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ARTICLE

Intersecting families without unique shadow

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Abstract
Let $\mathcal{F}$ be an intersecting family. A $(k-1)$-set E is called a unique shadow if it is contained in exactly one member of $\mathcal{F}$. Let $\mathcal{A}=\{A\in\binom{[n]}{k}\colon |A\cap\{1,2,3\}|\geq 2\}$. In the present paper, we show that for $n\geq 28k$, $\mathcal{A}$ is the unique family attaining the maximum size among all intersecting families without unique shadow. Several other results of a similar flavour are established as well.

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1. Introduction

Let $n>k>t$ be positive integers and let $[n]=\{1,2,\dots,n\}$ be the standard n-element set. For $1\leq i<j\leq n$, let $[i,j]=\{i,i+1,\dots,j\}$. Let $\binom{[n]}{k}$ denote the collection of all k-subsets of $[n]$. Subsets of $\binom{[n]}{k}$ are called *k-uniform hypergraphs* or *k-graphs* for short. A k-graph $\mathcal{F}$ is called *t-intersecting* if $|F\cap F'|\geq t$ for all $F,F'\in\mathcal{F}$. In case of $t=1$ we often use the term *intersecting* instead of 1-intersecting. Investigating various properties of t-intersecting families is one of the central topics of extremal set theory (cf. the recent book of Gerbner and Patkós [13]). Let us state the quintessential result of this topic.

Erdős-Ko-Rado Theorem ([3]). Suppose that $n\geq n_0(k,t)$ and $\mathcal{F}\subset\binom{[n]}{k}$ is t-intersecting. Then

$$|\mathcal{F}|\leq\binom{n-t}{k-t}. \tag{1}$$

Remark 1. For $t=1$ the exact value $n_0(k,t)=(k-t+1)(t+1)$ was proved in [3]. For $t\geq 15$ it is due to [5]. Finally Wilson [21] closed the gap $2\leq t\leq 14$ with a proof valid for all t.

Let us note that the *full t-star*, $\left\{F\in\binom{[n]}{k}\colon [t]\subset F\right\}$ shows that (1) is best possible. In general, for a set $T\subset[n]$ let $\mathcal{S}_T=\left\{S\in\binom{[n]}{k}\colon T\subset S\right\}$ denote the *star of T*.

For $t=1$, there is a strong stability for the Erdős-Ko-Rado Theorem.

Theorem 1.1 (Hilton-Milner Theorem [14]). *Suppose that $n>2k\geq 4$, $\mathcal{F}\subset\binom{[n]}{k}$ is intersecting and $\mathcal{F}$ is not a star, then*

$$|\mathcal{F}|\leq\binom{n-1}{k-1}-\binom{n-k-1}{k-1}+1. \tag{2}$$

Let us define the Hilton-Milner Family

$$\mathcal{H}(n,k)=\left\{F\in\binom{[n]}{k}:\ 1\in F,\ F\cap[2,k+1]\neq\emptyset\right\}\cup\{[2,k+1]\},$$

showing that (2) is best possible.

Let us recall the notion of *immediate shadow*, $\partial\mathcal{F}$: For $\mathcal{F}\subset\binom{[n]}{k}$,

$$\partial\mathcal{F}=\left\{G\in\binom{[n]}{k-1}:\ \exists F\in\mathcal{F}, G\subset F\right\}.$$

If for some $G\in\partial\mathcal{F}$ there is only one choice of $F\in\mathcal{F}$ satisfying $G\subset F$ then G is called *unique* or a *unique shadow*. Note that in the full star $\mathcal{S}_{\{x\}}$ for each member S, $S\setminus\{x\}$ is unique. In the Hilton-Milner family $\mathcal{H}(n,k)$, each member $H\in\mathcal{H}(n,k)\setminus\{[2,k+1]\}$ contains a unique shadow $H\setminus\{1\}$. Just for curiosity let us mention that if each member of $\mathcal{F}\subset\binom{[n]}{k}$ contains a unique shadow then $|\mathcal{F}|\leq\binom{n-1}{k-1}$.

Let us introduce the central notion of the present paper.

Definition 1.2. For an integer $r\geq 2$ and a family $\mathcal{F}\subset\binom{[n]}{k}$, we say that $\mathcal{F}$ is r-complete if every $G\in\partial\mathcal{F}$ is contained in at least r members of $\mathcal{F}$.

Note that $\mathcal{F}$ is r-complete if and only if the minimum non-zero co-degree of $\mathcal{F}$ is at least r. This notion has been introduced and used by Kostochka et al. [17–19] to determine hypergraph Turán numbers for paths, cycles and trees.

Clearly, if $\mathcal{F}\subset\binom{[n]}{k}$ is r-complete with $r\geq 2$, then $\mathcal{F}$ is far from a star. It is natural to ask for the maximum size of an r-complete intersecting family. Let us define the function:

$$f(n,k,r)=\max\left\{|\mathcal{F}|:\ \mathcal{F}\subset\binom{[n]}{k}\text{ is intersecting and } r\text{-complete}\right\}.$$

Let us give some examples. For $1\leq r<k$ the complete k-graph $\binom{[k+r]}{k}$ is intersecting and $(r+1)$-complete. This shows in particular that

$$f(n,k,k)\geq\binom{2k-1}{k}. \tag{3}$$

Example 1.3. For $n\geq k\geq r\geq 1$ define

$$\mathcal{L}(n,k,r)=\left\{F\in\binom{[n]}{k}:\ |F\cap[2r-1]|\geq r\right\}.$$

Clearly $\mathcal{L}(n,k,r)$ is intersecting, r-complete and

$$|\mathcal{L}(n,k,r)|=\sum_{r\leq i\leq 2r-1}\binom{2r-1}{i}\binom{n-2r+1}{k-i}.$$

Our main result shows that this example is best possible for $n\geq n_0(k,r)$.

Theorem 1.4. *For $n\geq 28k$,*

$$f(n,k,2)=|\mathcal{L}(n,k,2)|. \tag{4}$$

Moreover, up to isomorphism $\mathcal{L}(n,k,2)$ is the only family attaining equality.

Theorem 1.5. *For $k\geq 3$, $r\geq 3$ and $n\geq n_0(k,r)$,*

$$f(n,k,r)=\begin{cases}|\mathcal{L}(n,k,r)|, & 3\leq r\leq k;\\ 0. & r\geq k+1.\end{cases} \tag{5}$$

For a positive integer ℓ and an ℓ-graph $\mathcal{H}$, define the *clique family*

$$\mathcal{K}(\mathcal{H}) = \left\{K \colon |K| = \ell + 1, \binom{K}{\ell} \subset \mathcal{H}\right\}.$$

Define $\nu(\mathcal{F})$, the *matching number* of $\mathcal{F}$ as the maximum number of pairwise disjoint edges in $\mathcal{F}$. Note that $\nu(\mathcal{F}) = 1$ iff $\mathcal{F}$ is intersecting. We are going to prove Theorem 1.4 using the following result exhibiting a surprising connection between the matching number and the size of the clique family. Define the Erdős-family

$$\mathcal{E}(n, k, s) = \left\{E \in \binom{[n]}{k} \colon E \cap [s] \neq \emptyset\right\}.$$

Note that

$$\mathcal{K}(\mathcal{E}(n, k, s)) = \left\{K \in \binom{[n]}{k+1} \colon |K \cap [s]| \geq 2\right\}.$$

Theorem 1.6. *Let $\mathcal{F} \subset \binom{[n]}{k}$ be a family with $\nu(\mathcal{F}) \leq s$. If $n \geq 5sk + 13k$ and $s \geq 3$, then*

$$|\mathcal{K}(\mathcal{F})| \leq |\mathcal{K}(\mathcal{E}(n, k, s))|.$$

Moreover, up to isomorphism $\mathcal{E}(n, k, s)$ is the only family attaining equality.

Let us define the notion of r-complete edges.

Definition 1.7. For an integer $r \geq 2$ and a family $\mathcal{F} \subset \binom{[n]}{k}$, we say that $F \in \mathcal{F}$ is r-complete if every $G \in \binom{F}{k-1}$ is contained in at least r members of $\mathcal{F}$.

Clearly, $\mathcal{F}$ is r-complete if and only if every $F \in \mathcal{F}$ is r-complete. One can also ask for the maximum number of r-complete edges in an intersecting family. For an intersecting family $\mathcal{F} \subset \binom{[n]}{k}$, define $\mathcal{F}_r^*(\mathcal{F})$ as the family of all r-complete edges in $\mathcal{F}$. Let

$$f^*(n, k, r) = \max\left\{|\mathcal{F}_r^*(\mathcal{F})| \colon \mathcal{F} \subset \binom{[n]}{k} \text{ is intersecting.}\right\}$$

If $\mathcal{F}$ is r-complete then we have $\mathcal{F}_r^*(\mathcal{F}) = \mathcal{F}$, implying that $f(n, k, r) \leq f^*(n, k, r)$. For $\mathcal{F}' \subset \mathcal{F}$, we say that $\mathcal{F}'$ is *relatively r-complete* with respect to $\mathcal{F}$ if every $F' \in \mathcal{F}'$ is an r-complete edge in $\mathcal{F}$. Clearly $\mathcal{F}_r^*(\mathcal{F})$ is a relatively r-complete family of the maximum size with respect to $\mathcal{F}$.

Our next result determines $f^*(n, k, r)$ for all $k \geq 3$ and $r \geq 2$, asymptotically.

Theorem 1.8. *For $k \geq 3$, $r \geq 2$ and $n \geq n_0(k, r)$,*

$$f^*(n, k, r) = \begin{cases} |\mathcal{L}(n, k, r)|, & r = 2, 3; \\ \binom{n-3}{k-3} + O(n^{k-r}), & 4 \leq r \leq k-1; \\ \binom{n-3}{k-3}, & r \geq k \geq 4. \end{cases} \tag{6}$$

The next proposition shows that the term $O(n^{k-r})$ in (6) cannot be removed for $k \geq 5$ and $4 \leq r \leq k-1$.

