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ON THE EDGEWORTH EXPANSION FOR ELEMENTARY POLYNOMIALS BASED ON TRIMMED SAMPLES

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This paper develops a one term Edgeworth expansion under minimal conditions for elementary symmetric polynomials of any degree based on trimmed samples. These statistics are special cases of trimmed U-statistics and natural extensions of the trimmed mean.

1. INTRODUCTION AND RESULTS

Let $X_1, \ldots, X_n$ be independent and identically distributed, real-valued, random variables with distribution function F and let $X_{n1} \leqslant \cdots \leqslant X_{nn}$ denote the order statistics of the $X_i's$. We shall consider the behaviour of trimmed versions of the elementary symmetric polynomials studied in [**12, 8, 9, 21**], among others.

Consider the trimmed U-statistic sum of the form

$$U(\alpha,\beta) = \binom{k_{\alpha\beta}}{m}^{-1} \sum_{k_\alpha+1 \leqslant i_1 < \cdots < i_m \leqslant k_\beta} h(X_{ni_1}, \ldots, X_{ni_m}) \tag{1}$$

with the kernel

$$h(x_1, \ldots, x_m) = x_1 \cdots x_m, \ m \geqslant 1 ,$$

where $0 \leqslant \alpha < \beta \leqslant 1$ are any fixed numbers, $k_{\alpha\beta} = k_\beta - k_\alpha, k_\alpha = [\alpha n], k_\beta = [\beta n]$ and $[\cdot]$ denotes the integer part.

If $\alpha = 0$ and $\beta = 1$ then $U(0,1)$ corresponds to the ordinary elementary symmetric polynomial of degree m which is a U-statistic with product kernel based on the full sample. Limit theorems, Berry-Esseen bounds, Edgeworth espansions and large deviation results have been established for this class of statistic. See, for example, [**8, 10, 4, 6**].

For $m = 1$, (1) gives the (α,β) trimmed sample mean

$$\overline{X}(\alpha,\beta) = (k_\beta - k_\alpha)^{-1} \sum_{i=k_\alpha+1}^{k_\beta} X_{ni}. \tag{2}$$

The asymptotic normality of the trimmed mean was established in [**3**] and many properties were established in the theory of robust estimation. The approximation problems

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connected with sharpening the rate of convergence to the normal distribution of $\overline{X}(\alpha,\beta)$ have been explored in many papers including [**11, 13, 14, 15, 19, 22, 23, 24, 25**].

For the general class of U-statistics, the asymptotic normality of the trimmed statistic was established in [**17**]. For fixed $m \geqslant 1$ we shall obtain conditions under which the 1-term Edgeworth expansion holds for the trimmed elementary symmetric polynomials given in (1).

We need the following notation. Write $F^{-1}(u) = \inf\{x : F(x) \geqslant u\}$, $0 < u \leqslant 1$, for the left-continuous inverse-function of F and $F_n(x)$ for the empirical distribution function. The γth quantile of F is $\xi_\gamma = F^{-1}(\gamma)$. The sample estimate of ξ_γ is $\overline{\xi}_\gamma = F_n^{-1}(\gamma) = X_{nk_\gamma}$.

Let

$$\mu = (\beta-\alpha)^{-1}\int_\alpha^\beta F^{-1}(u)du \tag{3}$$

and $w_\gamma = n\gamma - [n\gamma]$ with $\gamma = \alpha,\beta$. Let $W_i, i = 1,\dots,n$, denote X_i winsorised outside of $(\xi_\alpha, \xi_\beta]$, that is

$$W_i = \xi_\alpha I(X_i \leqslant \xi_\alpha) + X_i I(\xi_\alpha < X_i \leqslant \xi_\beta) + \xi_\beta I(X_i > \xi_\beta)\ , \tag{4}$$

where $I(A)$ is the indicator of the event A. Then $W_i \overset{d}{=} Q(U_i), i = 1,\dots,n$, where U_i are independent random variables uniformly distributed on $(0,1)$ and

$$Q(u) = \xi_\alpha I(u \leqslant \alpha) + F^{-1}(u) I(\alpha < u \leqslant \beta) + \xi_\beta I(u > \beta)\ . \tag{5}$$

