'|\leqslant K} |d^j_{m,m'}| \leqslant D$, $\operatorname{supp} \Psi^{j,L}_{\bar{F},m'} \subset B(x^j_r, c_2 2^{-j})$ (boundary wavelets).

The following theorem was proved by Triebel in [**14**, Theorem 2.33].

Theorem 2.6. *Let Ω be an arbitrary domain in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$. For any $u \in \mathbb{N}$ there are u-wavelet systems (with respect to Ω) that are orthonormal bases in $L_2(\Omega)$.*

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We sketch the construction. Let

$$Q^0_{\ell r} \subset Q^1_{\ell r} \subset Q^2_{\ell r} \subset Q^3_{\ell r} \subset Q_{\ell r}, \quad \ell \in \mathbb{N}_0,\ r = 1, 2, \dots, \tag{2.5}$$

be concentric open cubes in $\mathbb{R}^n$, with sides parallel to the axes of coordinates centred at $2^{-\ell} m^r$, $\ell \in \mathbb{N}_0$, $m^r \in \mathbb{Z}^n$, with the respective side lengths $2^{-\ell}$, $5 \cdot 2^{-\ell-2}$, $6 \cdot 2^{-\ell-2}$, $7 \cdot 2^{-\ell-2}$, $2^{-\ell+1}$. According to the Whitney decomposition, there are disjoint cubes $Q^0_{\ell r}$ of the above type such that

$$\Omega = \bigcup_{\ell, r} \bar{Q}^0_{\ell r}, \quad \operatorname{dist}(Q^0_{\ell r}, \partial\Omega) \sim 2^{-\ell} \quad \text{if } \ell \in \mathbb{N} \tag{2.6}$$

and

$$\operatorname{dist}(Q^0_{0r}, \partial\Omega) \geqslant c \quad \text{for some } c > 0.$$

Moreover, we assume that $|\ell - \ell'| \leqslant 1$ for any two admissible cubes such that $Q^1_{\ell r} \cap Q^1_{\ell' r'} \neq \emptyset$.

Let $\Psi^{j,L}_{G,m}$ be given by (2.1), where $L \in \mathbb{N}_0$, and, consequently, $\varepsilon > 0$ is fixed such that

$$\operatorname{supp} \Psi^{j,L}_{G,m} \subset Q^3_{\ell r} \quad \text{if } 2^{-j-L} m \in Q^2_{\ell r} \text{ for } \ell \in \mathbb{N}_0 \text{ and } j \geqslant \ell,$$

and

$$2^{-j-L} m \in Q^2_{\ell r} \quad \text{if } \operatorname{supp} \Psi^{j,L}_{G,m} \cap Q^1_{\ell r} \neq \emptyset \text{ for } \ell \in \mathbb{N}_0 \text{ and } j \geqslant \ell.$$

For $j \in \mathbb{N}$ we set

$$S^{\Omega,1}_j = \{F, M\}^{n*} \times \{m \in \mathbb{Z}^n \colon 2^{-j-L} m \in Q^2_{\ell r} \text{ for some } \ell < j \text{ and some } r\}.$$

Moreover, we set $S^{\Omega,1}_0 = \emptyset$. In a similar way, for $j \in \mathbb{N}_0$ we define

$$S^{\Omega,2}_j = \{F, M\}^n \times \{m \in \mathbb{Z}^n \colon 2^{-j-L} m \in Q^2_{jr} \text{ for some } r\} \setminus S^{\Omega,1}_j.$$

We consider the wavelet system related to the above index sets, setting

$$\Psi^{\Omega,1} = \{\Psi^{j,L}_{G,m} \colon (j, G, m) \in S^{\Omega,1}\}, \quad S^{\Omega,1} = \bigcup_{j=1}^{\infty} S^{\Omega,1}_j,$$

and

$$\Psi^{\Omega,2} = \{\Psi^{j,L}_{G,m} \colon (j, G, m) \in S^{\Omega,2}\}, \quad S^{\Omega,2} = \bigcup_{j=0}^{\infty} S^{\Omega,2}_j.$$

The system $\Psi^{\Omega,1}$ is orthonormal in $L_2(\Omega)$ and any element of $\Psi^{\Omega,1}$ is orthogonal to any element of $\Psi^{\Omega,2}$. Moreover,

$$L_2(\Omega) = L^{(1)}_2(\Omega) \oplus L^{(2)}_2(\Omega),$$

with

$$L_2^{(1)}(\Omega) = \overline{\operatorname{span} \Psi^{\Omega,1}} \quad \text{and} \quad L_2^{(2)}(\Omega) = \overline{\operatorname{span} \Psi^{\Omega,2}},$$

where the closure is taken in $L_2(\Omega)$. The set $\Psi^{\Omega,2}$ consists of two types of elements: $\Psi^{0,L}_{G,m}$ and $\Psi^{j,L}_{G,m}$ for $j > 0$. The elements $\Psi^{0,L}_{G,m}$ are pairwise orthogonal and orthogonal to the rest of the elements of $\Psi^{\Omega} = \Psi^{\Omega,1} \cup \Psi^{\Omega,2}$, so they cause no problem. But, for the rest we need an orthonormalization process that does not destroy the localization of the elements. The orthogonalization process results in boundary wavelets. We refer the reader to §§ 2.3 and 2.4, in particular to the proof of [**14**, Theorem 2.33] for details.

Remark 2.7. It follows from the above construction that we should take

$$N_j = \#S_j^{\Omega,1} + \#S_j^{\Omega,2} \tag{2.7}$$

in (2.2). Here, $\#X$ denotes the cardinality of the set X. The definitions of the sets $S_j^{\Omega,1}$, $S_j^{\Omega,2}$ and the Whitney decomposition (2.6) link the values of N_j with the geometry of the set Ω. In particular, if Ω is a bounded domain or a domain with finite Lebesgue measure, then all N_j are finite and $N_j \sim 2^{jn}$. On the other hand, if the domain Ω contains infinitely many dyadic cubes of size $2^{-\ell}$, then the $N_j = \infty$ for $j > \ell$. In this paper, we are mainly interested in the domains with infinite Lebesgue measure, but such that all the numbers N_j are finite.

2.3. Wavelet characterization of function spaces on E-porous and E-thick domains

Let $\beta = \{\beta_j\}_{j=0}^{\infty}$ be a sequence of positive numbers and let $N_j \in \bar{\mathbb{N}}$, $j \in \mathbb{N}_0$, and $\bar{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. We will work with the sequence spaces

$$\ell_q(\beta_j \ell_p^{N_j}) := \Bigg\{ \lambda = \{\lambda_{j,k}\}_{j,k} \colon \lambda_{j,k} \in \mathbb{C}, \qquad \|\lambda \mid \ell_q(\beta_j \ell_p^{N_j})\| = \Bigg(\sum_{j=0}^{\infty} \beta_j^q \Bigg(\sum_{k=1}^{N_j} |\lambda_{j,k}|^p \Bigg)^{q/p} \Bigg)^{1/q} < \infty \Bigg\}$$

(with the usual modification if $p = \infty$ and/or $q = \infty$). If $N_j = \infty$, then $\ell_p^{N_j} = \ell_p$. Moreover, if $\beta_j = 1$ for any j, then we will write $\ell_q(\ell_p^{N_j})$.

Theorem 2.8. *Let Ω be a uniformly E-porous domain in $\mathbb{R}^n$, $\Omega \neq \mathbb{R}^n$. Let*

$$\{\Phi_r^j \colon j \in \mathbb{N}_0;\ r = 1, \dots, N_j\}, \quad \textit{with } \operatorname{supp} \Phi_r^j \subset B(x_r^j, c_2 2^{-j}),\ j \in \mathbb{N}_0,$$

be a u-wavelet system that is an orthonormal basis in $L_2(\Omega)$ and $u > \max(s, \sigma_p - s)$.