Proposition 1.9. *For $k \geq 5$, $4 \leq r \leq k-1$ and $n \geq k + r - 1$,*

$$f^*(n, k, r) \geq \binom{n-3}{k-3} + \binom{n-r-2}{k-r-1}.$$

Proof. For $4 \leq r \leq k-1$, let

$$\mathcal{B}(r) = \{\{1, 2\}\} \cup \left\{\{3, i, j\} \colon i = 1 \text{ or } 2,\ 4 \leq j \leq r+2\right\} \cup \{[r+2] \setminus \{u, 3\} \colon u = 1, 2\}$$

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and let

$$\mathcal{I}(n,k,r)=\bigcup_{B\in\mathcal{B}(r)}\mathcal{S}_B,\ \mathcal{I}^*(n,k,r)=\mathcal{S}_{\{1,2,3\}}\cup\mathcal{S}_{[r+2]\setminus\{3\}}.$$

It is easy to check that $B(r)$ is intersecting, implying that $\mathcal{I}(n,k,r)$ is intersecting. Since $\mathcal{S}_{\{1,2,3\}}\subset\mathcal{S}_{\{1,2\}}$ and $\mathcal{S}_{[r+2]\setminus\{3\}}\subset\mathcal{S}_{\{1,2\}}$, $\mathcal{I}^*(n,k,r)\subset\mathcal{I}(n,k,r)$. In the rest of the proof, we show that $\mathcal{I}^*(n,k,r)$ is relatively r-complete with respect to $\mathcal{I}(n,k,r)$.

For any $F\in\mathcal{S}_{\{1,2,3\}}$, since $\mathcal{S}_{\{1,2\}}\subset\mathcal{I}(n,k,r)$ and $n\geq k+r-1$, we see that for each $x\in F\setminus\{1,2\}$, $F\setminus\{x\}$ is covered by at least r members of $\mathcal{I}(n,k,r)$. Moreover, since $\mathcal{S}_{\{3,i,j\}}\subset\mathcal{I}(n,k,r)$ for $i\in\{1,2\}$ and $j\in\{4,5,\dots,r+2\}$, we infer that $F\setminus\{3-i\}$ is covered by at least r members of $\mathcal{I}(n,k,r)$ for $i=1,2$. Thus $\mathcal{S}_{\{1,2,3\}}$ is relatively r-complete with respect to $\mathcal{I}(n,k,r)$.

Let $F\in\mathcal{S}_{[r+2]\setminus\{3\}}$ and $G\in\binom{F}{k-1}$. If $\{1,2\}\subset G$, then by $\mathcal{S}_{\{1,2\}}\subset\mathcal{I}(n,k,r)$ and $n\geq k+r$ we infer that G is covered by at least r members of $\mathcal{I}(n,k,r)$. If $i\notin G$ for $i=1,2$, since $\mathcal{S}_{[r+2]\setminus\{i,3\}}\subset\mathcal{I}(n,k,r)$ and $n\geq k+r-1$, then G is also covered by at least r members of $\mathcal{I}(n,k,r)$. Hence $\mathcal{S}_{[r+2]\setminus\{3\}}$ is relatively r-complete with respect to $\mathcal{I}(n,k,r)$. Therefore, $\mathcal{I}^*(n,k,r)$ is relatively r-complete with respect to $\mathcal{I}(n,k,r)$ and

$$f^*(n,k,r)\geq|\mathcal{I}^*(n,k,r)|=\binom{n-3}{k-3}+\binom{n-r-2}{k-r-1}.$$

□

The rest of the paper is organized as follows. We list some results that are needed in Section 2. We prove Theorems 1.4 and 1.6 in Section 3 and prove Theorem 1.5 in Section 4. The proof of Theorem 1.8 splits into two parts. In Section 5, we prove it for $3\leq r<k$. In Section 6, we prove it for $r\geq k$. Finally, we give some concluding remarks in Section 7.

2. Preliminaries

In this section, we list some notions and results that are needed for the proofs.

For a family $\mathcal{F}\subset\binom{[n]}{k}$ define the family of *transversals*, $\mathcal{T}(\mathcal{F})$ by

$$\mathcal{T}(\mathcal{F})=\left\{T\subset[n]\colon |T|\leq k,\ T\cap F\neq\emptyset \text{ for all } F\in\mathcal{F}\right\}.$$

Note that $\mathcal{F}$ is intersecting iff $\mathcal{F}\subset\mathcal{T}(\mathcal{F})$. Note also that $\mathcal{T}(\mathcal{F})$ is not uniform in general. Set $\mathcal{T}^{(k)}(\mathcal{F})=\{T\in\mathcal{T}(\mathcal{F})\colon |T|=k\}$. If $\mathcal{F}=\mathcal{T}^{(k)}(\mathcal{F})$ then $\mathcal{F}$ is called *saturated*. It is equivalent to the fact that $\mathcal{F}\cup\{H\}$ is no longer intersecting for $H\in\binom{[n]}{k}\setminus\mathcal{F}$. It should be clear that in the definition of $f^*(n,k,r)$ it is sufficient to consider saturated intersecting families $\mathcal{F}$.

Let us recall a special case of the Katona Intersecting Shadow Theorem [15].

Theorem 2.1 ([15]). *Suppose that $\mathcal{F}\subset\binom{[n]}{k}$ is intersecting. Then*

$$|\partial\mathcal{F}|\geq|\mathcal{F}| \text{ with equality iff } \mathcal{F}=\binom{X}{k} \text{ for some } (2k-1)\text{-set } X. \tag{7}$$

We need the following generalization of (7) as well.

Theorem 2.2 ([8]). *Suppose that $\mathcal{F}\subset\binom{[n]}{k}$. Then*

$$|\partial\mathcal{F}|\nu(\mathcal{F})\geq|\mathcal{F}|. \tag{8}$$

We need also a classical result of Bollobás, the so-called Bollobás Set-pair Inequality.

Theorem 2.3 ([1]). *Let a,b be positive integers, $A_1,\dots,A_m$ a-element sets, $B_1,\dots,B_m$ b-element sets such that $A_i\cap B_j=\emptyset$ iff $i=j$. Then*

$$m\leq\binom{a+b}{b}\text{(cf. [16] for a very slick proof.)} \tag{9}$$

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There is a very important operation on families of sets which was discovered by Erdős et al. [3]. It is called shifting and it is known not to increase the matching number $\nu(\mathcal{F})$ ([7]) and not to decrease the size of $\mathcal{K}(\mathcal{F})$ (cf. [20]).

Let us define the *shifting partial order* $\prec$. For two k-sets A and B where $A = \{a_1, \ldots, a_k\}$, $a_1 < \ldots < a_k$ and $B = \{b_1, \ldots, b_k\}$, $b_1 < \ldots < b_k$ we say that A precedes B and denote it by $A \prec B$ if $a_i \leq b_i$ for all $1 \leq i \leq k$.

A family $\mathcal{F} \subset \binom{[n]}{k}$ is called *shifted* (or *initial*) if $A \prec B$ and $B \in \mathcal{F}$ always imply $A \in \mathcal{F}$. By repeated shifting one can transform an arbitrary k-graph into a shifted k-graph with the same number of edges.

We need the following inequality generalizing the case $t = 1$ of the Erdős-Ko-Rado Theorem.

Proposition 2.4 ([7]). *Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ then*

$$|\mathcal{F}| \leq \nu(\mathcal{F})\binom{n-1}{k-1}. \tag{10}$$

Finally we need the following stability theorem concerning the Erdős-Ko-Rado Theorem.

Hilton-Milner-Frankl Theorem ([6,14]). Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ is t-intersecting, $\mathcal{F}$ is not a t-star and $n \geq (k-t+1)(t+1)$. Then

$$|\mathcal{F}| \leq \max\left\{|\mathcal{A}(n,k,t)|, \binom{n-t}{k-t} - \binom{n-k-1}{k-t} + t\right\} < \binom{n-t-1}{k-t-1}\max\{t+2, k-t+1\}. \tag{11}$$

3. Intersecting families without unique shadow

In this section, we first prove Theorem 1.4 by assuming Theorem 1.6. Then by using the decomposition method of a shifted family introduced in [9], we give a proof of Theorem 1.6.

Actually, we shall prove the following version of Theorem 1.4, which also gives the $r = 2$ case of Theorem 1.8.

Theorem 3.1. *For $n \geq 28k$,*

$$f(n,k,2) = f^*(n,k,2) = |\mathcal{L}(n,k,2)|. \tag{12}$$

Moreover, up to isomorphism $\mathcal{L}(n,k,2)$ is the only family attaining equality.

Proof of Theorem 1.4. Recall that $\mathcal{L}(n,k,2)$ is 2-complete intersecting and $f(n,k,2) \leq f^*(n,k,2)$. It follows that $|\mathcal{L}(n,k,2)| \leq f(n,k,2) \leq f^*(n,k,2)$. Thus we are left to show $f^*(n,k,2) \leq |\mathcal{L}(n,k,2)|$.

Let $\mathcal{F} \subset \binom{[n]}{k}$ be an intersecting family. Let $\mathcal{F}^*$ be the family of 2-complete sets in $\mathcal{F}$ and let $\mathcal{H} = \partial\mathcal{F}^*$. Note that this guarantees that every member of $\mathcal{H}$ is contained in at least two members of $\mathcal{F}$.

Claim 1. $\nu(\mathcal{H}) \leq 3$.

Proof. Suppose for contradiction that $D_i = F_i \cap G_i$, $1 \leq i \leq 4$, are pairwise disjoint sets in $\mathcal{H}$ and $F_i, G_i \in \mathcal{F}$. Define x_i, y_i by $F_i \setminus D_i = \{x_i\}$, $G_i \setminus D_i = \{y_i\}$. Since $|\{x_i, y_i\}| = 2$, by symmetry we may assume that $(x_1, y_1) \cap D_4 = \emptyset$. This implies $F_1 \cap D_4 = \emptyset, G_1 \cap D_4 = \emptyset$. From $F_4 \cap F_1 \neq \emptyset$, $F_4 \cap G_1 \neq \emptyset$, $G_4 \cap F_1 \neq \emptyset$ and $G_4 \cap G_1 \neq \emptyset$, we infer $(x_4, y_4) \subset D_1$. Consequently $F_4 \cap D_p = \emptyset$, $G_4 \cap D_p = \emptyset$ for $p = 2, 3$. This implies as above $(x_p, y_p) \subset D_4$. Now $x_2 \neq x_3$ or $x_2 \neq y_3$ (or both) hold. By symmetry $x_2 \neq x_3$. Then $F_2 \cap F_3 = \emptyset$, a contradiction. $\square$

By Theorem 1.6 and Claim 1, for $n \geq (5 \times 3 + 13)k = 28k$ we have $|\mathcal{F}^*| \leq |\mathcal{K}(\mathcal{H})| \leq |\mathcal{K}(\mathcal{E}(n,k,3))| = |\mathcal{L}(n,k,2)|$. The uniqueness follows from Theorem 1.6. $\square$

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The families $\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_s$ are called *overlapping* if there is no choice of $F_i \in \mathcal{F}_i$ such that $F_0, F_1, \ldots, F_s$ are pairwise disjoint. For the proof of Theorem 1.6 the following lemma is needed. A similar lemma was proved in [11], although without characterization of the case of equality.