Furthermore, let $W_{ni}, i = 1,\dots,n$ denote the order statistics corresponding to $W_1,\dots,W_n$. Then

$$W_{ni} = \xi_\alpha I(i \leqslant N_\alpha) + X_{ni} I(N_\alpha < i \leqslant N_\beta) + \xi_\beta I(i > N_\beta)\ , \tag{6}$$

where $N_\gamma = \sum_{i=1}^n I(X_i \leqslant \xi_\gamma)$ with $\gamma = \alpha,\beta$. Note that
$N_\gamma \overset{d}{=} \sum_{i=1}^n I(U_i \leqslant \gamma)$. Let

$$v = \int_0^1 Q(u)du, \quad \sigma^2 = \int_0^1 (Q(u)-v)^2 du\ ,$$

$$\gamma_2 = -\alpha^2 \frac{1}{f(\xi_\alpha)}(v-\xi_\alpha)^2 + (1-\beta)^2 \frac{1}{f(\xi_\beta)}(v-\xi_\beta)^2 + \frac{m-1}{\beta-\alpha}\frac{\sigma^4}{\mu}\ ,$$

$$\gamma_3 = \int_0^1 (Q(u)-v)^3 du\ , \quad \lambda_1 = \gamma_3/\sigma^3, \quad \lambda_2 = \gamma_2/\sigma^3,$$

$$\lambda_3 = 6\sigma^{-1}\Big\{(\xi_\alpha-\mu)w_\alpha - (\xi_\beta-\mu)w_\beta - \frac{\alpha(1-\alpha)}{2f(\xi_\alpha)} + \frac{\beta(1-\beta)}{2f(\xi_\beta)} + \frac{m-1}{2(\beta-\alpha)\mu}\big[\alpha(\xi_\alpha-\mu)^2 + (1-\beta)(\xi_\beta-\mu)^2 - (v-\mu)^2\big]\Big\}\ .$$

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For $x \in R$ let

$$F_\sigma(x) = P\Big(\frac{\sqrt{n}(\beta-\alpha)}{m\mu^{m-1}\sigma}(U(\alpha,\beta)-\mu^m) \leqslant x\Big) ,$$
$$G(x) = \Phi(x) - \frac{\phi(x)}{6\sqrt{n}}(\lambda_1 + 3\lambda_2(x^2-1)+\lambda_3) ,$$

where Φ is the standard normal distribution function, $\phi = \Phi'$.

Theorem 1. *Assume that $f = F'$ exists in neighbourhoods of ξ_α and ξ_β where it satisfies a Lipschitz condition. Further assume that $f(\xi_\alpha) > 0$ and $f(\xi_\beta) > 0$. Then*

$$\sup_x \big|F_\sigma(x) - G(x)\big| = O\big((\ln n)^{5/4} n^{-3/4}\big)$$

as $n \to \infty$.

To studentise $U(\alpha,\beta)$ we need an estimate of σ^2. We shall use

$$S_n^2 = \frac{k_\alpha}{n} X_{nk_\alpha}^2 + \frac{1}{n}\sum_{i=k_\alpha+1}^{k_\beta-1} X_{ni}^2 + \frac{n-k_\beta+1}{n} X_{nk_\beta}^2 - v_n^2 ,$$

where

$$v_n = \frac{k_\alpha}{n} X_{nk_\alpha} + \frac{1}{n}\sum_{i=k_\alpha+1}^{k_\beta-1} X_{ni} + \frac{n-k_\beta+1}{n} X_{nk_\beta}.$$

For $x \in R$ let

(7) $$F_s(x) = P\Big(\frac{\sqrt{n}(\beta-\alpha)}{m\mu^{m-1}S_n}(U(\alpha,\beta)-\mu^m) \leqslant x\Big) ,$$
$$H(x) = \Phi(x) + \frac{\phi(x)}{6\sqrt{n}}\big((2x^2+1)\lambda_1 + 3(x^2+1)\lambda_2 - \lambda_3\big) .$$

Theorem 2. *Assume that the conditions of Theorem 1 are satisfied. Then*

$$\sup_x \big|F_s(x) - H(x)\big| = O\big((\ln n)^{5/4} n^{-3/4}\big),$$

as $n \to \infty$.

2. Proofs

We begin with following lemma which gives a useful representation of $U(\alpha,\beta)$.

Lemma 1.

(8) $$U(\alpha,\beta) - \mu^m = \sum_{r=1}^{m} \frac{m(m-1)\cdots(m-r+1)}{k_{\alpha\beta}(k_{\alpha\beta}-1)\cdots(k_{\alpha\beta}-r+1)} \cdot \mu^{m-r} \cdot S_r(\alpha,\beta) ,$$

where

(9) $$S_r(\alpha,\beta) = \sum (-1)^{r+i_1+\cdots+i_r} \prod_{\ell=1}^{r} (\ell^{i_\ell} i_\ell !)^{-1} \prod_{\ell=1}^{r} \pi_\ell^{i_\ell} ,$$