Then, $f \in D'(\Omega)$ is an element of $\bar{B}^s_{p,q}(\Omega)$ if and only if it can be represented as

$$f = \sum_{j=0}^{\infty} \sum_{r=1}^{N_j} \lambda_r^j 2^{-jn/2} \Phi_r^j, \quad \lambda \in \ell_q(2^{j(s-(n/p))} \ell_p^{N_j}), \tag{2.8}$$

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unconditional convergence being in $D'(\Omega)$. Furthermore, if $f \in \bar{B}^s_{p,q}(\Omega)$, then the representation (2.8) is unique with $\lambda = \lambda(f)$:

$$\lambda^j_r = \lambda^j_r(f) = 2^{jn/2}(f, \Phi^j_r),$$

where $(\cdot,\cdot)$ is a dual pairing and

$$I\colon \bar{B}^s_{p,q}(\Omega) \ni f \mapsto \lambda(f) \in \ell_q(2^{j(s-(n/p))}\ell_p^{N_j})$$

is an isomorphism. If, in addition, $\max(p,q) < \infty$, then $\{\Phi^j_r\}$ is an unconditional basis in $\bar{B}^s_{p,q}(\Omega)$.

Remark 2.9. The above theorem was proved by Triebel; see [**14**, Theorem 3.23]. If we assume that the domain Ω is E-thick, then the theorem holds for $\bar{B}^s_{p,q}(\Omega)$ with $s \neq 0$; see [**14**, Theorem 3.13]. Similar results hold for $\bar{F}^s_{p,q}(\Omega)$ spaces. One can find the remarks concerning convergences, duality and other technicalities in, for example, the paragraph before Theorem 3.13 in [**14**].

3. Continuity and compactness of embeddings

3.1. Continuity and compactness of embeddings of sequence spaces

First, we formulate sufficient and necessary conditions for boundedness and compactness of embeddings of sequence spaces. This will be complemented by two simple corollaries concerning function spaces on arbitrary domains. A more detailed study of embeddings of function spaces will be postponed till the next subsection.

We recall that $a_+ = \max(a, 0)$ for any real number a. We also set

$$\frac{1}{p^*} := \left(\frac{1}{p_2} - \frac{1}{p_1}\right)_+ \quad \text{and} \quad \frac{1}{q^*} := \left(\frac{1}{q_2} - \frac{1}{q_1}\right)_+.$$

In the next theorem, we give the sufficient and necessary conditions for boundedness and compactness of embeddings of the sequence spaces.

Theorem 3.1. *Let $0 < p_1,\ p_2 \leqslant \infty$ and $0 < q_1,\ q_2 \leqslant \infty$.*

(i) *If there exists j such that $N_j = \infty$, then the embedding $\ell_{q_1}(\beta_j \ell_{p_1}^{N_j}) \hookrightarrow \ell_{q_2}(\ell_{p_2}^{N_j})$ holds if and only if*

$$p_1 \leqslant p_2 \quad \textit{and} \quad \{\beta_j^{-1}\}_j \in \ell_{q*}. \tag{3.1}$$

Moreover, in that case, the embedding $\ell_{q_1}(\beta_j \ell_{p_1}^{N_j}) \hookrightarrow \ell_{q_2}(\ell_{p_2}^{N_j})$ is not compact.

(ii) *If for any j the index N_j is finite, then the embedding $\ell_{q_1}(\beta_j \ell_{p_1}^{N_j}) \hookrightarrow \ell_{q_2}(\ell_{p_2}^{N_j})$ holds if and only if*

$$\{\beta_j^{-1} N_j^{1/p^*}\}_j \in \ell_{q*}. \tag{3.2}$$

The embedding $\ell_{q_1}(\beta_j \ell_{p_1}^{N_j}) \hookrightarrow \ell_{q_2}(\ell_{p_2}^{N_j})$ is compact if and only if (3.2) holds and, in addition,

$$\lim_{j\to\infty} \beta_j^{-1} N_j^{1/p^*} = 0 \quad \textit{if } q^* = \infty. \tag{3.3}$$

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(iii) *In both cases, it holds that*

$$\|\mathrm{id} \mid \ell_{q_1}(\beta_j \ell_{p_1}^{N_j}) \to \ell_{q_2}(\ell_{p_2}^{N_j})\| = \|\{\beta_j^{-1} N_j^{1/p^*}\}_j \mid \ell_{q^*}\|,$$

where $N_j^{1/p^*} = 1$ *if* $N_j = p^* = \infty$.

Proof. Proving the sufficiency of (3.1) and (3.2) is a simple exercise, using Hölder's inequality, the monotonicity of the ℓ_p spaces and the norm of the embedding $\ell_{p_1}^N \to \ell_{p_2}^N$.

The necessity of the conditions can be proved in a similar way as in the proof of [**9**, Theorem 3.1]; see also [**10**,**11**]. First, we assume that there exists a j_0 such that $N_{j_0} = \infty$. We consider the following commutative diagram:

$$\begin{array}{ccc} \ell_{p_1} & \xrightarrow{\ \mathrm{id}_1\ } & \ell_{p_2} \\ {\scriptstyle T}\big\downarrow & & \big\uparrow{\scriptstyle S} \\ \ell_{q_1}(\beta_j \ell_{p_1}^{N_j}) & \xrightarrow{\ \mathrm{id}\ } & \ell_{q_2}(\ell_{p_2}^{N_j}) \end{array}$$

Here,

$$(T\eta)_{j,i} := \begin{cases} \eta_i & \text{if } j = j_0, \\ 0 & \text{otherwise}, \end{cases} \qquad \eta \in \ell_{p_1},$$
$$(S\lambda)_i := \lambda_{j_0,i}, \quad \lambda \in \ell_{q_2}(\ell_{p_2}^{N_j}).$$

It follows from the above diagram that the embedding $\ell_{q_1}(\beta_j \ell_{p_1}^{N_j}) \hookrightarrow \ell_{q_2}(\ell_{p_2}^{N_j})$ implies that $p_1 \leqslant p_2$. Moreover, compactness of id implies compactness of id_1, so the embedding is not compact.

As for the conditions involving β_j, we employ the fact that the best constant c_j in

$$\|\tau_k \mid \ell_{p_2}^{N_j}\| \leqslant c_j \|\tau_k \mid \ell_{p_1}^{N_j}\|, \quad \{\tau_k\}_k \in \ell_{p_1},$$

is $c_j = N_j^{1/p^*}$, with $c_j = 1$ if $p^* = \infty$ and $N_j \in \mathbb{N} \cup \{\infty\}$. Hence, for all $\varepsilon > 0$ and all $j \in \mathbb{N}_0$ there exists a sequence $\{\tau_{j,k}\}_k \in \ell_{p_1}^{N_j}$ such that $\|\{\tau_{j,k}\}_k \mid \ell_{p_1}^{N_j}\| = 1$ and

$$\|\{\tau_{j,k}\}_k \mid \ell_{p_2}^{N_j}\| \geqslant (1 - 2^{-j}\varepsilon) N_j^{1/p^*} \|\{\tau_{j,k}\}_k \mid \ell_{p_1}^{N_j}\|.$$

We set $\mu_j := \beta_j^{-1} N_j^{1/p^*}$, $j \in \mathbb{N}_0$. Then,

$$\|\{\tau_{j,k}\}_k \mid \ell_{p_2}^{N_j}\| \geqslant (1 - 2^{-j}\varepsilon) \mu_j \beta_j.$$

By the same reasoning as before, we obtain that for all $\varepsilon > 0$ there exists a sequence $\{\gamma_j\}_j$ such that $\|\{\gamma_j\}_j \mid \ell_{q_2}\| = 1$ and

$$\|\{\gamma_j\}_j \mid \ell_{q_2}\| \geqslant (1 - \varepsilon) \|\{\mu_j\}_j \mid \ell_{q^*}\| \|\{\mu_j^{-1}\gamma_j\}_j \mid \ell_{q_1}\|.$$

Defining $\delta_j := \gamma_j/(\mu_j\beta_j)$, $j \in \mathbb{N}_0$, we arrive at

$$\begin{aligned}\biggl(\sum_{j=0}^{M}(\delta_j\|\{\tau_{j,k}\}_k \mid \ell_{p_2}^{N_j}\|)^{q_2}\biggr)^{1/q_2} &\geqslant (1-\varepsilon)\|\{\delta_j\mu_j\beta_j\}_j \mid \ell_{q_2}\| \\ &\geqslant (1-\varepsilon)^2\|\{\mu_j\}_j \mid \ell_{q^*}\|\|\{\delta_j\beta_j\}_j \mid \ell_{q_1}\| \\ &\geqslant (1-\varepsilon)^2\|\{\mu_j\}_j \mid \ell_{q^*}\|\|\{\delta_j\tau_{j,k}\}_{j,k} \mid \ell_{q_1}(\beta_j\ell_{p_1}^{N_j})\|\end{aligned}$$

if M is chosen sufficiently large. This proves that (3.1) and (3.2) are necessary for the embedding of the first sequence space into the second.