Lemma 3.2. *Let $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_s \subset \binom{Y}{\ell}$ be overlapping families and let $p_0 \geq p_1 \geq \ldots \geq p_s$ be positive reals. Let*

$$d_{\vec{p}} = \frac{s(p_0 + \cdots + p_s)}{p_1 + \cdots + p_s}.$$

For $|Y| \geq (d_{\vec{p}} + 1)\ell$,

$$\sum_{0 \leq i \leq s} p_i |\mathcal{F}_i| \leq (p_1 + \cdots + p_s)\binom{|Y|}{\ell}, \tag{13}$$

where the equality holds iff $\mathcal{F}_1 = \cdots = \mathcal{F}_s = \binom{Y}{\ell}$, $\mathcal{F}_0 = \emptyset$.

Proof. Let $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_s \subset \binom{Y}{\ell}$ be overlapping families. Let $t = \lfloor |Y|/\ell \rfloor \geq \lfloor d_{\vec{p}} \rfloor + 1 \geq s + 1$ and choose a random matching $F_1, F_2, \ldots, F_t$ from $\binom{Y}{\ell}$. Consider the weighted bipartite graph G on partite sets $\{F_1, F_2, \ldots, F_t\}$ and $\{\mathcal{F}_0, \mathcal{F}_1, \cdots, \mathcal{F}_s\}$ where we have an edge $(F_i, \mathcal{F}_j)$ iff $F_i \in \mathcal{F}_j$. This edge gets weight p_j.

Since $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_s$ are overlapping, G has matching number at most s. Applying the König-Hall Theorem we can find s vertices covering all edges of the bipartite graph G. Let $F_1, \ldots, F_q$ be the vertices of the covering set chosen from the random matching and $\mathcal{F}_{q+1}, \ldots, \mathcal{F}_s$ the remaining $s - q$ chosen from the families.

The total weight of the edges covered by F_i is at most $p_0 + \ldots + p_s$. The total weight of the edges covered by $\mathcal{F}_j$ is at most tp_j. Thus, the total weight of the edges in G is at most

$$\begin{aligned} q(p_0 + \cdots + p_s) &+ t(p_{q+1} + \cdots + p_s) \\ &= t(p_1 + \cdots + p_s) - t(p_1 + \cdots + p_q) + q(p_0 + \cdots + p_s). \end{aligned} \tag{14}$$

Note that $p_1 \geq \ldots \geq p_s$ implies

$$\frac{i+1}{p_1 + \cdots + p_{i+1}} \geq \frac{i}{p_1 + \cdots + p_i}.$$

It follows that

$$\frac{q}{p_1 + \cdots + p_q}(p_0 + \cdots + p_s) \leq \frac{s}{p_1 + \cdots + p_s}(p_0 + \cdots + p_s) = d_{\vec{p}} < t. \tag{15}$$

By (14) and (15), the total weight of the edges in G is at most $t(p_1 + \ldots + p_s)$.

Since the probability

$$Pr(F_i \in \mathcal{F}_j) = \frac{|\mathcal{F}_j|}{\binom{|Y|}{\ell}},$$

the expected value of the total weight of the edges in G is $\sum_{j=0}^{s} tp_j \frac{|\mathcal{F}_j|}{\binom{|Y|}{\ell}}$. Thus (13) follows. In case of equality $q = 0$. Then for every t-matching $F_1, F_2, \ldots, F_t$ in Y, $\mathcal{F}_0$ has degree 0 and $\mathcal{F}_i$ has degree t in G for $i = 1, \ldots, s$. Hence the equality holds iff $\mathcal{F}_1 = \cdots = \mathcal{F}_s = \binom{Y}{\ell}$, $\mathcal{F}_0 = \emptyset$. □

For the proof of Theorem 1.6 we also need the following proposition, which is proved in [20]. Here we include a short proof for self-containedness.

Proposition 3.3. *For $\mathcal{F} \subset \binom{[n]}{k}$ and $1 \leq i < j \leq n$, $|\mathcal{K}(S_{ij}(\mathcal{F}))| \geq |\mathcal{K}(\mathcal{F})|$.*

Proof. We prove the statement by defining an injective map σ from $\mathcal{K}(\mathcal{F}) \setminus \mathcal{K}(S_{ij}(\mathcal{F}))$ to $\mathcal{K}(S_{ij}(\mathcal{F})) \setminus \mathcal{K}(\mathcal{F})$. Let $K \in \mathcal{K}(\mathcal{F}) \setminus \mathcal{K}(S_{ij}(\mathcal{F}))$. Clearly $j \in K$ and $i \notin K$, and we define $\sigma(K) = K' =$

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$(K \setminus \{j\}) \cup \{i\}$. We show that σ is well-defined by checking $K' \in \mathcal{K}(S_{ij}(\mathcal{F})) \setminus \mathcal{K}(\mathcal{F})$. Firstly, suppose that $K' \notin \mathcal{K}(S_{ij}(\mathcal{F}))$ and let $F' \in \binom{K'}{k}$ be an edge not in $S_{ij}(\mathcal{F})$. If $i \notin F'$ then $F' = K \setminus \{j\}$ and $S_{ij}(F') = F'$, implying that $F' \in S_{ij}(\mathcal{F})$, a contradiction. If $i \in F'$, then $F = (F' \setminus \{i\}) \cup \{j\} \subset K$ is an edge of $\mathcal{F}$ since $K \in \mathcal{K}(\mathcal{F})$. Hence after shifting we have $F' \in S_{ij}(\mathcal{F})$, a contradiction. This shows $K' \in \mathcal{K}(S_{ij}(\mathcal{F}))$. Secondly, if $K' \in \mathcal{K}(\mathcal{F})$ then $K \in \mathcal{K}(\mathcal{F})$ implies $K \in \mathcal{K}(S_{ij}(\mathcal{F}))$, contradicting the assumption that $K \notin \mathcal{K}(S_{ij}(\mathcal{F}))$. Thus $K' \in \mathcal{K}(S_{ij}(\mathcal{F})) \setminus \mathcal{K}(\mathcal{F})$ and σ is indeed a map from $\mathcal{K}(\mathcal{F}) \setminus \mathcal{K}(S_{ij}(\mathcal{F}))$ to $\mathcal{K}(S_{ij}(\mathcal{F})) \setminus \mathcal{K}(\mathcal{F})$. Clearly, σ is injective and the proposition follows. □

Proof of Theorem 1.6. Since the shifting operator does not increase the matching number and does not decrease the size of $\mathcal{K}(\mathcal{F})$, we may assume that $\mathcal{F}$ is shifted. Let $\mathcal{K} = \mathcal{K}(\mathcal{F})$ and $\mathcal{K}^* = \mathcal{K}(\mathcal{E}(n, k, s))$. For any $S \subset [s+1]$ and a family $\mathcal{H} \subset \binom{[n]}{h}$, define

$$\mathcal{H}(S) := \{H \setminus [s+1] \colon H \in \mathcal{H},\ H \cap [s+1] = S\}.$$

Clearly $\mathcal{H}(S) \subset \binom{[s+2,n]}{h-|S|}$.

For $|S| \geq 3$, we have $\mathcal{K}^*(S) = \binom{[s+2,n]}{k+1-|S|}$. It follows that

$$\sum_{S \subset [s+1], |S| \geq 3} |\mathcal{K}(S)| \leq \sum_{S \subset [s+1], |S| \geq 3} |\mathcal{K}^*(S)|. \tag{16}$$

We are left to compare $|\mathcal{K}(S)|$ with $|\mathcal{K}^*(S)|$ for all $S \subset [s+1]$ with $|S| \leq 2$.

Claim 2. *$\mathcal{K}(\{i\}) = \mathcal{F}(\emptyset)$ for $i = 1, 2, \ldots, s+1$ and $\mathcal{K}(\{i, j\}) = \mathcal{F}(\{j\})$ for $1 \leq i < j \leq s+1$.*

Proof. For $F \in \mathcal{K}(\{i\})$, $F \cup \{i\} \in \mathcal{K}$ implies that $F \in \mathcal{F}(\emptyset)$. Let $F \in \mathcal{F}(\emptyset)$. Since $x \geq s+2 > i$ each $x \in F$, by shiftedness $(F \setminus \{x\}) \cup \{i\} \in \mathcal{F}$. It follows that $\binom{F \cup \{i\}}{k} \subset \mathcal{F}$ and $F \cup \{i\} \in \mathcal{K}$. Thus $F \in \mathcal{K}(\{i\})$. Therefore $\mathcal{K}(\{i\}) = \mathcal{F}(\emptyset)$.