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where the summation is over all non-negative integers $i_1, \ldots, i_r$ *satisfying* $\sum_{\ell=1}^{r} \ell i_\ell = r$, *in addition,*

$$\pi_\ell = \sum_{p=0}^{\ell} \binom{\ell}{p} (-\mu)^{\ell-p} \sigma_p(\alpha,\beta) , \tag{10}$$

where

$$\sigma_p(\alpha,\beta) = \sum_{i=1}^{n} W_i^p - k_\alpha \xi_\alpha^p - (n-k_\beta)\xi_\beta^p + J_p(\alpha) - \overline{J}_p(\alpha) - J_p(\beta) + \overline{J}_p(\beta) \tag{11}$$

with, for $\gamma = \alpha, \beta$,

$$J_p(\gamma) = I(k_\gamma < N_\gamma) \sum_{i=k_\gamma+1}^{N_\gamma} (X_{ni}^p - \xi_\gamma^p) \ , \overline{J}_p(\gamma) = I(k_\gamma > N_\gamma) \sum_{i=N_\gamma+1}^{k_\gamma} (X_{ni}^p - \xi_\gamma^p) \ . \tag{12}$$

PROOF: First we have Hoeffding's decomposition

$$U(\alpha,\beta) - \mu^m = \sum_{r=1}^{m} \frac{m(m-1)\cdots(m-r+1)}{k_{\alpha\beta}(k_{\alpha\beta}-1)\cdots(k_{\alpha\beta}-r+1)} \mu^{m-r} S_r(\alpha,\beta) ,$$

where

$$S_r(\alpha,\beta) = \sum_{k_\alpha+1 \leqslant i_1 < \cdots < i_r \leqslant k_\beta} (X_{ni_1} - \mu) \cdots (X_{ni_r} - \mu) \ .$$

Further, by Waring's formula, (see for example, [7]), we obtain for $S_r(\alpha,\beta)$ the representation (9) with

$$\pi_\ell = \sum_{i=k_\alpha+1}^{k_\beta} (X_{ni} - \mu)^\ell.$$

Hence, it is necessary to prove that this π_ℓ has the form (10). Indeed,

$$\pi_\ell = \sum_{i=k_\alpha+1}^{k_\beta} \sum_{p=0}^{\ell} \binom{\ell}{p} (-\mu)^{\ell-p} X_{ni}^p = \sum_{p=0}^{\ell} \binom{\ell}{p} (-\mu)^{\ell-p} \sigma_p(\alpha,\beta)$$

where

$$\sigma_p(\alpha,\beta) = \sum_{i=k_\alpha+1}^{k_\beta} X_{ni}^p \ . \tag{13}$$

Using (4) and (6) we can write

$$\sum_{i=1}^{n} W_i^p = \sum_{i=1}^{n} W_{ni}^p = N_\alpha \xi_\alpha^p + \sum_{i=N_\alpha+1}^{N_\beta} X_{ni}^p + (n - N_\beta)\xi_\beta^p ,$$

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that is,

$$\sum_{i=N_\alpha+1}^{N_\beta} X_{ni}^p = \sum_{i=1}^{n} W_i^p - N_\alpha \xi_\alpha^p - (n - N_\beta)\xi_\beta^p . \tag{14}$$

Furthermore, we have

$$\sum_{i=k_\alpha+1}^{k_\beta} X_{ni}^p = \sum_{i=N_\alpha+1}^{N_\beta} X_{ni}^p + J_p(\alpha) - \overline{J}_p(\alpha) + \overline{J}_p(\beta) - J_p(\beta) \tag{15}$$
$$+ (N_\alpha - k_\alpha)\xi_\alpha^p + (k_\beta - N_\beta)\xi_\beta^p .$$

From (13)–(15) we obtain (11) and hence (10). Lemma 1 is proved. □

Let $U_1 = 1/(\sigma\sqrt{n})\sum_{i=1}^{n} g(X_i)$, where

$$g(X_i) = W_i - v,\ U_2 = 1/(\sigma n\sqrt{n}) \sum_{1\leqslant i<j\leqslant n} h(X_i, X_j),$$

and

$$h(X_i, X_j) = -(I(X_i \leqslant \xi_\alpha) - \alpha)(I(X_j \leqslant \xi_\alpha) - \alpha)\frac{1}{f(\xi_\alpha)}$$
$$+ (I(X_i \leqslant \xi_\beta) - \beta)(I(X_j \leqslant \xi_\beta) - \beta)\frac{1}{f(\xi_\beta)} + \frac{(m-1)}{(\beta-\alpha)\mu}(W_i - v)(W_j - v).$$