The sufficiency and necessity of the conditions for compactness in case (ii) was proved in [**10**]. □

We recall that an unbounded domain Ω in $\mathbb{R}^n$ is called quasi-bounded if

$$\lim_{x\in\Omega,\,|x|\to\infty} \operatorname{dist}(x,\partial\Omega) = 0.$$

An unbounded domain is not quasi-bounded if and only if it contains infinitely many pairwise disjoint congruent balls; see [**1**, p. 173].

As a simple consequence of the above theorem and the wavelet characterization of the function spaces, we have the following two corollaries.

Corollary 3.2. *Let Ω be an unbounded, uniformly E-porous domain in $\mathbb{R}^n$.*
If Ω is not quasi-bounded, then there exists an embedding

$$\bar{B}^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow \bar{B}^{s_2}_{p_2,q_2}(\Omega)$$

if and only if $p_1 \leqslant p_2$ and

$$s_1 - \frac{n}{p_1} - s_2 + \frac{n}{p_2} \geqslant 0 \quad \text{if } q^* = \infty,$$
$$s_1 - \frac{n}{p_1} - s_2 + \frac{n}{p_2} > 0 \quad \text{if } q^* < \infty.$$

If Ω is not quasi-bounded, then the embedding

$$\bar{A}^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow \bar{A}^{s_2}_{p_2,q_2}(\Omega)$$

is not compact.

Proof. The domain Ω is not quasi-bounded. Therefore it contains infinitely many pairwise disjoint congruent cubes. Thus, the assumption implies that for sufficiently large j the indices N_j are infinite: $N_j = \infty$ if $j > j_0$. So, the spaces $\ell_{q_i}(\beta_j\ell_{p_i}^{N_j})$, $i = 1, 2$, satisfy the assumptions of (i) of Theorem 3.1, with $\beta_j = 2^{j\delta}$ and $\delta = s_1 - (n/p_1) - s_2 + (n/p_2)$, if the above conditions on δ are satisfied. Thus, for Besov spaces the corollary follows from Theorems 2.8 and 3.1. The second statement for the F-scale follows from the case of the B-scale by elementary embeddings

$$\bar{B}^s_{p,\min(p,q)}(\Omega) \hookrightarrow \bar{F}^s_{p,q}(\Omega) \quad \text{and} \quad \bar{F}^s_{p,q}(\Omega) \hookrightarrow \bar{B}^s_{p,\max(p,q)}(\Omega).$$

□

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Corollary 3.3. *Let Ω be a uniformly E-porous domain in $\mathbb{R}^n$ with finite Lebesgue measure.*

Then, there exists an embedding

$$\bar{B}^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow \bar{B}^{s_2}_{p_2,q_2}(\Omega)$$

if and only if

$$s_1 - s_2 - \left(\frac{n}{p_1} - \frac{n}{p_2}\right)_+ \geqslant 0 \quad \text{if } q^* = \infty,$$
$$s_1 - s_2 - \left(\frac{n}{p_1} - \frac{n}{p_2}\right)_+ > 0 \quad \text{if } q^* < \infty.$$

The embedding

$$\bar{A}^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow \bar{A}^{s_2}_{p_2,q_2}(\Omega)$$

is compact if and only if

$$s_1 - s_2 - \left(\frac{n}{p_1} - \frac{n}{p_2}\right)_+ > 0.$$

Proof. The domain Ω is an open set. Therefore it contains a dyadic cube of $Q_{j_0 m}$ for some sufficiently large $j_0 \in \mathbb{N}$. On the other hand, it is a set of finite measure. Therefore we have that

$$2^{-j_0 n} \leqslant 2^{-jn} N_j \leqslant |\Omega| \quad \text{if } j > j_0;$$

see Remark 2.7. Consequently, there exist $c_1, c_2 > 0$ depending on Ω such that

$$c_1 2^{jn} \leqslant N_j \leqslant c_2 2^{jn}, \quad j \in \mathbb{N}_0.$$

Now, the corollary follows from (ii) of Theorem 3.1. □

Remark 3.4. For a set of finite Lebesgue measure we get the same conditions for continuity and compactness as for bounded smooth domains. On the other hand, if the domain has infinite measure and satisfies the assumptions of Corollary 3.2, then the Sobolev embeddings behave in the same way as on the whole space $\mathbb{R}^n$.

So, the most interesting cases are the quasi-bounded domains with infinite measure. If Ω is such a domain, then all numbers N_j are finite, since by the construction all cubes related to our wavelets are contained in Ω and the domain does not contain infinitely many congruent dyadic cubes. But, in contrast to the domain with finite measure, the numbers N_j are not asymptotically equivalent to 2^{jn}.

3.2. Box packing of open sets

Corollaries 3.2 and 3.3 suggest that the ball or cube coverings of an open set influence the behaviour of Sobolev embeddings. To describe this influence, we introduce the box-packing number $b(\Omega)$ of an open set Ω.

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Let $\Omega \subset \mathbb{R}^n$ be a non-empty open set $\Omega \neq \mathbb{R}^n$. Let

$$b_j(\Omega) = \sup\Big\{k\colon \bigcup_{\ell=1}^{k} Q_{j,m_\ell} \subset \Omega,\ Q_{j,m_\ell} \text{ being pairwise disjoint}$$
$$\text{dyadic cubes of side length } 2^{-j},\ m_\ell \in \mathbb{Z}^n\Big\}, \quad j = 0,1,\dots.$$

The following properties of the sequence $(b_j(\Omega))_{j=0,1,2,\dots}$ are obvious.

- $0 \leqslant b_j(\Omega) \leqslant \infty$ for any $j \in \mathbb{N}_0$ and $0 < b_j(\Omega)$ for sufficiently large j.
- If $b_{j_0}(\Omega) = \infty$, then $b_j(\Omega) = \infty$ for any $j \geqslant j_0$.
- If $b_{j_0}(\Omega) > 0$, then $b_j(\Omega) > 0$ for any $j \geqslant j_0$.
- If $\Omega_0 \subset \Omega$, then $b_j(\Omega_0) \leqslant b_j(\Omega)$ for any $j \in \mathbb{N}_0$.
- There exists a constant $j_0 = j_0(\Omega) \in \mathbb{N}_0$ such that for any $j \geqslant j_0$ we have that
$$2^{-j_0 n} \leqslant b_j(\Omega)2^{-jn}. \tag{3.4}$$
- If $|\Omega| < \infty$, then
$$b_j(\Omega)2^{-jn} \leqslant |\Omega|. \tag{3.5}$$

It follows from (3.4) that if $0 < s < n$, then $\lim_{j\to\infty} b_j(\Omega)2^{-js} = \infty$. Moreover, if $s_1 < s_2$ and the sequence $b_j(\Omega)2^{-js_1}$ is bounded, then $\lim_{j\to\infty} b_j(\Omega)2^{-js_2} = 0$. Thus, there exists at most one number $b \in \mathbb{R}$ such that $\limsup_{j\to\infty} b_j(\Omega)2^{-js} = \infty$ if $s < b$ and $\lim_{j\to\infty} b_j(\Omega)2^{-js} = 0$ if $s > b$. We set

$$b(\Omega) = \sup\Big\{t \in \mathbb{R}_+\colon \limsup_{j\to\infty} b_j(\Omega)2^{-jt} = \infty\Big\}. \tag{3.6}$$

Remark 3.5. For any non-empty open set $\Omega \subset \mathbb{R}^n$ we have that $n \leqslant b(\Omega) \leqslant \infty$. If Ω is unbounded and not quasi-bounded, then $b(\Omega) = \infty$. But, there are quasi-bounded domains such that $b(\Omega) = \infty$; see Example 3.12. Moreover, it follows from (3.4) and (3.5) that if the measure $|\Omega|$ is finite, then $b(\Omega) = n$.