For any $E \in \mathcal{K}(\{i, j\})$ we have $\binom{E \cup \{i,j\}}{k} \subset \mathcal{F}$. It follows that $E \cup \{j\} \in \mathcal{F}$. Thus $E \in \mathcal{F}(\{j\})$. Let $E \in \mathcal{F}(\{j\})$. By shiftedness and $i < j$, $E \cup \{i\} \in \mathcal{F}$. Moreover, $E \cup \{i, j\} \setminus \{x\} \in \mathcal{F}$ for each $x \in E$. That is, $\binom{E \cup \{i,j\}}{k} \subset \mathcal{F}$ and $E \cup \{i, j\} \in \mathcal{K}$. Thus $E \in \mathcal{K}(\{i, j\})$. Therefore $\mathcal{K}(\{i, j\}) = \mathcal{F}(\{j\})$. □

Note that for any $K \in \mathcal{K}(\emptyset)$ we have $\binom{K}{k} \subset \mathcal{F}(\emptyset)$. It follows that $\partial \mathcal{K}(\emptyset) \subset \mathcal{F}(\emptyset)$. Since $\nu(\mathcal{K}(\emptyset)) \leq s$, by (8) we have

$$s|\mathcal{F}(\emptyset)| \geq s|\partial \mathcal{K}(\emptyset)| \geq |\mathcal{K}(\emptyset)|. \tag{17}$$

By Claim 2,

$$\sum_{1 \leq i \leq s+1} |\mathcal{K}(\{i\})| = (s+1)|\mathcal{F}(\emptyset)|$$

and

$$\sum_{1 \leq i < j \leq s+1} |\mathcal{K}(\{i, j\})| = \sum_{2 \leq j \leq s+1} (j-1)|\mathcal{F}(\{j\})|.$$

It follows that

$$\begin{aligned}
\sum_{S \in [n], |S| \leq 2} |\mathcal{K}(S)| &= |\mathcal{K}(\emptyset)| + \sum_{1 \leq i \leq s+1} |\mathcal{K}(\{i\})| + \sum_{1 \leq i < j \leq s+1} |\mathcal{K}(\{i, j\})| \\
&\overset{(17)}{\leq} s|\mathcal{F}(\emptyset)| + (s+1)|\mathcal{F}(\emptyset)| + \sum_{2 \leq j \leq s+1} (j-1)|\mathcal{F}(\{j\})| \\
&\leq (2s+1)|\mathcal{F}(\emptyset)| + \sum_{2 \leq j \leq s+1} (j-1)|\mathcal{F}(\{j\})|.
\end{aligned} \tag{18}$$

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Again by shiftedness $\partial\mathcal{F}(\emptyset)\subset\mathcal{F}(\{s+1\})$, and using (8) we infer

$$s|\mathcal{F}(\{s+1\})|\geq s|\partial\mathcal{F}(\emptyset)|\geq|\mathcal{F}(\emptyset)|. \tag{19}$$

Substituting (19) into (18), we arrive at

$$\begin{aligned}\sum_{S\in[n],|S|\leq 2}|\mathcal{K}(S)| &\leq \sum_{2\leq j\leq s}(j-1)|\mathcal{F}(\{j\})|+s|\mathcal{F}(\{s+1\})|+(2s+1)s|\mathcal{F}(\{s+1\})|\\ &=|\mathcal{F}(\{2\})|+\sum_{3\leq j\leq s}(j-1)|\mathcal{F}(\{j\})|+2s(s+1)|\mathcal{F}(\{s+1\})|\\ &\leq \frac{1}{2}|\mathcal{F}(\{1\})|+\frac{1}{2}|\mathcal{F}(\{2\})|+\sum_{3\leq j\leq s}(j-1)|\mathcal{F}(\{j\})|+2s(s+1)|\mathcal{F}(\{s+1\})|.\end{aligned}$$

By shiftedness, $\mathcal{F}(\{1\})\supset\cdots\supset\mathcal{F}(\{s+1\})$ are overlapping families. Set

$$\mathcal{F}_0=\mathcal{F}(\{s+1\}),\ \mathcal{F}_1=\mathcal{F}(\{s\}),\ \ldots,\mathcal{F}_{s-2}=\mathcal{F}(\{3\}),\ \mathcal{F}_{s-1}=\mathcal{F}(\{2\}),\ \mathcal{F}_s=\mathcal{F}(\{1\})$$

and set

$$p_0=2s(s+1),\ p_1=s-1,\ldots,p_{s-2}=2,\ p_{s-1}=\frac{1}{2},\ p_s=\frac{1}{2}.$$

Then $p_0\geq p_1\geq\ldots\geq p_s$ and by $s\geq 3$,

$$d_{\vec{p}}=\frac{s(p_0+\cdots+p_s)}{(p_1+\cdots+p_s)}=\frac{4s(s+1)+s(s-1)}{s-1}=5s+8+\frac{8}{s-1}\leq 5s+12.$$

By Lemma 3.2, for $n-s-1\geq(5s+13)(k-1)\geq(d_{\vec{p}}+1)(k-1)$ we have

$$\begin{aligned}\sum_{S\subset[s+1],|S|\leq 2}|\mathcal{K}(S)| &\leq (p_1+p_2+\ldots+p_s)\binom{n-s-1}{k-1}\\ &=\binom{s}{2}\binom{n-s-1}{k-1}\\ &=\sum_{S\subset[s+1],|S|\leq 2}|\mathcal{K}^*(S)|.\end{aligned} \tag{20}$$

Adding (16) and (20), we conclude that

$$|\mathcal{K}(\mathcal{F})|=\sum_{S\subset[s+1]}|\mathcal{K}(S)|\leq\sum_{S\subset[s+1]}|\mathcal{K}^*(S)|=|\mathcal{K}(\mathcal{E}(n,k,s))|.$$

Let $\mathcal{F}$ be a family with $\nu(\mathcal{F})\leq s$ and $|\mathcal{K}(\mathcal{F})|=|\mathcal{K}(\mathcal{E}(n,k,s))|$. If $\mathcal{F}$ is shifted, then by Lemma 3.2 we have $\mathcal{F}(\{s+1\})=\emptyset$. It follows that $\mathcal{F}=\mathcal{E}(n,k,s)$. Now assume that $\mathcal{F}$ is not shifted. Then it changes to $\mathcal{E}(n,k,s)$ by applying shifting repeatedly. Let $\mathcal{G}$ be the last family that is not isomorphic to $\mathcal{E}(n,k,s)$ in this process. That is, $\mathcal{G}$ is not isomorphic to $\mathcal{E}(n,k,s)$ but $S_{ij}(\mathcal{G})$ is isomorphic to $\mathcal{E}(n,k,s)$ for some $1\leq i<j\leq n$. By symmetry, we may assume that $\mathcal{G}\neq\mathcal{E}(n,k,s)$ and $S_{s,s+1}(\mathcal{G})=\mathcal{E}(n,k,s)$. Let

$$\mathcal{G}(s\overline{(s+1)})=\left\{E\in\binom{[n]\setminus\{s,s+1\}}{k-1}: E\cup\{s\}\in\mathcal{G}\right\},$$

$$\mathcal{G}(\bar{s}(s+1))=\left\{E\in\binom{[n]\setminus\{s,s+1\}}{k-1}: E\cup\{s+1\}\in\mathcal{G}\right\}.$$

Since $S_{s,s+1}(\mathcal{G})=\mathcal{E}(n,k,s)$, we see $\mathcal{G}(s\overline{(s+1)})\cup\mathcal{G}(\bar{s}(s+1))=\binom{[n]\setminus\{s,s+1\}}{k-1}$ and $\mathcal{G}(s\overline{(s+1)})\cap\mathcal{G}(\bar{s}(s+1))=\emptyset$. It follows that for each $E\in\binom{[n]\setminus\{s,s+1\}}{k-1}$, exactly one of $E\cup\{s\}\in\mathcal{G}$ and

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$E \cup \{s+1\} \in \mathcal{G}$ holds. Now consider a graph G on the vertex set $\binom{[n]\setminus\{s,s+1\}}{k-1}$ where (E_1, E_2) forms an edge if and only if $|E_1 \cap E_2| = k-2$. It is easy to see that G is a connected graph. Since $\mathcal{G}$ is not isomorphic to $\mathcal{E}(n,k,s)$, we infer that $\mathcal{G}(s\overline{(s+1)}) \neq \emptyset$ and $\mathcal{G}(\overline{s}(s+1)) \neq \emptyset$. Then there exists an edge (E_1, E_2) in G such that $E_1 \cup \{s\} \in \mathcal{G}$ and $E_2 \cup \{s+1\} \in \mathcal{G}$. Let $F := E_1 \cup E_2 \in \binom{[n]\setminus\{s,s+1\}}{k}$. Then $F \cup \{s+1\} \notin \mathcal{K}(\mathcal{G})$ and $F \cup \{s\} \notin \mathcal{K}(\mathcal{G})$. But $S_{s,s+1}(\mathcal{G}) = \mathcal{E}(n,k,s)$ implies $F \cup \{s\} \in \mathcal{K}(S_{s,s+1}(\mathcal{G}))$. Moreover, for any $K \in \mathcal{K}(\mathcal{G}) \setminus \mathcal{K}(S_{s,s+1}(\mathcal{G}))$, we have $(K \setminus \{s+1\}) \cup \{s\} \in \mathcal{K}(S_{s,s+1}(\mathcal{G})) \setminus \mathcal{K}(\mathcal{G})$ by the injective map defined in Proposition 3.3. Hence $|\mathcal{K}(S_{s,s+1}(\mathcal{G}))| > |\mathcal{K}(\mathcal{G})| \geq |\mathcal{K}(\mathcal{F})| = |\mathcal{K}(\mathcal{E}(n,k,s))|$, a contradiction. Thus up to isomorphism $\mathcal{E}(n,k,s)$ is the only family attaining equality. □

4. The maximum size of an *r*-complete intersecting family

In this section, we determine $f(n,k,r)$ for all $k, r \geq 3$ and $n \geq n_0(k,r)$, thereby proving Theorem 1.5.

Proposition 4.1. *For $r \geq k+1$, $f(n,k,r) = 0$. For $n \geq 2k-1$,*

$$f(n,k,k) = \binom{2k-1}{k}.$$

Moreover, the unique family satisfying the condition is $\binom{X}{k}$ with $|X| = 2k-1$.

Proof. Suppose that $\mathcal{F}$ is an intersecting k-graph and each $F \in \mathcal{F}$ is k-wise covered. Consider the bipartite graph with partite sets $\mathcal{F}$, $\partial\mathcal{F}$ and an edge between F and G iff $G \subset F$. It is clear that each $F \in \mathcal{F}$ has degree k. On the other hand, the condition implies that each $G \in \partial\mathcal{F}$ has degree at least k. Consequently, $|\mathcal{F}| \geq |\partial\mathcal{F}|$. In view of (7), we see $|\mathcal{F}| = |\partial\mathcal{F}|$ and equality holds iff $\mathcal{F} = \binom{X}{k}$ with $|X| = 2k-1$. The same argument implies $f(n,k,r) = 0$ for $r \geq k+1$. □

We need a notion of basis for an intersecting family inspired by [6]. For any intersecting family $\mathcal{F} \subset \binom{[n]}{k}$, we define a *basis* $\mathcal{B}(\mathcal{F})$ which is not necessarily unique by the following process. We start with $\mathcal{F}^0 = \mathcal{F}$. Note that $\mathcal{F}^0$ is an antichain. A collection of sets $F_0, \ldots, F_k$ is called a *sunflower of size $k+1$ with centre C* if $F_i \cap F_j = C$ for all distinct $i, j \in \{0, 1, \ldots, k\}$. Note that in this case $F_0 \setminus C, \ldots, F_k \setminus C$ are pairwise disjoint. At the i-th step try and find in the current family $\mathcal{F}^i$ a sunflower $F_0, \ldots, F_k$ of size $k+1$ (the size of F_j may be distinct). Let C_i be the centre of the sunflower. Then let $\mathcal{F}^{i+1}$ be the family obtained from $\mathcal{F}^i$ by deleting all sets containing C_i and adding C_i. Clearly $\mathcal{F}^{i+1}$ is also an antichain.