Lemma 2. *Suppose that the conditions of Theorem 1 are satisfied. Then*

$$\frac{\sqrt{n}(\beta-\alpha)}{m\mu^{m-1}\sigma}(U(\alpha,\beta) - \mu^m) = U_1 + U_2 + \frac{\lambda_3}{6\sqrt{n}}$$
$$+ \frac{1}{n\sqrt{n}}\left|\sum_{i=1}^{n}(I(X_i \leqslant \xi_\alpha) - \alpha)\right|^{3/2} R_n + \frac{1}{n\sqrt{n}}\left|\sum_{i=1}^{n}(I(X_i \leqslant \xi_\beta) - \beta)\right|^{3/2} R_n + \overline{R}_n,$$

where R_n and $\overline{R}_n$ satisfy

$$P\big(|R_n| > c\sqrt{\ln n}\big) = O(n^{-d}), \quad P\big(|\overline{R}_n| > c(\ln n)^{3/2} n^{-1}\big) = O(n^{-d})$$

as $n \to \infty$ for some sufficiently large, positive constants c and d not depending on n.

Proof: We shall follow the approach in [**13**] and [**11**] to obtain sharp approximations for $J_p(\gamma)$ and $\overline{J}_p(\gamma)$ in (12) by functions of N_γ for $0 < \gamma < 1$ and any integer $p \geqslant 1$. Let $U_{n1} \leqslant \cdots \leqslant U_{nn}$ be the order statistics corresponding to the independent random variables $U_1, \ldots, U_n$ uniformly distributed on $(0, 1)$.

Estimating $J_p(\gamma)$. Under the conditions of the theorem

$$J_p(\gamma) \stackrel{d}{=} I(k_\gamma < N_\gamma) \sum_{i=k_\gamma+1}^{N_\gamma} \Big[(F^{-1}(U_{ni}))^p - (F^{-1}(\gamma))^p\Big]$$
$$= I(k_\gamma < N_\gamma)\Big\{\frac{p\,\xi_\gamma^{p-1}}{f(\xi_\gamma)} \sum_{i=k_\gamma+1}^{N_\gamma} (U_{ni} - \gamma) + r(\gamma)\Big\} , \tag{16}$$

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where

$$\left|r(\gamma)\right| \leqslant c \sum_{i=k_\gamma+1}^{N_\gamma} (U_{ni} - \gamma)^2$$

and the constant c can depend on p, γ and F. Conditional on N_γ the order statistics U_{ni}, $1 \leqslant i \leqslant N_\gamma$, are distributed as the order statistics from a sample of size N_γ from the uniform distribution on $(0, \gamma)$. Therefore for $i = 1, \ldots, N_\gamma$

$$\mu_i(\gamma) = E(U_{ni} \mid N_\gamma) = \frac{\gamma i}{N_\gamma + 1}, \quad \sigma_i^2(\gamma) = E\Big(\big(U_{ni} - \mu_i(\gamma)\big)^2 \mid N_\gamma\Big) = \frac{\gamma^2 i(N_\gamma - i + 1)}{(N_\gamma + 1)^2 (N_\gamma + 2)}$$

and in (16)

$$\text{(17)} \quad \sum_{i=k_\gamma+1}^{N_\gamma} (U_{ni} - \gamma) = -\frac{\gamma}{2(N_\gamma + 1)}(N_\gamma - k_\gamma)(N_\gamma - k_\gamma + 1) + \sum_{i=k_\gamma+1}^{N_\gamma} \big(U_{ni} - \mu_i(\gamma)\big),$$

$$\sum_{i=k_\gamma+1}^{N_\gamma} (U_{ni} - \gamma)^2 \leqslant 2\gamma^2 \frac{(N_\gamma - k_\gamma)^3}{(N_\gamma + 1)^2} + 2 \sum_{i=k_\gamma+1}^{N_\gamma} \big(U_{ni} - \mu_i(\gamma)\big)^2.$$

Denote for $i = k_\gamma + 1, \ldots, N_\gamma$

$$\eta_i = \big(U_{ni} - \mu_i(\gamma)\big)/\sigma_i(\gamma)$$

and note that

$$\sigma_i^2(\gamma) \leqslant \gamma^2 \frac{N_\gamma - k_\gamma}{(N_\gamma + 1)^2}.$$