Theorem 3.6. *Let Ω be a uniformly E-porous quasi-bounded domain in $\mathbb{R}^n$ and let $s_1 > s_2$.*

(i) *Let $p^* = \infty$, i.e. $p_1 \leqslant p_2$. Then, there exists an embedding*
$$\bar{B}^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow \bar{B}^{s_2}_{p_2,q_2}(\Omega) \tag{3.7}$$
if and only if
$$s_1 - \frac{n}{p_1} - s_2 + \frac{n}{p_2} \geqslant 0 \quad \text{if } q^* = \infty, \tag{3.8}$$
$$s_1 - \frac{n}{p_1} - s_2 + \frac{n}{p_2} > 0 \quad \text{if } q^* < \infty. \tag{3.9}$$

(ii) *Let $p^* < \infty$, i.e. $p_1 > p_2$.*
If $b(\Omega) = \infty$, then no embedding (3.7) holds.
If $b(\Omega) < \infty$ and

$$0 < \limsup_{j\to\infty} b_j(\Omega) 2^{-jb(\Omega)} < \infty, \tag{3.10}$$

then there exists an embedding (3.7) if and only if

$$s_1 - \frac{n}{p_1} - s_2 + \frac{n}{p_2} \geqslant \frac{b(\Omega)}{p^*} \quad \text{if } q^* = \infty,$$
$$s_1 - \frac{n}{p_1} - s_2 + \frac{n}{p_2} > \frac{b(\Omega)}{p^*} \quad \text{if } q^* < \infty.$$

(iii) *Let $p^* = \infty$. The embedding*

$$\bar{A}^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow \bar{A}^{s_2}_{p_2,q_2}(\Omega) \tag{3.11}$$

is compact if and only if

$$s_1 - \frac{n}{p_1} - s_2 + \frac{n}{p_2} > 0. \tag{3.12}$$

(iv) *Let $p^* < \infty$ and $b(\Omega) < \infty$. Then, the embedding (3.11) is compact if*

$$s_1 - \frac{n}{p_1} - s_2 + \frac{n}{p_2} > \frac{b(\Omega)}{p^*}. \tag{3.13}$$

Conversely, if the embedding (3.11) is compact, then $s_1 - (n/p_1) - s_2 + (n/p_2) \geqslant b(\Omega)/p^*$.

Proof. Due to the wavelet characterization of the function spaces one can reduce the problem to the embeddings of sequence spaces and use Theorem 3.1, with $\beta_j = 2^{j\delta}$, $\delta = s_1 - s_2 - n(1/p_1 - 1/p_2)$. Moreover, it follows from the construction of the wavelet basis that $N_j \leqslant b_j(\Omega)$; see (2.3) and (2.4). On the other hand, if there exists a dyadic cube $Q_{j-2,m} \subset \Omega$, then there are elements of the wavelet basis $\Psi^{j,L}_{G,m_r}$ such that $\operatorname{supp} \Psi^{j,L}_{G,m_r} \subset Q_{j-1,m_r} \subset Q_{j-2,m} \subset \Omega$; see (2.5), (2.6) and the construction described on page 834. Hence,

$$b_{j-2}(\Omega) \leqslant N_j \leqslant b_j(\Omega). \tag{3.14}$$

In particular, $N_j = \infty$ for large j if and only if $b_j(\Omega) = \infty$ for large j. Since the domain is quasi-bounded, we have that $b_j(\Omega) < \infty$ and, consequently, $N_j < \infty$ for any $j \in \mathbb{N}$.

We consider first the case $b(\Omega) = \infty$. Now, (3.6) and (3.14) imply that for any $s > 0$ there exist an increasing sequence j_k and a positive constant $c > 0$ such that $c2^{j_k s} \leqslant N_{j_k}$. This implies that if $q^* < \infty$, then (3.2) is fulfilled if and only if $\delta > 0$ and $1/p^* = 0$. In a similar way, if $q^* = \infty$, then (3.2) is fulfilled if and only if $\delta \geqslant 0$ and $1/p^* = 0$. Concerning compactness, (3.3) is fulfilled if and only if $\delta > 0$ and $1/p^* = 0$. Together with the elementary embeddings for B- and F-spaces, this proves the theorem in the case $b(\Omega) = \infty$.

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Let $b(\Omega) < \infty$. First, we consider the compactness. We choose $t > b(\Omega)$ such that $\delta > t/p*$. Then, it follows from (3.6) that $\limsup_{j\to\infty} b_j(\Omega)2^{-jt} < \infty$. This and (3.14) imply that there exists a constant M such that $N_j \leqslant M2^{jt}$. Thus, (3.2) holds if $\delta > t/p^*$. The last condition also implies (3.3) if $q^* = \infty$.

Conversely, let the embedding (3.11) be compact. If $1/p^* = 0$, then the inequality $\delta > 0$ follows directly from (3.2) and (3.3). If $1/p^* > 0$, then for any $t < b(\Omega)$ we have that $\limsup_{j\to\infty} b_j(\Omega)2^{-jt} = \infty$. But (3.3) implies that

$$2^{-j\delta}N_j \to 0 \quad \iff \quad 2^{-j(\delta-(t/p^*))}(2^{-jt}b_j(\Omega))^{1/p^*} \to 0.$$

Moreover, $\limsup_{j\to\infty} b_j(\Omega)2^{-jt} = \infty$, so $2^{-j(\delta-(t/p^*))} \to 0$. Thus, $\delta \geqslant b(\Omega)/p^*$.

Now, we consider the continuity of the embeddings for $b(\Omega) < \infty$. If $p^* = \infty$, we have exactly the same situation as in the case $b(\Omega) = \infty$. We assume that $p^* < \infty$. If $q^* < \infty$, then to prove the sufficiency of the conditions we can follow exactly the argument for the compactness and do not need (3.10). But, in the case $q^* = \infty$ we have to use the inequality $\limsup_{j\to\infty} b_j(\Omega)2^{-jb(\Omega)} < \infty$; see (3.10). Then, a similar argument to the one above proves the continuity of the embeddings.

Let (3.7) hold. As a consequence, $2^{-j\delta}N_j^{1/p^*} \in \ell_{q^*}$; see (3.2). But (3.14) implies that

$$c2^{-(j-2)\delta}b_{j-2}(\Omega) \leqslant 2^{-j\delta}N_j \leqslant 2^{-j\delta}b_j(\Omega).$$

So,

$$2^{-j\delta}N_j^{1/p^*} \in \ell_{q^*} \quad \iff \quad 2^{-j(\delta-(t/p^*))}(2^{-jt}b_j(\Omega))^{1/p^*} \in \ell_{q^*}. \tag{3.15}$$

If $q^* = \infty$, we choose t such that $\delta p^* < t < b(\Omega)$. Then, $2^{-jt}b_j(\Omega) \notin \ell_\infty$ and (3.15) imply that $\delta > t/p^*$. As a consequence, $\delta \geqslant b(\Omega)/p^*$.