Claim 3. *If $\mathcal{F}^i$ is intersecting, then $\mathcal{F}^{i+1}$ is also intersecting.*

Proof. Take an arbitrary set F from $\mathcal{F}^i$. Since $|F| \leq k$, we have $F \cap (F_j \setminus C_i) = \emptyset$ for some j, $0 \leq j \leq k$. Then $F \cap C_i = F \cap F_j \neq \emptyset$. □

Continue this process until no more sunflowers of size $k+1$ can be formed. Let $\mathcal{B}(\mathcal{F})$ be the final family. Clearly, $\mathcal{B}(\mathcal{F})$ is an antichain and for all $F \in \mathcal{F}$ there exists $B \in \mathcal{B}(\mathcal{F})$ with $B \subset F$. By Claim 3, $\mathcal{B}(\mathcal{F})$ is intersecting. In view of the Erdős-Rado sunflower lemma [4],

$$|\mathcal{B}^{(\ell)}| \leq \ell! k^{\ell}, \ \forall 1 \leq \ell \leq k. \tag{21}$$

Proof of Theorem 1.5. By proposition 4.1, we may assume $3 \leq r \leq k$. Let $\mathcal{F}$ be an r-complete intersecting family of maximal size and let $\mathcal{B} = \mathcal{B}(\mathcal{F})$ be its basis. Let $X = \bigcup\limits_{B \in \mathcal{B}} B$. By (21) we have

$$|X| \leq \sum_{1 \leq \ell \leq k} \ell! k^{\ell} \leq 2k! k^k \leq k^{2k}.$$

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By the definition of $\mathcal{B}$, for any $F \in \mathcal{F}$ there exists $B \in \mathcal{B}$ such that $B \subset F$. Then for $F, F' \in \mathcal{F}$, there exist $B, B' \in \mathcal{B}$ such that $B \subset F$ and $B' \subset F'$. Since $\mathcal{B}$ is intersecting, $\emptyset \neq B \cap B' \subset F \cap F' \cap X$. Thus, for all $F, F' \in \mathcal{F}$, $F \cap F' \cap X \neq \emptyset$.

Let us define $p = \min\{|F \cap X| \colon F \in \mathcal{F}\}$ and choose an arbitrary pair (F, P_0), $P_0 \in \binom{X}{p}$, $F \cap X = P_0$. Set $H = F \setminus P_0$ and define

$$\mathcal{P}(H) = \left\{P \in \binom{X}{p} \colon H \cup P \in \mathcal{F}\right\}.$$

Note that $P_0 \in \mathcal{P}(H)$.

Claim 4. *$\mathcal{P}(H)$ is intersecting and r-complete.*

Proof. For $P, P' \in \mathcal{P}(H)$ fix $B, B' \in \mathcal{B}(\mathcal{F})$ satisfying $B \subset H \cup P$, $B' \subset H \cup P'$. Since $H \cap X = \emptyset$, $B \subset P$ and $B' \subset P'$. Consequently, $P \cap P' \supset B \cap B' \neq \emptyset$.

Let us prove the r-completeness of $\mathcal{P}(H)$ next. Fix $P \in \mathcal{P}(H)$ and $R \in \binom{P}{p-1}$. Using the r-completeness of $\mathcal{F}$ there are r distinct elements $x_1, x_2, \ldots, x_r$ such that $(H \cup R \cup \{x_i\}) \in \mathcal{F}$. The minimal choice of p implies $|(H \cup R \cup \{x_i\}) \cap X| \geq p$, whence $x_i \in X$, $1 \leq i \leq r$. Thus $R \cup \{x_i\} \in \mathcal{P}(H)$, proving the r-completeness of $\mathcal{P}(H)$. □

If $p < r$, by Claim 4 and Proposition 4.1 we have $1 \leq |\mathcal{P}(H)| \leq f(|X|, p, r) = 0$, a contradiction. Thus $p \geq r$. Define

$$\mathcal{F}_0 = \{F \in \mathcal{F} \colon |F \cap X| \geq r+1\}.$$

Then

$$|\mathcal{F}_0| \leq \sum_{r+1 \leq i \leq k} \binom{|X|}{i}\binom{n-|X|}{k-i} \leq \sum_{r+1 \leq i \leq k} \binom{k^{2k}}{i}\binom{n-k^{2k}}{k-i} < 2\binom{k^{2k}}{r+1}\binom{n-2r}{k-r-1}.$$

If $p \geq r+1$, then

$$|\mathcal{F}| = |\mathcal{F}_0| \leq 2\binom{k^{2k}}{r+1}\binom{n-2r}{k-r-1} \leq \binom{2r-1}{r}\binom{n-2r+1}{k-r} < |\mathcal{L}(n,k,r)|.$$

Thus we assume $p = r$.

If $|\mathcal{P}(H)| \leq \binom{2r-1}{r} - 1$ holds for all $H \in \binom{[n]\setminus X}{k-r}$, then

$$\begin{aligned}
|\mathcal{F}| &\leq \sum_{H \in \binom{[n]\setminus X}{k-r}} |\mathcal{P}(H)| + |\mathcal{F}_0| \\
&\leq \left(\binom{2r-1}{r} - 1\right)\binom{n-|X|}{k-r} + 2\binom{k^{2k}}{r+1}\binom{n-2r}{k-r-1} \\
&\leq \binom{2r-1}{r}\binom{n-2r+1}{k-r} \quad (\text{for } n \geq n_0(k,r)) \\
&< |\mathcal{L}(n,k,r)|.
\end{aligned}$$

Assume now that for some $H \in \binom{[n]\setminus X}{k-r}$, $|\mathcal{P}(H)| = \binom{2r-1}{r}$. By Proposition 4.1 we may assume that $\mathcal{P}(H) = \binom{Y}{r}$, $Y \in \binom{X}{2r-1}$. We claim that $|F \cap Y| \geq r$ for all $F \in \mathcal{F}$. Indeed the opposite would mean that $F \cap P = \emptyset$ for some $P \in \binom{Y}{r}$. Then $F \cap (H \cup P) \cap X = F \cap P = \emptyset$, a contradiction. Consequently $\mathcal{F} \subset \{F \in \binom{[n]}{k} \colon |F \cap Y| \geq r\}$, i.e., $\mathcal{F}$ is contained in an isomorphic copy of $\mathcal{L}(n,k,r)$. □

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5. Maximizing the number of r-complete sets in an intersecting family

In this section, we prove Theorem 1.8 for $3 \le r < k$ and $n \ge n_0(k, r)$. We need a different notion of basis. For a saturated intersecting family $\mathcal{F}$, define $\mathcal{B}(\mathcal{F})$ be the family of minimal (for containment) sets in $\mathcal{T}(\mathcal{F})$. Define $X = \bigcup_{B \in \mathcal{B}} B$ the *support* of $\mathcal{B}$. The following properties of $\mathcal{B}(\mathcal{F})$ were proved in [10].

Lemma 5.1 ([10]). *Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ is a saturated intersecting family and $\mathcal{B} = \mathcal{B}(\mathcal{F})$. Then*

(i) $\mathcal{B}$ is an intersecting antichain,

(ii) $\mathcal{F} = \{H \in \binom{[n]}{k} : \exists B \in \mathcal{B}, B \subset H\}$,

(iii) for all $F, F' \in \mathcal{F}$,

$$F \cap F' \cap X \ne \emptyset. \tag{22}$$

The following lemma is essentially proved in [10]. For self-containedness we include its proof as well.

Lemma 5.2 ([10]). *Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ is a saturated intersecting family. Then $|\mathcal{B}(\mathcal{F})| \le k^k$.*

Proof. Let $\mathcal{B} = \mathcal{B}(\mathcal{F})$. For the proof we use a branching process. During the proof *a sequence* $S = (x_1, x_2, \ldots, x_\ell)$ is an ordered sequence of distinct elements of $[n]$ and we use $\widehat{S}$ to denote the underlying unordered set $\{x_1, x_2, \ldots, x_\ell\}$. At the beginning, we assign weight 1 to the empty sequence $S_\emptyset$. At the first stage, we choose $B_1 \in \mathcal{B}$ with $|B_1|$ minimal. For any vertex $x \in B_1$, define one sequence (x) and assign the weight $|B_1|^{-1}$ to it.

In each subsequent stage, we pick a sequence $S = (x_1, \ldots, x_p)$ and denote its weight by $w(S)$. If $\widehat{S} \cap B \ne \emptyset$ holds for all $B \in \mathcal{B}$ then we do nothing. Otherwise we pick $B \in \mathcal{B}$ satisfying $\widehat{S} \cap B = \emptyset$ and replace S by the $|B|$ sequences $(x_1, \ldots, x_p, y)$ with $y \in B$ and assign weight $\frac{w(S)}{|B|}$ to each of them. Clearly, the total weight is always 1.

We continue until $\widehat{S} \cap B \ne \emptyset$ for all sequences S and all $B \in \mathcal{B}$. Since $[n]$ is finite, each sequence has length at most n and eventually the process stops. Let $\mathcal{S}$ be the collection of sequences that survived in the end of the branching process and let $\mathcal{S}^{(\ell)}$ be the collection of sequences in $\mathcal{S}$ with length ℓ.