For η_i we can write (see, for example, Lemma 3.1.1 in [**20**]),

$$\text{(18)} \qquad P\big(|\eta_i| > c\sqrt{\ln n} | N_\gamma\big) = O(n^{-d})$$

uniformly for $k_\gamma + 1 \leqslant i \leqslant N_\gamma$ with some positive constants c and d which do not depend on n. Furthermore

$$\text{(19)} \quad \begin{aligned} \left|\sum_{i=k_\gamma+1}^{N_\gamma} \big(U_{ni} - \mu_i(\gamma)\big)\right| &\leqslant (N_\gamma - k_\gamma) \max_{k_\gamma+1 \leqslant i \leqslant N_\gamma} \left|U_{ni} - \mu_i(\gamma)\right| \\ &\leqslant \gamma \frac{1}{(N_\gamma + 1)} (N_\gamma - k_\gamma)^{3/2} \max_{k_\gamma+1 \leqslant i \leqslant N_\gamma} |\eta_i| \end{aligned}$$

and

$$\text{(20)} \quad \begin{aligned} \sum_{i=k_\gamma+1}^{N_\gamma} \big(U_{ni} - \mu_i(\gamma)\big)^2 &\leqslant (N_\gamma - k_\gamma) \max_{k_\gamma+1 \leqslant i \leqslant N_\gamma} \big(U_{ni} - \mu_i(\gamma)\big)^2 \\ &\leqslant \gamma^2 \frac{1}{(N_\gamma + 1)^2} (N_\gamma - k_\gamma)^2 \max_{k_\gamma+1 \leqslant i \leqslant N_\gamma} \eta_i^2. \end{aligned}$$

Combining (16)–(20) we find

$$J_p(\gamma) = -I(k_\gamma < N_\gamma)\frac{(N_\gamma - k_\gamma)^2}{n}\frac{p}{2}\frac{\xi_\gamma^{p-1}}{f(\xi_\gamma)} + \frac{1}{n}|N_\gamma - k_\gamma|^{3/2} r_n + \frac{1}{n^2}(N_\gamma - k_\gamma)^2 r_n^2 \tag{21}$$

with $P(|r_n| > c\sqrt{\ln n}) = O(n^{-d})$.

ESTIMATING $\overline{J}_p(\gamma)$. By analogy with (16) we write

$$\overline{J}_p(\gamma) \overset{d}{=} I(k_\gamma > N_\gamma)\left\{\frac{p\,\xi_\gamma^{p-1}}{f(\xi_\gamma)} \sum_{i=N_\gamma+1}^{k_\gamma} (U_{ni} - \gamma) + \overline{r}(\gamma)\right\}, \tag{22}$$

where

$$|\overline{r}(\gamma)| \leqslant c \sum_{i=N_\gamma+1}^{k_\gamma} (U_{ni} - \gamma)^2.$$

Note that now conditional on N_γ the order statistics $U_{ni}, N_\gamma + 1 \leqslant i \leqslant n$ are distributed as the order statistics of a sample of size $n - N_\gamma$ from a uniform distribution on $(\gamma, 1)$. Hence for $i = N_\gamma + 1, \dots, n$

$$\overline{\mu}_i(\gamma) = E(U_{ni} \mid N_\gamma) = \gamma + (1 - \gamma)\frac{(i - N_\gamma)}{n - N_\gamma + 1},$$

$$\overline{\sigma}_i^2(\gamma) = E\Big(\big(U_{ni} - \overline{\mu}_i(\gamma)\big)^2 \mid N_\gamma\Big) = \frac{(1-\gamma)^2(i - N_\gamma)(n - i + 1)}{(n - N_\gamma + 1)^2(n - N_\gamma + 2)}$$

and if $N_\gamma + 1 \leqslant i \leqslant k_\gamma$ then

$$\overline{\sigma}_i^2(\gamma) \leqslant (1-\gamma)^2 \frac{k_\gamma - N_\gamma}{(n - N_\gamma + 1)^2}$$

and

$$P\big(|\overline{\eta}_i| > c\sqrt{\ln n} | N_\gamma\big) = O(n^{-d}),$$

where $\overline{\eta}_i = \big(U_{ni} - \overline{\mu}_i(\gamma)\big)/\overline{\sigma}_i(\gamma)$. Further, by analogy with (17)–(21) we obtain from (22)

$$\overline{J}_p(\gamma) = I(k_\gamma > N_\gamma)\frac{(N_\gamma - k_\gamma)^2}{n}\frac{p}{2}\frac{\xi_\gamma^{p-1}}{f(\xi_\gamma)} + \frac{1}{n}|N_\gamma - k_\gamma|^{3/2}\overline{r}_n + \frac{1}{n^2}(N_\gamma - k_\gamma)^2 \overline{r}_n^2 \tag{23}$$

with $P(|r_n| > c\sqrt{\ln n}) = O(n^{-d})$.