If $q^* < \infty$, one can find a sequence $(j_k)_k$ such that $\lim_{k\to\infty} b_{j_k}(\Omega)2^{-j_k b(\Omega)} > 0$; see (3.10). Thus, (3.15) implies that $\delta > b(\Omega)/p^*$. □

Remark 3.7. Using the same method, one can prove that (3.13) is a sufficient and necessary condition for compactness if $\limsup_{j\to\infty} b_j(\Omega)2^{-jb(\Omega)} > 0$.

Remark 3.8. Let Ω be a uniformly E-porous quasi-bounded domain in $\mathbb{R}^n$. Let $\tilde{W}_p^k(\Omega)$, $1 < p < \infty$, $k = 1, 2, \dots$, be the completion of the space $C_0^\infty(\Omega)$ in the norm

$$\|f \mid \tilde{W}_p^k(\Omega)\| = \sum_{|\alpha|\leqslant k} \|\partial^\alpha f \mid L_p(\Omega)\|.$$

Then, $\tilde{W}_p^k(\Omega) = \tilde{F}_{p,2}^k(\Omega) = \bar{F}_{p,2}^k(\Omega)$ and the corresponding norms are equivalent; see [**14**, Theorem 4.30]. We set $\tilde{W}_p^0(\Omega) = L_p(\Omega)$ as well.

Corollary 3.9. *Let Ω be a uniformly E-porous quasi-bounded domain in $\mathbb{R}^n$. If $k_1 - k_2 - n(1/p_1 - 1/p_2) > b(\Omega)/p^*$, then the embedding*

$$\tilde{W}_{p_1}^{k_1}(\Omega) \hookrightarrow \tilde{W}_{p_2}^{k_2}(\Omega)$$

is compact. Here, $b(\Omega)/p^* = 0$ if $b(\Omega) = p^* = \infty$.

Corollary 3.10. *Let Ω be a uniformly E-porous quasi-bounded domain in $\mathbb{R}^n$. If $b(\Omega) < \infty$ and $s_1 - s_2 > \frac{1}{2}b(\Omega)$, then the embedding*

$$\bar{F}^{s_1}_{2,2}(\Omega) \hookrightarrow \bar{F}^{s_2}_{2,2}(\Omega) \tag{3.16}$$

is a Hilbert–Schmidt operator.

If the embedding (3.16) is a Hilbert–Schmidt operator, then $b(\Omega) < \infty$ and the inequality $s_1 - s_2 \geqslant \frac{1}{2}b(\Omega)$ holds.

Proof. Let $\{\Phi^j_r\}_{j,r}$, $j = 0, 1, 2, \dots$, $r = 1, \dots, N_j$, be an orthonormal u-wavelet basis in $L_2(\Omega)$. If $u > |s|$, then the system $\{2^{-js}\Phi^j_r\}_{j,r}$ is an orthogonal basis in $\bar{B}^s_{2,2}(\Omega) = \bar{F}^s_{2,2}(\Omega)$ such that

$$\|2^{-js}\Phi^j_r \mid \bar{B}^s_{2,2}(\Omega)\| \sim 1;$$

see Theorem 2.8.

If $u > s_1$, then $\{\Phi^j_r\}_{j,r}$ is an orthogonal basis in $\bar{F}^{s_1}_{2,2}(\Omega)$ as well as in $\bar{F}^{s_2}_{2,2}(\Omega)$. Moreover,

$$\|2^{-js_1}\Phi^j_r \mid \bar{B}^{s_2}_{2,2}(\Omega)\| \sim 2^{-j(s_1-s_2)}$$

for any $r = 1, \dots, N_j$. Thus, the embedding is a Hilbert–Schmidt operator if and only if

$$\sum_{j=0}^{\infty} 2^{-j2(s_1-s_2)} N_j < \infty.$$

Now, arguments similar to those used in the proof of Theorem 3.6, in particular the inequality (3.14), prove the corollary. □

Remark 3.11. Once more, if $\limsup_{j\to\infty} b_j(\Omega)2^{-jb(\Omega)} > 0$, then we can prove that the assumption that the inclusion (3.16) is a Hilbert–Schmidt operator implies that $s_1 - s_2 > \frac{1}{2}b(\Omega)$. If Ω is a domain with finite measure, then the condition means that $s_1 - s_2 > \frac{1}{2}n$. For bounded domains and the Sobolev spaces $\tilde{W}^k_2(\Omega)$, the sufficiency of the condition was proved by Maurin; see [**12**] or [**1**, p. 202].

Example 3.12. Let $\{r_j\}_j$, $j = 0, 1, 2, \dots$, be an increasing sequence of positive numbers. We assume that $r_0 > 1$ and set $R_j = R_{j-1} + 2^{j(n-1)}r_j$, with $R_{-1} = 0$. The sequence $\{R_j - R_{j-1}\}_j$ is also increasing. For any $j \in \mathbb{N}_0$ we define

$$\mathcal{A}_j = \{m\colon m = (\ell, 0, \dots, 0),\ 3j + R_{j-1} \leqslant 2^{-j}\ell < 3j + R_j,\ \ell \in \mathbb{N}\},$$

$$\Omega_j = \bigcup_{m\in\mathcal{A}_j} \bar{Q}_{j,m} \quad \text{and} \quad \Omega = \Big(\bigcup_{j\in\mathbb{N}_0} \Omega_j\Big)^{\circ}.$$

The set Ω is an open proper subset of $\mathbb{R}^n$. The cubes $Q_{j,m}$ and $Q_{k,n}$ are disjoint if $m \in \mathcal{A}_j$, $n \in \mathcal{A}_k$ and $j \neq k$. Moreover, Ω is E-thick, since $\operatorname{dist}(\Omega_j, \Omega_{j+1}) > 2$ and $\operatorname{dist}(x, \partial\Omega_j) \leqslant 2^{-j-1}$, $x \in \Omega_j$. Using the fact that the cardinality of the set $\mathcal{A}_j$ satisfies $\operatorname{card} \mathcal{A}_j \sim 2^j(R_j - R_{j-1})$, one can easily calculate that

$$b_j(\Omega) \sim 2^{jn} \sum_{k=0}^{j} r_k.$$

- If $r_k = 2^{\alpha k}$, $\alpha > 0$, then $0 < \lim_{j\to\infty} b_j(\Omega)2^{-j(n+\alpha)} < \infty$. As a consequence, $b(\Omega) = n + \alpha$.
- If $r_k = 2^{\alpha k}\max(1,k)$, $\alpha > 0$, then
$$\lim_{j\to\infty} b_j(\Omega)2^{-js} = \begin{cases}\infty & \text{if } s \leqslant n+\alpha,\\ 0 & \text{if } s > n+\alpha.\end{cases}$$
Thus, $b(\Omega) = n + \alpha$, but $\lim_{j\to\infty} b_j(\Omega)2^{-jb(\Omega)} = \infty$.
- If $r_k = 2^{\alpha k}\max(1,k)^{-1}$, $\alpha > 0$, then
$$\lim_{j\to\infty} b_j(\Omega)2^{-js} = \begin{cases}\infty & \text{if } s < n+\alpha,\\ 0 & \text{if } s > n+\alpha.\end{cases}$$
Thus, $b(\Omega) = n + \alpha$, but $\lim_{j\to\infty} b_j(\Omega)2^{-jb(\Omega)} = 0$.
- If $r_k = 2^{k^\alpha}$, $\alpha > 1$, then $b_j(\Omega)$ is finite, but $b(\Omega) = \infty$.