Claim 5. *For each $B \in \mathcal{B}^{(\ell)}$, there is some sequence $S \in \mathcal{S}^{(\ell)}$ with $\widehat{S} = B$.*

Proof. Let us suppose the contrary and let $S = (x_1, \ldots, x_p)$ be a sequence of maximal length that occurred at some stage of the branching process satisfying $\widehat{S} \subsetneq B$. Since $\mathcal{B}$ are intersecting, $B_1 \cap B \ne \emptyset$, implying that $p \ge 1$. Since $\widehat{S}$ is a proper subset of B and $B \in \mathcal{B}$, it follows that $\widehat{S} \notin \mathcal{T}(\mathcal{F})$. Thereby there exists $F \in \mathcal{F}$ with $\widehat{S} \cap F = \emptyset$. In view of Lemma 5.1 (ii), we can find $B' \in \mathcal{B}$ such that $\widehat{S} \cap B' = \emptyset$. Thus at some point we picked S and some $\tilde{B} \in \mathcal{B}$ with $\widehat{S} \cap \tilde{B} = \emptyset$. Since $\mathcal{B}$ is intersecting, $B \cap \tilde{B} \ne \emptyset$. Consequently, for each $y \in B \cap \tilde{B}$ the sequence $(x_1, \ldots, x_p, y)$ occurred in the branching process. This contradicts the maximality of p. Hence there is an S at some stage satisfying $\widehat{S} = B$. Since $\mathcal{B}$ is intersecting, $\widehat{S} \cap B' = B \cap B' \ne \emptyset$ for all $B' \in \mathcal{B}$. Thus $\widehat{S} \in \mathcal{S}$ and the claim holds. □

By Claim 5, we see that $|\mathcal{B}^{(\ell)}| \le |\mathcal{S}^{(\ell)}|$ for all $\ell \ge 1$. Let $S = (x_1, \ldots, x_\ell) \in \mathcal{S}^{(\ell)}$ and let $S_i = (x_1, \ldots, x_i)$ for $i = 1, \ldots, \ell$. At the first stage, $w(S_1) = 1/|B_1|$. Assume that B_i is the selected set when replacing S_{i-1} in the branching process for $i = 2, \ldots, \ell$. Then

$$w(S) = \prod_{i=1}^{\ell} \frac{1}{|B_i|} \ge k^{-\ell}.$$

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It follows that

$$k^{-k}\sum_{1\le\ell\le k}|\mathcal{B}^{(\ell)}|\le\sum_{1\le\ell\le k}k^{-\ell}|\mathcal{B}^{(\ell)}|\le\sum_{1\le\ell\le k}k^{-\ell}|\mathcal{S}^{(\ell)}|\le\sum_{1\le\ell\le k}\sum_{S\in\mathcal{S}^{(\ell)}}w(S)\le\sum_{S\in\mathcal{S}}w(S)=1.$$

Thus $|\mathcal{B}|=\sum_{1\le\ell\le k}|\mathcal{B}^{(\ell)}|\le k^k$. □

Proposition 5.3. *If $\partial\mathcal{F}$ is intersecting, then $\mathcal{F}$ is 3-intersecting.*

Proof. Suppose that $|F\cap F'|\le 2$ and $F,F'\in\mathcal{F}$. If $F\cap F'=\{x,x'\}$ then $F\setminus\{x\},F'\setminus\{x'\}\in\partial\mathcal{F}$ and they are disjoint, a contradiction. The case $F\cap F'=\{x\}$ is even easier. □

Proposition 5.4. *Let $\mathcal{F}\subset\binom{[n]}{3}$ be a saturated intersecting family and let $\mathcal{F}^*$ be the family of r-complete sets in $\mathcal{F}$. For $r=3$, $|\mathcal{F}^*|\le\binom{5}{3}$ with equality holding iff $\mathcal{F}=\binom{[5]}{3}$ up to isomorphism. For $r\ge 4$, $|\mathcal{F}^*|\le 1$ with equality holding iff $\mathcal{F}=\mathcal{L}(n,3,2)$ up to isomorphism.*

Proof. Let $r=3$. Suppose that there exist two edges intersecting in one vertex, say $(x_1,x_2,z),(y_1,y_2,z)\in\mathcal{F}^*$, since (x_1,x_2) is 3-fold covered and $\mathcal{F}$ is intersecting, we have $(x_1,x_2,y_i)\in\mathcal{F}$, $i=1,2$. Similarly, $(y_1,y_2,x_i)\in\mathcal{F}$, $i=1,2$. Since $(x_1,x_2,y_2)\in\mathcal{F}$ and (z,y_1) is 3-fold covered, $(z,y_1,x_1),(z,y_1,x_2)\in\mathcal{F}$. Similarly, $(z,y_2,x_1),(z,y_2,x_2)\in\mathcal{F}$. Hence $\{x_1,x_2,y_1,y_2,z\}$ spans a complete 3-graph in $\mathcal{F}$. Since $\mathcal{F}$ is intersecting, we conclude that $\mathcal{F}=\binom{[5]}{3}$ up to isomorphism and $|\mathcal{F}^*|=10$. Suppose next that there are two edges intersecting in two vertices say $(x,z_1,z_2),(y,z_1,z_2)\in\mathcal{F}^*$, since $(x,z_1),(y,z_2)$ are 3-fold covered and $\mathcal{F}$ is intersecting, there exists w such that $(x,z_1,y),(y,z_2,x),(x,z_1,w),(y,z_2,w)\in\mathcal{F}$. Arguing with (x,z_2) and (y,z_1), we infer that $(x,z_2,y),(y,z_1,x),(x,z_2,w),(y,z_1,w)\in\mathcal{F}$. Hence $\{x,y,z_1,z_2,w\}$ spans a complete 3-graph in $\mathcal{F}$. Since $\mathcal{F}$ is intersecting, we conclude that $\mathcal{F}=\binom{[5]}{3}$ up to isomorphism and $|\mathcal{F}^*|=10$. If $\mathcal{F}^*$ is 3-intersecting, then $|\mathcal{F}^*|\le 1$ holds trivially.

For $r\ge 4$, we claim that each member in $\partial\mathcal{F}^*$ is a transversal of $\mathcal{F}$. Otherwise, let $G\in\partial\mathcal{F}^*$ be a 2-set that is not a transversal. Then there exists $F\subset\mathcal{F}$ such that $F\cap G=\emptyset$. Since $G\in\partial\mathcal{F}^*$ and $r\ge 4>|F|$, there exists x such that $G\cup\{x\}\in\mathcal{F}$ and $F\cap(G\cup\{x\})=\emptyset$, a contradiction.

Thus $\partial\mathcal{F}^*\subset\mathcal{T}(\mathcal{F})$. By Lemma 5.1 (i) $\partial\mathcal{F}^*$ is intersecting. In view of Proposition 5.3, $\mathcal{F}^*$ is 3-intersecting. For $n\ge 6$, by Proposition 5.3 and (1) we have $|\mathcal{F}^*|\le\binom{n-3}{3-3}=1$. In the case of equality, by symmetry we may assume that $\mathcal{F}^*=[3]$.

Then we claim $|F\cap[3]|\ge 2$ for all $F\in\mathcal{F}$. Indeed, otherwise $|F\cap[3]|=1$ for some $F\in\mathcal{F}$, without loss of generality assume $F\cap[3]=\{1\}$, then we can find an $F'\in\mathcal{F}$ disjoint to F since $(2,3)$ is r-fold covered with $r\ge 4>|F|$, a contradiction. Thus $|F\cap[3]|\ge 2$ for all $F\in\mathcal{F}$. Using that $\mathcal{F}$ is saturated, we conclude that $\mathcal{F}=\mathcal{L}(n,3,2)$ up to isomorphism. For $n\le 5$, clearly $\mathcal{F}\subset\binom{[5]}{3}$. Since no 2-set is contained in 4 or more 3-sets in $\binom{[5]}{3}$, we have $|\mathcal{F}^*|=0$. □

The following proposition proves Theorem 1.8 for $3\le r\le k-1$ and $n\ge n_0(k,r)$.

Proposition 5.5. *$f^*(n,k,r)=\binom{n-3}{k-3}+O(n^{k-r})$ for $4\le r\le k-1$ and $n\ge n_0(k,r)$; $f^*(n,k,3)=|\mathcal{L}(n,k,3)|$ for $n\ge n_0(k)$.*

Proof. Let $\mathcal{F}\subset\binom{[n]}{k}$ be a saturated intersecting family and let $\mathcal{F}^*$ be the family of r-complete sets in $\mathcal{F}$. Let $\mathcal{B}=\mathcal{B}(\mathcal{F})$ and $X=\bigcup_{B\in\mathcal{B}}B$. By Lemma 5.2, we have $|X|\le k\cdot k^k=k^{k+1}$. Let

$$p=\min\left\{|F\cap X|\colon F\in\mathcal{F}^*\right\}.$$

If $p\ge 4$, then for $n\ge n_0(k,r)$ we have

$$|\mathcal{F}^*|\le\sum_{4\le i\le k}\binom{|X|}{i}\binom{n-|X|}{k-i}\le\sum_{4\le i\le k}\binom{k^{k+1}}{i}\binom{n-k^{k+1}}{k-i}\le 2\binom{k^{k+1}}{4}\binom{n-6}{k-4}<\binom{n-3}{k-3}$$

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and we are done. If $p=1$, then there exists $F^*\in\mathcal{F}^*$ such that $F^*\cap X=\{x\}$. It follows that $\{x\}\in\mathcal{B}$. By saturatedness we have $\mathcal{F}=\mathcal{S}_x$ and $|\mathcal{F}^*|=0$. If $p=2$ then for some $F^*\in\mathcal{F}^*$, $F^*\cap X=\{x,y\}\in\mathcal{B}$. Using r-completeness we find $F_x\in\mathcal{F}$, $F_x\supset F^*\setminus\{y\}$ and $F_y\in\mathcal{F}$, $F_y\supset F^*\setminus\{x\}$ and $F_x\cap F_y\cap X=\emptyset$, contradicting (22). Thus we may assume $p=3$.

Define the 3-graph

$$\mathcal{T}^*=\left\{T\in\binom{X}{3}: \exists F^*\in\mathcal{F}^*, F^*\cap X=T\right\},\ \mathcal{T}=\left\{T\in\binom{X}{3}: \exists F\in\mathcal{F}, F\cap X=T\right\}.$$

Note that $p=3$ implies $\mathcal{T}\neq\emptyset$. By (22) we infer that $\mathcal{T}$ is intersecting. We distinguish two cases.

Case 1. $r=3$.