Combining (10), (11), (21) and (23) we find

$$\begin{aligned} \pi_\ell = \sum_{i=1}^{n}(W_i - \mu)^\ell - k_\alpha(\xi_\alpha - \mu)^\ell - (n - k_\beta)(\xi_\beta - \mu)^\ell \\ - \frac{(N_\alpha - k_\alpha)^2}{n}\frac{\ell(\xi_\alpha - \mu)^{\ell-1}}{2f(\xi_\alpha)} + \frac{(N_\beta - k_\beta)^2}{n}\frac{\ell(\xi_\beta - \mu)^{\ell-1}}{2f(\xi_\beta)} \\ + \Big(\frac{1}{n}|N_\alpha - k_\alpha|^{3/2} + \frac{1}{n}|N_\beta - k_\beta|^{3/2}\Big) r_{n\ell} \\ + \Big(\frac{1}{n^2}(N_\alpha - k_\alpha)^2 + \frac{1}{n^2}(N_\beta - k_\beta)^2\Big) r_{n\ell}^2, \end{aligned} \tag{24}$$

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where $r_{n\ell}$ satisfies $P\{\max_{1\leqslant\ell\leqslant m}|r_{n\ell}| > c\sqrt{\ln n}\} = O(n^{-d})$. Note that in (24) by Bernstein's inequality

$$(25) \qquad P\big(|N_\gamma - k_\gamma| > c\sqrt{n\ln n}\big) = O(n^{-d}).$$

Furthermore, from (8)

$$(26) \qquad U(\alpha,\beta) - \mu^m = \frac{m}{k_{\alpha\beta}}\mu^{m-1}\pi_1 + \frac{m(m-1)}{k_{\alpha\beta}(k_{\alpha\beta}-1)}\mu^{m-2}\frac{1}{2}(\pi_1^2 - \pi_2) + T_n(\alpha,\beta)\,,$$

where

$$T_n(\alpha,\beta) = \sum_{r=3}^{m}\frac{m(m-1)\cdots(m-r+1)}{k_{\alpha\beta}(k_{\alpha\beta}-1)\cdots(k_{\alpha\beta}-r+1)}\mu^{m-r}S_r(\alpha,\beta).$$

ESTIMATING $T_n(\alpha,\beta)$. We shall show that

$$(27) \qquad P\Big(\big|\sqrt{n}T_n(\alpha,\beta)\big| > c(\ln n)^{3/2}n^{-1}\Big) = O(n^{-d}).$$

According to (9) $S_r(\alpha,\beta)$ is a polynomial of degree r on r variables $\pi_1,\ldots,\pi_r$. Each of these variables we can estimate, with the help of the representation (24). At first let $\ell = 1$. Since $EW_1 = v = \alpha\xi_\alpha + (\beta-\alpha)\mu + (1-\beta)\xi_\beta$, then in (24)

$$(28) \quad \pi_1 = \sum_{i=1}^{n}(W_i - v) + (\xi_\alpha - \mu)w_\alpha - (\xi_\beta - \mu)w_\beta - \frac{(N_\alpha - k_\alpha)^2}{n}\frac{1}{2f(\xi_\alpha)} + \frac{(N_\beta - k_\beta)^2}{n}\frac{1}{2f(\xi_\beta)} + \rho_n\,,$$

where

$$\rho_n = \Big(\frac{1}{n}|N_\alpha - k_\alpha|^{3/2} + \frac{1}{n}|N_\beta - k_\beta|^{3/2}\Big)r_{n1} + \Big(\frac{1}{n^2}(N_\alpha - k_\alpha)^2 + \frac{1}{n^2}(N_\beta - k_\beta)^2\Big)r_{n1}^2\,.$$

By Bernstein's inequality $P\big(|\pi_1| > c\sqrt{n\ln n}\big) = O(n^{-d})$, as $n\to\infty$. If in (24) $\ell \geqslant 2$ then we can clearly bound $|\pi_\ell|$ by cn for some positive constant c not depending on n. This argument shows that for any $r \geqslant 3$ and all non-negative integers $i_1,\ldots,i_r$ satisfying $\sum_{\ell=1}^{r}\ell i_\ell = r$

$$P\Big(\sqrt{n}\Big|\prod_{\ell=1}^{r}(n^{-\ell}\pi_\ell)^{i_\ell}\Big| > c(\ln n)^{3/2}n^{-1}\Big) = O(n^{-d})$$

as $n\to\infty$. This proves (27).

Further consider $\pi_1^2 - \pi_2$ in (26). From (24) and (28) we have

$$(29) \quad \frac{1}{n\sqrt{n}}(\pi_1^2 - \pi_2) = \frac{2}{n\sqrt{n}}\sum_{1\leqslant i<j\leqslant n}(W_i - v)(W_j - v) + \frac{1}{\sqrt{n}}\big[\alpha(\xi_\alpha - \mu)^2 + (1-\beta)(\xi_\beta - \mu)^2 - (v-\mu)^2\big] + \overline{\rho}_n\,,$$

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where $P(|\overline{\rho}_n| > c(\ln n)^{3/2}n^{-1}) = O(n^{-d})$.