Example 3.13. Let $\alpha > 0$. We consider the open sets $\omega_\alpha, \Omega_\alpha \subset \mathbb{R}^2$ defined as
$$\omega_\alpha = \{(x,y) \in \mathbb{R}^2 \colon |y| < x^{-\alpha},\ x > 1\} \quad \text{and} \quad \Omega_\alpha = \{(x,y) \in \mathbb{R}^2 \colon |y| < |x|^{-\alpha}\}.$$
One can easily calculate that
$$b_j(\omega_\alpha) \sim \begin{cases} 2^{j(\alpha^{-1}+1)} & \text{if } 0 < \alpha < 1,\\ j2^{2j} & \text{if } \alpha = 1,\\ 2^{2j} & \text{if } \alpha > 1.\end{cases}$$
As a consequence,
$$b(\omega_\alpha) = \begin{cases}\alpha^{-1}+1 & \text{if } 0 < \alpha < 1,\\ 2 & \text{if } \alpha \geqslant 1.\end{cases}$$
The limit $\lim_{j\to\infty} b_j(\omega_\alpha)2^{-jb(\omega_\alpha)}$ is a positive finite number if $\alpha \neq 1$. If $\alpha = 1$, then the limit equals infinity. As a consequence,
$$b_j(\Omega_\alpha) \sim \begin{cases} 2^{j(\alpha^{-1}+1)} & \text{if } 0 < \alpha < 1,\\ j2^{2j} & \text{if } \alpha = 1,\\ 2^{j(\alpha+1)} & \text{if } \alpha > 1,\end{cases} \quad \text{and} \quad b(\Omega_\alpha) = \begin{cases}\alpha^{-1}+1 & \text{if } 0 < \alpha < 1,\\ \alpha+1 & \text{if } \alpha \geqslant 1.\end{cases}$$

4. Entropy numbers of embeddings

We recall the definition of entropy numbers.

Definition 4.1. Let $T \colon X \to Y$ be a bounded linear operator between complex quasi-Banach spaces, and let $k \in \mathbb{N}$. Then, the kth entropy number of T is defined as
$$e_k(T \colon X \to Y) := \inf\{\varepsilon > 0 \colon T(B_X) \text{ can be covered by } 2^{k-1} \text{ balls in } Y \text{ of radius } \varepsilon\},$$
where $B_X := \{x \in X \colon \|x\|_X \leqslant 1\}$ denotes the closed unit ball of X.

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In particular, $T\colon X \to Y$ is compact if and only if $\lim_{k\to\infty} e_k(T) = 0$. For details and basic properties like multiplicativity, additivity, behaviour under interpolation and the relation to eigenvalues of the compact operator we refer the reader to the monographs [**2**, **5**, **6**].

We recall that the definition of the relation '$\sim$' is given in § 1.

Theorem 4.2. *Let Ω be a uniformly E-porous domain in $\mathbb{R}^n$, with $\Omega \neq \mathbb{R}^n$, and let $b(\Omega) < \infty$. Let $\bar{A}^{s_1}_{p_1,q_1}(\Omega)$ and $\bar{A}^{s_2}_{p_2,q_2}(\Omega)$ be the function spaces defined in Definition 2.1. Let $s_1 - s_2 - n(1/p_1 - 1/p_2) > b(\Omega)/p^*$. If*

$$0 < \liminf_{j\to\infty} b_j(\Omega)2^{-jb(\Omega)} \leqslant \limsup_{j\to\infty} b_j(\Omega)2^{-jb(\Omega)} < \infty, \tag{4.1}$$

then, for $k \in \mathbb{N}$,

$$e_k(\bar{A}^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow \bar{A}^{s_2}_{p_2,q_2}(\Omega)) \sim k^{-\gamma}, \tag{4.2}$$

with

$$\gamma = \frac{s_1 - s_2}{b(\Omega)} + \frac{b(\Omega) - n}{b(\Omega)}\left(\frac{1}{p_1} - \frac{1}{p_2}\right).$$

Proof. It is sufficient to consider Besov spaces. The rest follows by the elementary embeddings of function spaces and the elementary properties of entropy numbers. By the wavelet characterization of the spaces $\bar{B}^s_{p,q}(\Omega)$ we have that

$$e_k(\bar{B}^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow \bar{B}^{s_2}_{p_2,q_2}(\Omega)) \sim e_k(\ell_{q_1}(2^{j\delta}\ell^{N_j}_{p_1}) \hookrightarrow \ell_{q_2}(\ell^{N_j}_{p_2})), \quad \delta = s_1 - s_2 - n\left(\frac{1}{p_1} - \frac{1}{p_2}\right),$$

for a suitable sequence $\{N_j\}_j$, $N_j \in \mathbb{N}$. Assumption (4.1) implies that there exist positive constants c_1 and c_2 such that

$$c_1 \leqslant b_j(\Omega)2^{-jb(\Omega)} \leqslant c_2 \quad \text{if } j > j_0.$$

As a consequence, there exists a positive constant $c \geqslant 2^{b(\Omega)}c_1^{-1}c_2$ such that

$$2^n b_j(\Omega) \leqslant b_{j+1}(\Omega) \leqslant c b_j(\Omega).$$

The left-hand inequality follows easily from the fact that any cube of size 2^{-j} contains 2^n cubes of size $2^{-(j+1)}$. In view of (3.14), we have that

$$2^n N_j(\Omega) \leqslant N_{j+1}(\Omega) \leqslant C N_j(\Omega).$$

Thus, both sequences $\{N_j\}_j$ and $\{2^{j\delta}\}_j$, $\delta > 0$, are strongly increasing and admissible in the sense of [**10**]. If $p_1 \leqslant p_2$, then [**10**, Theorem 3] implies that

$$e_{2N_j}(\ell_{q_1}(2^{j\delta}\ell^{N_j}_{p_1}) \hookrightarrow \ell_{q_2}(\ell^{N_j}_{p_2})) \sim 2^{-j\delta}N_j^{-(1/p_1 - 1/p_2)}. \tag{4.3}$$

But, (4.1) and (3.14) imply that

$$N_j \sim 2^{jb(\Omega)}. \tag{4.4}$$

Now, (4.3) and (4.4) imply (4.2).

If $p_2 < p_1$ and $\delta > b(\Omega)/p^*$, then the sequence $\{2^{j\delta}N_j^{1/p_1 - 1/p_2}\}_j$ is also a strongly increasing admissible sequence. Then, [**10**, Theorem 4] implies (4.3) and, as a consequence, (4.2). □

Remark 4.3. We always have that

$$\left(\frac{1}{p_1}-\frac{1}{p_2}\right)_+ < \gamma \leqslant \frac{s_1-s_2}{n}.$$

The first inequality follows from $\gamma \geqslant 1/p_1 - 1/p_2 + 1/p^*$ and the second from

$$\frac{(s_1-s_2)}{n} \geqslant \max\left\{\frac{b(\Omega)-n}{np^*}, \frac{1}{p_1}-\frac{1}{p_2}\right\}$$

and $b(\Omega) \geqslant n$.

Corollary 4.4. *Let Ω be a uniformly E-porous domain in $\mathbb{R}^n$, with $\Omega \neq \mathbb{R}^n$. Let Ω be of finite Lebesgue measure. If the embedding $\bar{A}^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow \bar{A}^{s_2}_{p_2,q_2}(\Omega)$ is compact, then*

$$e_k(\bar{A}^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow \bar{A}^{s_2}_{p_2,q_2}(\Omega)) \sim k^{-(s_1-s_2)/n}, \quad k \in \mathbb{N}. \tag{4.5}$$

Remark 4.5. It was mentioned by Triebel that one needs $|\Omega| < \infty$, but not that Ω is bounded, to prove the estimates (4.5); see [**14**, p. 125]. In the next theorem, we answer, at least in part, a question posed in [**14**, p. 128].