If there exist $(x_1,y_1,z),(x_2,y_2,z)\in\mathcal{T}^*$, let $H_i\in\binom{[n]\setminus X}{k-3}$ such that $H_i\cup\{x_i,y_i,z\}\in\mathcal{F}^*$, $i=1,2$. By 3-completeness and (22), we have

$$(x_1,y_1,x_2),(x_1,y_1,y_2),(x_2,y_2,x_1),(x_2,y_2,y_1)\in\mathcal{T}.$$

Then there exists $H_3\in\binom{[n]\setminus X}{k-3}$ such that $H_3\cup\{x_1,y_1,x_2\}\in\mathcal{F}$. Since $H_2\cup\{y_2,z\}$ is covered by at least 3 members of $\mathcal{F}$, by (22) we infer $H_2\cup(x_1,y_2,z),H_2\cup(y_1,y_2,z)\in\mathcal{F}$. Similarly, we have $H_2\cup(x_2,y_1,z),H_2\cup(x_1,x_2,z)\in\mathcal{F}$. Hence $\{x_1,x_2,y_1,y_2,z\}$ spans a complete 3-graph in $\mathcal{T}$. We claim that $|F\cap\{x_1,x_2,y_1,y_2,z\}|\geq 3$ for all $F\in\mathcal{F}$. Indeed, otherwise suppose that there is $F\in\mathcal{F}$ with $|F\cap\{x_1,x_2,y_1,y_2,z\}|\leq 2$. Without loss of generality assume that $F\cap\{y_1,y_2,z\}=\emptyset$. Since $\{y_1,y_2,z\}\in\mathcal{T}$, there exists $H\in\binom{[n]\setminus X}{k-3}$ such that $H\cup\{y_1,y_2,z\}=:F'\in\mathcal{F}$. But then $F\cap F'\cap X=\emptyset$, contradicting (22). By saturatedness, we conclude that $\mathcal{F}=\mathcal{L}(n,k,3)$ up to isomorphism and $|\mathcal{F}^*|=|\mathcal{L}(n,k,3)|$.

If there exist $(x_1,y,z),(x_2,y,z)\in\mathcal{T}^*$, let $H_i\in\binom{[n]\setminus X}{k-3}$ such that $H_i\cup\{x_i,y,z\}\in\mathcal{F}^*$, $i=1,2$. Since $H_1\cup\{x_1,y\},H_2\cup\{x_2,z\}$ are 3-fold covered, by (22) there exists $w\in X$ such that $H_1\cup\{x_1,y,x_2\},H_1\cup\{x_1,y,w\},H_2\cup\{x_2,z,x_1\},H_2\cup\{x_2,z,w\}\in\mathcal{F}$. Similarly, $H_1\cup\{x_1,z,x_2\},H_1\cup\{x_1,z,w\},H_2\cup\{x_2,y,x_1\},H_2\cup\{x_2,y,w\}\in\mathcal{F}$. Then $\{x_1,x_2,y,z,w\}$ spans a complete 3-graph in $\mathcal{T}$. By the same argument and saturatedness, we conclude that $\mathcal{F}=\mathcal{L}(n,k,3)$ up to isomorphism and $|\mathcal{F}^*|=|\mathcal{L}(n,k,3)|$.

Now we may assume that $\mathcal{T}^*$ is 3-intersecting. Since $\mathcal{T}^*$ is a 3-graph, we trivially have $|\mathcal{T}^*|\leq 1$. Then for $n\geq n_0(k)$ we obtain that

$$|\mathcal{F}^*|\leq\binom{n-|X|}{k-3}+\sum_{4\leq i\leq k}\binom{|X|}{i}\binom{n-|X|}{k-i}\leq 10\binom{n-5}{k-3}\leq|\mathcal{L}(n,k,3)|.$$

Case 2. $r\geq 4$.

Claim 6. *For all $F\in\mathcal{F}^*$ and $T\in\mathcal{T}^*$,*

$$|F\cap T|\geq 2. \tag{23}$$

Proof. Suppose the contrary. By symmetry let $T=\{1,2,3\}$, $F\cap T=\{3\}$ ($F\cap T\neq\emptyset$ by (22)). By r-completeness there are distinct elements $y_1,\ldots,y_r$ such that $(F\setminus\{3\})\cup\{y_i\}\in\mathcal{F}$. Since $r\geq 4$, without loss of generality, assume $y_r\notin\{1,2,3\}$. Then $((F\setminus\{3\})\cup\{y_r\})\cap T=\emptyset$ contradicting (22). □

Claim 7. $|\mathcal{T}^*|=1$.

Proof. Otherwise using Claim 6, without loss of generality, $\{1,2,3\},\{1,2,4\}\in\mathcal{T}^*$. Let $H_i\in\binom{[n]\setminus X}{k-3}$ such that $H_i\cup\{1,2,i\}\in\mathcal{F}^*$, $i=3,4$. Let $x_1,\ldots,x_r$ be such that $H_3\cup\{1,3,x_j\}\in\mathcal{F}$, $j=1,\ldots,r$. Let $y_1,\ldots,y_r$ be such that $H_4\cup\{2,4,y_j\}\in\mathcal{F}$, $j=1,\ldots,r$. By $r\geq 4$, without loss of generality assume $x_1\notin\{2,4\}$ and $y_1\notin\{1,3,x_1\}$. Then

$$(H_3\cup\{1,3,x_1\})\cap(H_4\cup\{2,4,y_1\})\cap X=\emptyset,$$

contradicting (22). □

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By Claim 7, we may assume that $\mathcal{T}^* = \{(1, 2, 3)\}$. Define

$$\mathcal{F}_i^* = \{F \in \mathcal{F}^* : F \cap [3] = [3] \setminus \{i\}\}, \ i = 1, 2, 3.$$

Claim 8. *$F \in \mathcal{F}_i^*$ implies $|F \cap X| \geq r$, $i = 1, 2, 3$.*

Proof. By symmetry assume $i = 1$ and set $S = F \cap X$. Suppose indirectly $|S| < r$. Let $\tilde{F} \in \mathcal{F}^*$ with $\tilde{F} \cap X = [3]$. By r-completeness there are $x_1, x_2, \ldots, x_r$ distinct elements with $(\tilde{F} \setminus \{3\}) \cup \{x_j\} \in \mathcal{F}$, $1 \leq j \leq r$. Also $F \setminus \{2\}$ is contained in $r > 3$ members of $\mathcal{F}$. Let $\hat{F}$ be one of them with $\hat{F} \cap [3] = \{3\}$ and let $\hat{S} = \hat{F} \cap X$. Clearly, $|\hat{S}| \leq |S| < r$. Consequently we can choose $x_j \notin \hat{S}$. Then

$$\left((\tilde{F} \setminus \{3\}) \cup \{x_j\}\right) \cap \hat{F} \cap X = \emptyset,$$

contradicting (22). □

By Claims 6, 7, 8, we have

$$\begin{aligned} |\mathcal{F}^*| &\leq |\mathcal{T}^*| \binom{n-|X|}{k-3} + \sum_{1 \leq i \leq 3} |\mathcal{F}_i^*| \\ &\leq \binom{n-|X|}{k-3} + \sum_{r \leq i \leq k} \binom{3}{2} \binom{|X|}{i-2} \binom{n-|X|}{k-i} \\ &= \binom{n-3}{k-3} + O(n^{k-r}) \end{aligned}$$

and the proposition is proven. □

6. Maximizing the number of *k*-complete sets in an intersecting family

In this section, we prove Theorem 1.8 for $r \geq k$. By using Bollobás Set-pairs Inequality (Theorem 2.3) and the Hilton-Milner-Frankl Theorem, we determine $f^*(n, k, k)$ for $k \geq 5$ and $n \geq n_0(k)$. The cases $r \geq k + 1$ and $r = k = 4$ of Theorem 1.8 will be proved separately.

First we show that if an intersecting family contains a relatively k-complete sunflower of given shape, then Theorem 1.8 holds.

Lemma 6.1. *Let $\mathcal{F} \subset \binom{[n]}{k}$ be an intersecting family and let $\mathcal{F}^*$ be the family of k-complete sets in $\mathcal{F}$. If $\mathcal{F}^*$ contains a sunflower with $k+1$ petals and centre C of size 3 and $k \geq 4$, then $C \subset F$ for all $F \in \mathcal{F}^*$. In particular,*

$$|\mathcal{F}^*| \leq \binom{n-3}{k-3}.$$

Proof. Suppose that $F_1, F_2, \ldots, F_{k+1}$ is a sunflower in $\mathcal{F}^*$ with centre $[3]$ and let $G_i = F_i \setminus [3]$, $i = 1, \ldots, k+1$.

If there exists $F \in \mathcal{F}^*$ with $|F \cap [3]| \leq 1$, pick $G \in \binom{F}{k-1}$ with $G \cap [3] = \emptyset$. Then $G \cap F_i \neq \emptyset$ can hold for at most $k-1$ values of i. Pick F_p, F_r disjoint to G. Now k-completeness and $k \geq 4$ imply that we can choose $z \notin [3]$, $G \cup \{z\} \in \mathcal{F}$. Then either F_p or F_r is disjoint to $G \cup \{z\}$, a contradiction.

If there exists $F \in \mathcal{F}^*$ with $|F \cap [3]| = 2$, without loss of generality, assume that $F \cap [3] = \{1, 2\}$ and let $G = F \setminus \{1, 2\}$. Pick F_p, F_q, F_r disjoint to G. Since $k \geq 4$, we can choose $z, w \notin [3]$ such that $G \cup \{1, z\}, G \cup \{1, w\} \in \mathcal{F}$. Then one of F_p, F_q, F_r, without loss of generality say F_p, is disjoint to both $G \cup \{z\}$ and $G \cup \{w\}$. Since $F_p \setminus \{1\}$ is covered by at least k members of $\mathcal{F}$, we can find $u \notin G \cup \{1\}$ such that $(F_p \setminus \{1\}) \cup \{u\} \in \mathcal{F}$. Then either $z \neq u$ or $w \neq u$ holds. Without loss of generality,

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assume that $z \neq u$, then $(F_p \setminus \{1\}) \cup \{u\}$ and $G \cup \{1, z\}$ are disjoint, a contradiction. Thus, $[3] \subset F$ for all $F \in \mathcal{F}^*$ and the lemma follows. □

We prove Theorem 1.8 for $r \geq k+1$ and $k \geq 4$ by the following proposition.

Proposition 6.2. *$f^*(n, k, r) = \binom{n-3}{k-3}$ for $r \geq k+1$ and $n \geq \max\{4(k-2), k+r-1\}$.*

Proof. Let $\mathcal{F} \subset \binom{[n]}{k}$ be a saturated intersecting family. Let $\mathcal{F}^*$ be the family of r-complete sets in $\mathcal{F}$ and let $\mathcal{G} = \partial\mathcal{F}^*$. We claim that each member in $\mathcal{G}$ is a transversal of $\mathcal{F}$. Otherwise, let $G \in \mathcal{G}$ be a $(k-1)$-set that is not a transversal. Then there exists $F \subset \mathcal{F}$ such that $F \cap G = \emptyset$. Since $G \in \partial\mathcal{F}^*$ and $r > |F|$, there exists x such that $G \cup \{x\} \in \mathcal{F}$ and $F \cap (G \cup \{x\}) = \emptyset$, a contradiction. Thus $\mathcal{G} \subset \mathcal{T}(\mathcal{F})$.