Finally, we obtain the representation for π_1 from (28):

$$
\begin{aligned}
\frac{1}{\sqrt{n}}\pi_1 = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(W_i - v) &- \frac{1}{n\sqrt{n}}\sum_{1\leqslant i<j\leqslant n}\big(I(X_i \leqslant \xi_\alpha) - \alpha\big)\big(I(X_j \leqslant \xi_\alpha) - \alpha\big)\frac{1}{f(\xi_\alpha)} \\
&+ \frac{1}{n\sqrt{n}}\sum_{1\leqslant i<j\leqslant n}(I(X_i \leqslant \xi_\beta) - \beta)(I(X_j \leqslant \xi_\beta) - \beta)\frac{1}{f(\xi_\beta)} \\
&+ \frac{1}{n\sqrt{n}}\left(\Big|\sum_{i=1}^{n}(I(X_i \leqslant \xi_\alpha) - \alpha)\Big|^{3/2} + \Big|\sum_{i=1}^{n}(I(X_i \leqslant \xi_\beta) - \beta)\Big|^{3/2}\right)r_{n1} \qquad (30) \\
&+ \frac{1}{\sqrt{n}}\left[(\xi_\alpha - \mu)w_\alpha - (\xi_\beta - \mu)w_\beta - \frac{\alpha(1-\alpha)}{2f(\xi_\alpha)} + \frac{\beta(1-\beta)}{2f(\xi_\beta)}\right] + \overline{r}_{n1},
\end{aligned}
$$

where $P(|\overline{r}_{n1}| > c(\ln n)^{3/2}n^{-1}) = O(n^{-d})$.

Combining (26)–(27) and (29)–(30) we obtain the proof of Lemma 2.

PROOF OF THEOREM 1: Using the notation of Lemma 2, let $\varphi(t) = E\exp(itg(X_1))$, $t \in R$,

$$
\begin{aligned}
\overline{F}_\sigma(x) &= P\{U_1 + U_2 \leqslant x\} \quad \text{and,} \\
\overline{G}(x) &= \Phi(x) - \frac{1}{\sigma^3\sqrt{n}}\frac{\kappa_3}{6}\phi(x)(x^2 - 1), \; x \in R\,,
\end{aligned}
$$

where $\kappa_3 = Eg^3(X_1) + 3Eg(X_1)g(X_2)h(X_1, X_2)$. Simple calculations show that

$$|\varphi(t)| \leqslant 1 - (\beta - \alpha) + 2|t|^{-1}\,,\; t \in R$$

and if $|t| > 2/(\beta - \alpha)$ then $|\varphi(t)| < 1$ and hence the Cramér condition is satisfied. Since the functions g and h are bounded then the theorem giving the asymptotic expansion for U-statistics holds (see, for example, [**2, 5, 18**])

$$\sup_x|\overline{F}_\sigma(x) - \overline{G}(x)| = O(n^{-1}). \qquad (31)$$

Now we shall apply Lemma 2. First we note that

$$P\left(\Big|\sum_{i=1}^{n}(I(X_i \leqslant \xi_\gamma) - \gamma)\Big| > c\sqrt{n\ln n}\right) = O(n^{-d})$$

for $\gamma = \alpha$ and β. Therefore

$$F_\sigma(x) = \overline{F}_\sigma\Big(x - \frac{\lambda_3}{6\sqrt{n}} + O((\ln n)^{5/4}n^{-3/4})\Big) + O(n^{-1})\,. \qquad (32)$$

And from

$$
\begin{aligned}
\sup_x|F_\sigma(x) - G(x)| &\leqslant \sup_x|\overline{F}_\sigma(x) - \overline{G}(x)| + O(n^{-1}) \\
&\qquad + \sup_x\Big|G(x) - \overline{G}\Big(x - \frac{\lambda_3}{6\sqrt{n}} + O((\ln n)^{5/4}n^{-3/4})\Big)\Big| \\
&= O((\ln n)^{5/4}n^{-3/4}) \qquad (33)
\end{aligned}
$$

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we obtain the proof of Theorem 1. ∎