Theorem 4.6. *Let $s_1, s_2 \in \mathbb{R}$, $0 < p_1$, $p_2 \leqslant \infty$ and $0 < q_1$, $q_2 \leqslant \infty$. We assume that $(s_1-s_2)/n > (1/p_1 - 1/p_2)_+$.*

For positive real γ, such that $(s_1-s_2)/n \geqslant \gamma > (1/p_1-1/p_2)_+$, there exists a uniformly E-porous quasi-bounded domain Ω in $\mathbb{R}^n$ such that

$$e_k(\bar{B}^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow \bar{B}^{s_2}_{p_2,q_2}(\Omega)) \sim k^{-\gamma}, \quad k \in \mathbb{N}. \tag{4.6}$$

If (4.6) holds for some uniformly E-porous quasi-bounded domain Ω in $\mathbb{R}^n$ and $b(\Omega) < \infty$, then $(s_1-s_2)/n \geqslant \gamma > (1/p_1-1/p_2)_+$.

Proof.

Step 1 (sufficiency). Since

$$\frac{s_1-s_2}{n} \geqslant \gamma > \left(\frac{1}{p_1}-\frac{1}{p_2}\right)_+ \geqslant \frac{1}{p_1}-\frac{1}{p_2}$$

we can choose θ, $0 < \theta \leqslant 1$, such that

$$\gamma = \theta\frac{s_1-s_2}{n} + (1-\theta)\left(\frac{1}{p_1}-\frac{1}{p_2}\right).$$

Let $b = n\theta^{-1}$. Then, $b \geqslant n$ and

$$\gamma = \frac{s_1-s_2}{b} + \frac{b-n}{b}\left(\frac{1}{p_1}-\frac{1}{p_2}\right).$$

If $b = n$, then one can take any uniformly E-porous domain with finite Lebesgue measure. If $b > n$, one can find an open set Ω in $\mathbb{R}^n$ such that Ω is a uniformly E-porous domain, $b(\Omega) = b$ and $0 < \lim_{j\to\infty} b_j(\Omega)2^{-jb} < \infty$; see Example 3.12. Now, Theorem 4.2 implies (4.6).

Step 2 (necessity). Let Ω be a uniformly E-porous domain in $\mathbb{R}^n$ such that (4.6) holds. Then, by the wavelet characterization the spaces $B^s_{p,q}(\Omega)$ are isomorphic to the sequence spaces $\ell_q(2^{js}\ell_p^{N_j})$ for a suitable sequence $(N_j)_j$. As a consequence,

$$e_k(\ell_{q_1}(2^{j\delta}\ell_{p_1}^{N_j}) \hookrightarrow \ell_{q_2}(\ell_{p_2}^{N_j})) \sim k^{-\gamma}. \tag{4.7}$$

The open set Ω contains a dyadic cube $Q_{j_o,m}$. Thus, by the construction of the wavelet basis, $2^{(-j_0-2)n}2^{jn} \leqslant N_j$ if $j > j_0 + 2$; see page 834. As a consequence, there exists a constant $c > 0$ such that the inequality $c2^{jn} \leqslant N_j$ holds for any $j \in \mathbb{N}_0$. Let $M_j = [c2^{jn}]$, where $[x]$ denotes the integer part of $x \in \mathbb{R}$. We have the following commutative diagram, where Id is the natural embedding and P is a projection:

$$\begin{array}{ccc}
\ell_{q_1}(2^{j\delta}\ell_{p_1}^{M_j}) & \xrightarrow{\mathrm{id}_1} & \ell_{q_2}(\ell_{p_2}^{M_j}) \\
{\scriptstyle \mathrm{Id}}\downarrow & & \uparrow{\scriptstyle P} \\
\ell_{q_1}(2^{j\delta}\ell_{p_1}^{N_j}) & \xrightarrow{\mathrm{id}} & \ell_{q_2}(\ell_{p_2}^{N_j})
\end{array}$$

Now, by [**10**, Theorem 3], the above diagram and the elementary properties of entropy numbers we have that

$$\begin{aligned}
c_1 k^{-(s_1-s_2)/n} &\leqslant e_k(\mathrm{id}_1 \colon \ell_{q_1}(2^{j\delta}\ell_{p_1}^{M_j}) \hookrightarrow \ell_{q_2}(\ell_{p_2}^{M_j})) \\
&\leqslant e_k(\mathrm{id} \colon \ell_{q_1}(2^{j\delta}\ell_{p_1}^{N_j}) \hookrightarrow \ell_{q_2}(\ell_{p_2}^{N_j})) \\
&\leqslant c_2 k^{-\gamma}.
\end{aligned}$$

Thus, $(s_1 - s_2)/n \geqslant \gamma$.

The necessity of the condition $\gamma > 0$ is clear, since otherwise the embedding is not compact. So it remains to show that (4.6) and $p_1 < p_2$ imply that $\gamma > 1/p_1 - 1/p_2$.

Let Ω be a uniformly E-porous domain in $\mathbb{R}^n$ such that (4.6) holds for some $\gamma > 0$. But, (4.6) implies (4.7) for suitable sequence spaces. We fix $b > b(\Omega)$. Then, by (3.14) and the definition of $b(\Omega)$ there exists a positive constant c such that

$$N_j \leqslant b_j(\Omega) \leqslant c2^{jb} \quad \text{if } j \geqslant j_0.$$

We recall that $b(\Omega) < \infty$.

So taking

$$\tilde{N}_j = \begin{cases} N_j & \text{if } j < j_0, \\ [c2^{jb}] + 1 & \text{if } j \geqslant j_0, \end{cases}$$

we get the following commutative diagram:

$$\begin{array}{ccc}
\ell_{q_1}(2^{j\delta}\ell_{p_1}^{N_j}) & \xrightarrow{\mathrm{id}_1} & \ell_{q_2}(\ell_{p_2}^{N_j}) \\
{\scriptstyle \mathrm{Id}}\downarrow & & \uparrow{\scriptstyle P} \\
\ell_{q_1}(2^{j\delta}\ell_{p_1}^{\tilde{N}_j}) & \xrightarrow{\mathrm{id}} & \ell_{q_2}(\ell_{p_2}^{\tilde{N}_j})
\end{array}$$

But the sequence $\{\tilde{N}_j\}_j$ is a strongly increasing admissible sequence in the sense of [**10**]. So the last commutative diagram implies that

$$c_1 k^{-\gamma} \leqslant e_k(\mathrm{id}_1\colon \ell_{q_1}(2^{j\delta}\ell_{p_1}^{N_j}) \hookrightarrow \ell_{q_2}(\ell_{p_2}^{N_j})) \leqslant e_k(\mathrm{id}\colon \ell_{q_1}(2^{j\delta}\ell_{p_1}^{\tilde{N}_j}) \hookrightarrow \ell_{q_2}(\ell_{p_2}^{\tilde{N}_j})) \leqslant c_2 k^{-\tilde{\gamma}},$$

where $\tilde{\gamma} > 1/p_1 - 1/p_2$; see [**10**] or Theorem 4.2.

This implies the assertion. □

5. Applications to spectral theory on unbounded domains

Let Ω be a quasi-bounded uniformly E-porous domain satisfying the assumptions of Theorem 4.2. Then, $\bar{B}^{2m}_{2,2}(\Omega) = \bar{F}^{2m}_{2,2}(\Omega) = \tilde{W}^{2m}_2(\Omega)$, $m \in \mathbb{N}$; see Remark 3.8.

Let

$$A(x, D) = \sum_{|\alpha| \leqslant 2m} a_\alpha(x)\partial^\alpha$$

be a formally self-adjoint, uniformly strongly elliptic differential operator of order $2m$, $m \in \mathbb{N}$, with real valued coefficients $a_\alpha \in C^\infty(\Omega)$, which are uniformly bounded and uniformly continuous for $|\alpha| \leqslant 2m$. Then, the operator $A = A(x, D)$, with domain

$$\mathcal{D}(A) = \bar{B}^{2m}_{2,2}(\Omega),$$

is a closed linear operator with discrete spectrum $\sigma(A)$ of eigenvalues having no finite accumulation point; see [**3**, **7**].

