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LETTERS TO THE EDITOR

A NOTE ON ONE-DIMENSIONAL CHAOTIC MAPS UNDER TIME REVERSAL

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Abstract

We have furnished further examples on the connection between some standard one-dimensional chaotic deterministic models and stochastic time series models via time reversal.

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Bartlett (1990) and the discussion therein (especially that given by Lawrance (1991)) have highlighted a link between some deterministic chaos and stochastic models by *reversing the time axis*. Specifically, consider the (deterministic) *Bernoulli shift* in reverse time:

$$X_{t-1} = 2X_t \pmod 1 \qquad t = 0, -1, -2, \cdots, \quad X_0 \in (0, 1). \tag{1}$$

It is obvious that this map has two branches, say A and B.

It is also well known that this map is ergodic with a uniform distribution on $[0, 1]$ as its invariant distribution. For simplicity, we assume that X_0 has a uniform distribution on $[0, 1]$. This makes $\{X_0, X_{-1}, X_{-2}, \cdots\}$ a stationary sequence. Suppose we reverse the time axis and consider the 'inverse' situation of mapping X_{t-1} to X_t. Clearly this is a 1–2 map. However, the uniform distribution implies that the two branches A and B would be chosen with equal probability. The following strictly stationary linear autoregressive model describes precisely this random selection of the branches and may therefore be considered the 'inverse' situation desired:

$$X_t = 0{\cdot}5X_{t-1} + \varepsilon_t, \; t = 1, 2, \cdots, \tag{2}$$

where ε_t equals 0 or $0{\cdot}5$ with equal probability. As usual, we assume that $\{\varepsilon_t\}$ is a sequence of independent and identically distributed random variables and ε_t is independent of X_s, $s < t$. To match the 'marginal distributions' of (1) and (2), we impose the initial condition that X_0 has a uniform distribution on $[0, 1]$ in (2).

The basic idea is one of assigning appropriate probabilities to the different branches of a multi-valued function. The same idea may be applied to the following well-known maps.

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The logistic map

$$X_{t-1} = 4X_t(1 - X_t), \qquad t = 0, -1, \cdots, \qquad X_0 \in (0, 1). \tag{3}$$

Here, the invariant distribution is a beta (0·5, 0·5) distribution, i.e. it has the probability density function $\{\pi x(1-x)\}^{-\frac{1}{2}}$ $(0 \leqq x \leqq 1)$, and, as above, we assume that X_0 has this distribution. There are two branches associated with the map: $X_{t-1} \to X_t$, namely

$$X_t = 0{\cdot}5[1 + (1 - X_{t-1})^{\frac{1}{2}}],$$

and

$$X_t = 0{\cdot}5[1 - (1 - X_{t-1})^{\frac{1}{2}}],$$

which may be 'synthesised' by the following non-linear autoregressive model with multiplicative noise:

$$X_t = 0{\cdot}5[1 + \varepsilon_t(1 - X_{t-1})^{\frac{1}{2}}], \qquad t = 1, 2, \cdots, \tag{4}$$

where ε_t equals 1 or -1 with equal probability. Again to match the marginal distributions, we assume that X_0 has the beta (0·5, 0·5) distribution.

In principle, the same procedure could be adopted to 'reverse' any model similar to (3) with 4 replaced by α, $0 < \alpha < 4$. However, the associated invariant distribution with general α is still an open problem in the chaos literature.

The tent map

$$X_{t-1} = \begin{cases} 2X_t, & \text{if } 0 \leqq X_t \leqq 0{\cdot}5 \\ 2(1 - X_t), & \text{if } 0{\cdot}5 \leqq X_t \leqq 1 \end{cases} \tag{5}$$

$t = 0, -1, \cdots$.

Repeating similar arguments, we get the bilinear model

$$X_t = 0{\cdot}5(1 - \varepsilon_t + \varepsilon_t X_{t-1}), \qquad t = 1, 2, \cdots, \tag{6}$$

where ε_t equals 1 or -1 with equal probability and X_0 is uniformly distributed on [0, 1].

It may be natural to enquire about higher dimensional cases. Interestingly for dimensions higher than 1, the extra 'degree of freedom' permits 'stretching' and 'folding' (essential for the generation of chaos) within the 1–1 maps. In this sense, the one-dimensional case is probably quite atypical.

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