PROOF OF THEOREM 2: By Bahadur's theorem (see, for example, [1], or [11])

$$X_{nk_\gamma} = \xi_\gamma - \frac{N_\gamma - \gamma n}{n} \frac{1}{f(\xi_\gamma)} + \rho_n , \tag{34}$$

where

$$P\big(|\rho_n| > c(\ln n/n)^{3/4}\big) = O(n^{-d}) \tag{35}$$

for some $c > 0$ and every $d > 0$ not depending on n. Furthermore, with the help of (11), (13), (21) and (23) for $p = 2$ and (34) with $\gamma = \alpha, \beta$ we obtain the following representation for S_n^2

$$S_n^2 = \sigma^2 + \frac{1}{n} \sum_{i=1}^{n} \psi(X_i) + \widetilde{\rho}_n, \tag{36}$$

where

$$\psi(X_i) = \big(I(X_i \leqslant \xi_\alpha) - \alpha\big)\frac{2\alpha}{f(\xi_\alpha)}(v - \xi_\alpha) + \big((W_i - v)^2 - \sigma^2\big) + \big(I(X_i \leqslant \xi_\beta) - \beta\big)\frac{2\beta}{f(\xi_\beta)}(v - \xi_\beta),$$

and the remainder term $\widetilde{\rho}_n$ satisfies (35). The details of the proof are similar to the proofs in [13] and [11]. Here we omit the details since the proof follows that of Lemma 2.

Recall

$$P\Big(\Big|\sum_{i=1}^{n}\big(I(X_i \leqslant \xi_\gamma) - \gamma\big)\Big| > c\sqrt{n \ln n}\Big) = O(n^{-d})$$

for $\gamma = \alpha$ and β. Applying this observation to the representation in Lemma 2 and using (36) we can write (7) as

$$F_s(x) = P\left(\frac{U_1 + U_2 + \lambda_3/(6\sqrt{n}) + O((\ln n)^{5/4} n^{-3/4})}{\sqrt{1 + (\sigma^2 n)^{-1} \sum_{i=1}^{n} \psi(X_i) + \sigma^{-2}\widetilde{\rho}_n}} \leqslant x\right) + O(n^{-1})$$

as $n \to \infty$. By the inequality from [16]

$$\left(1 + (\sigma^2 n)^{-1} \sum_{i=1}^{n} \psi(X_i) + \sigma^{-2}\widetilde{\rho}_n\right)^{-1/2} = 1 - \frac{1}{2\sigma^2 n} \sum_{i=1}^{n} \psi(X_i) + O\big((\ln n/n)^{3/4}\big)$$

with probability $1 - O(n^{-d})$ for every $d > 0$. Thus we can write

$$F_s(x) = P\left((U_1 + U_2)\Big(1 - \frac{1}{2\sigma^2 n}\sum_{i=1}^{n}\psi(X_i)\Big) + \frac{\lambda_3}{6\sqrt{n}} + O\big((\ln n)^{5/4} n^{-3/4}\big) \leqslant x\right) + O(n^{-1}), \quad x \in R .$$

Here with probability $1 - O(n^{-d})$ for every $d > 0$

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$$(U_1 + U_2)\Big(1 - \frac{1}{2\sigma^2 n}\sum_{i=1}^{n}\psi(X_i)\Big) = U_1 + U_3 - \frac{1}{2\sigma^3\sqrt{n}}Eg(X_1)\psi(X_1) + O\big((\ln n)^3 n^{-1}\big),$$

where

$$U_3 = \frac{1}{\sigma n\sqrt{n}}\sum_{1\leqslant i<j\leqslant n}\overline{h}(X_i, X_j),$$

and

$$\overline{h}(X_i, X_j) = h(X_i, X_j) - \frac{1}{2\sigma^2}\big(g(X_i)\psi(X_j) + g(X_j)\psi(X_i)\big).$$

Therefore

$$F_s(x) = P(U_1 + U_3 + \lambda \leqslant x) + O(n^{-1})$$

with $\lambda = -\dfrac{1}{2\sigma^3\sqrt{n}}Eg(X_1)\psi(X_1) + \dfrac{\lambda_3}{6\sqrt{n}} + O\big((\ln n)^{5/4}n^{-3/4}\big)$. Denote

$$\overline{H}(x) = \Phi(x) - \frac{1}{\sqrt{n}}\frac{\overline{\kappa}_3}{6}\phi(x)(x^2-1), \quad x \in R$$
$$\overline{\kappa}_3 = \big[Eg^3(X_1) + 3Eg(X_1)g(X_2)\overline{h}(X_1, X_2)\big]\sigma^{-3}\,.$$

Further by analogy with (31) - (33) we have after simple calculations

$$\begin{aligned}\sup_x\big|F_s(x) - H(x)\big| &= \sup_x\big|P(U_1 + U_3 \leqslant x) - H(x+\lambda)\big| + O(n^{-1})\\ &\leqslant \sup_x\big|P(U_1 + U_3 \leqslant x) - \overline{H}(x)\big| + \sup_x\big|\overline{H}(x) - H(x+\lambda)\big| + O(n^{-1})\\ &= O\big((\ln n)^{5/4}n^{-3/4}\big).\end{aligned}$$

This proves Theorem 2. ∎

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