We assume that A is a positive self-adjoint operator in $L_2(\Omega)$.

Theorem 5.1. *Let Ω be a uniformly E-porous domain in $\mathbb{R}^n$, $\Omega \neq \mathbb{R}^n$, such that $b(\Omega) < \infty$ and*

$$0 < \liminf_{j\to\infty} b_j(\Omega)2^{-jb(\Omega)} \leqslant \limsup_{j\to\infty} b_j(\Omega)2^{-jb(\Omega)} < \infty. \tag{5.1}$$

Let $\lambda_1, \lambda_2, \dots$ be eigenvalues of A ordered by their magnitude and counted according to their multiplicities. Then,

$$\lambda_k \sim k^{2m/b(\Omega)}, \quad k \in \mathbb{N}.$$

Proof. The proof is standard. Since A is a positive self-adjoint operator with compact resolvent, λ_k is an eigenvalue of A if and only if $\mu_k = \lambda_k^{-1}$ is an eigenvalue of A^{-1}. But A is a bounded operator mapping $\bar{B}^{2m}_{2,2}(\Omega)$ onto $L_2(\Omega)$. So we can factorize A^{-1} through the compact embedding $\bar{B}^{2m}_{2,2}(\Omega) \hookrightarrow L_2(\Omega)$. Using Carl's inequality (see [**6**, Theorem 1.3.4]) and Theorem 4.2, we get that

$$\mu_k \leqslant Ck^{-2m/b(\Omega)}.$$

On the other hand, let ψ be a smooth function such that $\operatorname{supp}\psi \subset (0,1)^n$ and $\|\psi \mid L_2(\Omega)\| = 1$. Let $\psi_j(x) = 2^{jn/2}\psi(2^j x)$, $j = 1, 2, \dots$, and let $\psi_{j,h}(x) = \psi_j(x-h)$, $h \in \mathbb{R}^n$.

If supp $\psi_{j,h} \subset \Omega$, then

$$\begin{aligned} C &\leqslant \|\psi_{j,h} \mid L_2(\Omega)\|^2 \\ &\leqslant \|A\psi_{j,h} \mid L_2(\Omega)\| \|A^{-1}\psi_{j,h} \mid L_2\| \\ &\leqslant C\|\psi_{j,h} \mid \tilde{W}_2^{2m}(\Omega)\| \|A^{-1}\psi_{j,h} \mid L_2(\Omega)\| \\ &\leqslant C2^{2mj} \|A^{-1}\psi_{j,h} \mid L_2(\Omega)\|. \end{aligned}$$

Please note that $\|\psi_{j,h} \mid \tilde{W}_2^{2m}(\Omega)\| \leqslant C2^{2mj}$, since the functions are defined by the translations and dilations by the factor 2^j of one fixed test function.

By translation one can find $b_j(\Omega)$ functions of the form $\psi_{j,h}$ with pairwise disjoint supports contained in Ω. These functions span the $b_j(\Omega)$-dimensional subspaces of $\bar{B}^{2m}_{2,2}(\Omega) \subset L_2(\Omega)$. So for any linear operator T of rank smaller than $b_j(\Omega)$ we can find $\psi_{j,h}$ such that $T(\psi_{j,h}) = 0$. But this implies that

$$\begin{aligned} a_{b_j(\Omega)}(A^{-1}) &= \inf\{\|A^{-1} - T \mid \mathcal{L}(L_2(\Omega))\|,\ \operatorname{rank} T < b_j(\Omega)\} \\ &\geqslant \|A^{-1}\psi_{j,h} \mid L_2(\Omega)\| \\ &\geqslant c2^{-2mj}. \end{aligned} \tag{5.2}$$

Now, using the properties of the approximation numbers $a_k(A^{-1})$ in Hilbert spaces, (5.1) and (5.2) we get that

$$\mu_k = a_k(A^{-1}) \geqslant ck^{-2m/b(\Omega)}.$$

This completes the proof. □

Example 5.2. Let $\alpha > 0$ and $\alpha \neq 1$. Once more we consider the open set $\Omega_\alpha \subset \mathbb{R}^2$ defined as

$$\Omega_\alpha = \{(x, y) \in \mathbb{R}^2 \colon |y| < |x|^{-\alpha}\}.$$

Since

$$b(\Omega_\alpha) = \begin{cases} \alpha^{-1} + 1 & \text{if } 0 < \alpha < 1, \\ \alpha + 1 & \text{if } \alpha > 1, \end{cases}$$

we have the following formula for the eigenvalues of the Dirichlet Laplacian on Ω_α:

$$\lambda_k(-\Delta) \sim \begin{cases} k^{2\alpha/(1+\alpha)} & \text{if } 0 < \alpha < 1, \\ k^{2/(1+\alpha)} & \text{if } \alpha > 1. \end{cases} \tag{5.3}$$

Remark 5.3. If $|\Omega| < \infty$, then we have that

$$\lambda_k \sim k^{2m/n}.$$

This formula is well known for bounded regular domains and goes back to Weyl; see [**15**]. On the other hand, König considered a similar problem for so-called quasi-bounded full C_1^ℓ-domains, $\ell > 0$. He proved that

$$\lambda_k \sim k^{(\ell/(\ell+1))(2m/n)};$$

see [**7**, **8**].

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We can also perturb A by a multiplication operator $f \mapsto Vf$, giving that

$$H_\alpha f = Af - \alpha V f, \quad V(x) > 0 \text{ almost everywhere in } \Omega, \ \alpha > 0,$$

and ask for the behaviour of the cardinality of the negative spectrum $\#\{\sigma(H_\alpha) \cap (-\infty, 0]\}$ as $\alpha \to \infty$. This is the usual question raised in the study of spectral properties of elliptic operators; see [**5**, **6**, **13**]. Using the entropy version of the Birman–Schwinger principle (see [**6**]), one can easily prove the following theorem.

Theorem 5.4. *Let Ω be a uniformly E-porous domain in $\mathbb{R}^n$, $\Omega \neq \mathbb{R}^n$, such that $b(\Omega) < \infty$, and let (5.1) hold.*

Suppose that $2 < r \leqslant \infty$, $mr > n$ and $V^{1/2} \in L_r(\Omega)$. Let $\varrho = b(\Omega)r/(2mr + b(\Omega) - n)$ if $r < \infty$ and let $\varrho = b(\Omega)/2m$ if $r = \infty$. Then,

$$\#\{\sigma(H_\alpha) \cap (-\infty, 0]\} \leqslant c(\alpha \|V^{1/2} \mid L_r(\Omega)\|)^\varrho$$

for some $c > 0$, which is independent of α.

Remark 5.5. If $|\Omega| < \infty$, then this estimate coincides with the estimate proved in [**6**] for bounded domains.

Example 5.6. Assumption (4.1) in Theorem 4.2 is sufficient but not necessary to get the estimates of corresponding entropy numbers. We consider the domain Ω_α from Example 3.13, with $\alpha = 1$. Then, $b_j(\Omega_1) \sim j2^{2j}$, $b(\Omega_1) = 2$, but $\limsup b_j(\Omega)2^{-jb(\Omega_1)} = \infty$, so we cannot apply Theorem 4.2. But, (3.14) implies that $(N_j)_{j=1}^\infty$ is an admissible strongly increasing sequence in the sense of [**10**] and

$$N_j \sim j2^{2j}.$$

Now, using [**10**, Theorem 3] we get that

$$e_k(\bar{A}^{s_1}_{p_1,q_1}(\Omega_1) \hookrightarrow \bar{A}^{s_2}_{p_2,q_2}(\Omega_1)) \sim k^{-(s_1-s_2)/2}(\log k)^{(s_1-s_2)/2-(1/p_1-1/p_2)}.$$

As a consequence, we get that

$$\lambda_k(-\Delta) \sim k \log k$$

as a counterpart of (5.3) for $\alpha = 1$.

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