Since $\mathcal{F}$ is saturated, all k-element supersets of any $G \in \mathcal{T}(\mathcal{F})$ are members of $\mathcal{F}$. By Lemma 5.1 (i) we see that $\mathcal{G}$ is intersecting. In view of Proposition 5.3, $\mathcal{F}^*$ is 3-intersecting. Since $n \geq 4(k-2)$, by (1) we have $|\mathcal{F}^*| \leq \binom{n-3}{k-3}$. □

By using the Bollobás Set-pairs Theorem and the Hilton-Milner-Frankl Theorem, we prove Theorem 1.8 for $r = k$ and $k \geq 5$.

Proposition 6.3. *$f^*(n, k, k) = \binom{n-3}{k-3}$ for $r = k \geq 5$ and $n \geq k^3\binom{2k-1}{k}$.*

Proof. Let $\mathcal{F} \subset \binom{[n]}{k}$ be a saturated intersecting family. Let $\mathcal{F}^*$ be the family of k-complete sets in $\mathcal{F}$ and let $\mathcal{G} = \partial\mathcal{F}^*$. Define

$$\mathcal{G}' = \{G \in \mathcal{G} \colon G \notin \mathcal{T}(\mathcal{F})\} \text{ and } \mathcal{E} = \left\{E \in \mathcal{F}^* \colon \binom{E}{k-1} \cap \mathcal{G}' \neq \emptyset\right\}.$$

Claim 9. *To every $G \in \mathcal{G}'$ there is a unique k-element set $H(G) \in \mathcal{F}$ which is disjoint to G.*

Proof. Let $G \cup \{x_i\} \in \mathcal{F}$, $i = 1, \ldots, k$, the existence of x_i is guaranteed by k-completeness. Since $G \notin \mathcal{T}(\mathcal{F})$, there is $F \in \mathcal{F}$ satisfying $G \cap F = \emptyset$. As $\mathcal{F}$ is intersecting, $F \cap (G \cup \{x_i\}) = \{x_i\}$ for $1 \leq i \leq k$. Using $|F| = k$, $F = \{x_1, \ldots, x_k\} =: H(G)$ is the unique possibility. □

From Claim 9 it is clear that $H(G) \neq H(G')$ imply $G \cap H(G') \neq \emptyset$. Define

$$\mathcal{H} = \{H(G) \colon G \in \mathcal{G}'\}.$$

Let $\mathcal{H} = \{H_1, \ldots, H_m\}$. To each $H_i \in \mathcal{H}$, fix $G_i \in \mathcal{G}'$ satisfying $H(G_i) = H_i$. Now $H_i \cap G_j = \emptyset$ iff $i = j$, By (9), we obtain

$$|\mathcal{H}| = m \leq \binom{2k-1}{k}. \tag{24}$$

For each $H \in \mathcal{H}$, let

$$\mathcal{G}'(H) = \{G \in \mathcal{G}' \colon H(G) = H\}.$$

Claim 10. *For each $H \in \mathcal{H}$, $\mathcal{G}'(H)$ is 2-intersecting.*

Proof. Suppose that there exist $G_1, G_2 \in \mathcal{G}'(H)$ with $G_1 \cap G_2 = \{x\}$. Let $H = \{x_1, \ldots, x_k\}$. Since $G_1 \in \partial\mathcal{F}^*$, there is $F_1 = G_1 \cup \{x_i\}$ such that $F_1 \in \mathcal{F}^*$. By symmetry we assume that $i = 1$. By k-completeness, we have $(F_1 \setminus \{x\}) \cup \{y_p\} \in \mathcal{F}$ for $p = 1, \ldots, k$. Since $|\{y_1, \ldots, y_k\}| > |G_2|$, there exist $y_{p_0} \notin G_2$. Since $k \geq 3$, we may assume that $x_2 \neq y_{p_0}$. Then $G_1 \cup \{x_1, y_{p_0}\} \setminus \{x\}$, $G_2 \cup \{x_2\}$ are disjoint, a contradiction. □

Now we distinguish two cases.

Case 1. There exists $H \in \mathcal{H}$ such that $|\mathcal{G}'(H)| > k(k-1)\binom{n-4}{k-4}$.

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Since $\mathcal{G}'(H)$ is a 2-intersecting $(k-1)$-graph, by (11) $\mathcal{G}'(H)$ is a 2-star. So let $\mathcal{G}'(H)$ be a star with centre $\{1,2\}$. Let $H=\{x_1,\ldots,x_k\}$. Define

$$\mathcal{G}_i^H=\left\{G\in\mathcal{G}'(H):\ G\cup\{x_i\}\in\mathcal{F}^*\right\}.$$

Note that $\mathcal{G}'(H)=\mathcal{G}_1^H\cup\ldots\cup\mathcal{G}_k^H$. Without loss of generality, we may assume that $|\mathcal{G}_1^H|\geq\ldots\geq|\mathcal{G}_k^H|$.

Claim 11. $\mathcal{G}_2^H=\cdots=\mathcal{G}_k^H=\emptyset$.

Proof. Suppose for contradiction that $\mathcal{G}_2^H\neq\emptyset$ and let $R_2\in\mathcal{G}_2^H([2])$. Since

$$|\mathcal{G}_1^H([2])|\geq\frac{1}{k}|\mathcal{G}'(H)|\geq(k-1)\binom{n-4}{k-4},$$

we have

$$\begin{aligned}\left|\mathcal{G}_1^H([2])\cap\binom{[n]\setminus R_2}{k-3}\right|&\geq|\mathcal{G}_1^H([2])|-|R_2|\binom{n-k-3}{k-4}\\&\geq(k-1)\binom{n-4}{k-4}-(k-3)\binom{n-4}{k-4}\\&=2\binom{n-4}{k-4}.\end{aligned}$$

By (10) we have $\nu(\mathcal{G}_1^H([2])\cap\binom{[n]\setminus R_2}{k-3})\geq 2$. It follows that there are $R_0,R_1\in\mathcal{G}_1^H([2])$ such that R_0,R_1,R_2 are pairwise disjoint sets. Set $G_i=R_i\cup[2]$ for $i=0,1,2$. Since $G_1\cup\{x_1\},\ G_2\cup\{x_2\}\in\mathcal{F}^*$, we know that $E_1=R_1\cup\{1,x_1\}$, $E_2=R_2\cup\{2,x_2\}$ are both covered by k members of $\mathcal{F}$. To avoid disjointness, we have $E_1\cup\{y_2\}\in\mathcal{F}$ for each $y_2\in E_2$ and $E_2\cup\{y_1\}\in\mathcal{F}$ for each $y_1\in E_1$. Moreover, there is an extra element z such that $E_1\cup\{z\},\ E_2\cup\{z\}\in\mathcal{F}$.

Let $E_0=R_0\cup\{1,x_1\}$ and clearly $E_0\cap E_2=\emptyset$. Since $E_0\subset G_0\cup\{x_1\}\in\mathcal{F}^*$, E_0 is covered by k members of $\mathcal{F}$, we can find $w\notin E_2$ such that $E_0\cup\{w\}\in\mathcal{F}$. For $k\geq 5$ we may choose $u\in R_1$, $u\neq w$. Then $E_2\cup\{u\}\in\mathcal{F}$ and $E_0\cup\{w\},\ E_2\cup\{u\}$ are disjoint, a contradiction. □

Claim 11 implies $\mathcal{G}'(H)=\mathcal{G}_1^H$ and $G\cup\{x_1\}\in\mathcal{F}^*$ for all $G\in\mathcal{G}'(H)$. Then by Lemma 6.1 we may assume that $\nu(\mathcal{G}_1^H([2]))\leq k$. By (10),

$$|\mathcal{G}'(H)|\leq k\binom{n-4}{k-4},$$

contradicting our assumption.

Case 2. For each $H\in\mathcal{H}$, $|\mathcal{G}'(H)|\leq k(k-1)\binom{n-4}{k-4}$.

By the definition of $\mathcal{E}$, we infer that each E in $\mathcal{E}$ contains a $(k-1)$-set $G\in\mathcal{G}'$, implying $|\mathcal{E}|\leq|\mathcal{G}'|$. By Claim 9 and (24),

$$|\mathcal{E}|\leq|\mathcal{G}'|\leq\sum_{H\in\mathcal{H}}|\mathcal{G}'(H)|\leq\binom{2k-1}{k}k(k-1)\binom{n-4}{k-4}.$$

Define $\mathcal{F}_1=\mathcal{F}^*\setminus\mathcal{E}$. Note that each member of $\partial\mathcal{F}_1$ is a transversal of $\mathcal{F}$. Since $\mathcal{F}$ is saturated, by Lemma 5.1 (i) the family $\partial\mathcal{F}_1$ is intersecting. By Proposition 5.3, $\mathcal{F}_1$ is 3-intersecting. If $|\mathcal{F}_1|\leq k\binom{n-4}{k-4}$, then

$$|\mathcal{F}^*|=|\mathcal{E}|+|\mathcal{F}_1|\leq\left(\binom{2k-1}{k}k(k-1)+k\right)\binom{n-4}{k-4}\leq\binom{n-3}{k-3}.$$

Otherwise, by (11) we have $[3] \subset F$ for all $F \in \mathcal{F}_1$. Then by Lemma 6.1, we may assume $\nu(\mathcal{F}_1([3])) \leq k$ and (10) implies

$$|\mathcal{F}_1| \leq k\binom{n-4}{k-4},$$

which contradicts the assumption and the proposition is proven. □

Let $g(\nu, \Delta)$ be the maximum number of edges in a graph $\mathcal{G}$ with $\nu(\mathcal{G}) \leq \nu$ and the maximum degree at most Δ. To determine $f^*(n, 4, 4)$, we need the following result due to Chvátal and Hanson [2].

Lemma 6.4 ([2]). *For every $\nu \geq 1$ and $\Delta \geq 1$,*

$$g(\nu, \Delta) = \nu\Delta + \left\lfloor \frac{\Delta}{2} \right\rfloor \left\lfloor \frac{\nu}{\lceil \Delta/2 \rceil} \right\rfloor \leq \nu\Delta + \nu. \tag{25}$$

Proposition 6.5. $f^*(n, 4, 4) = n - 3$ *for* $n \geq 63$.

Proof. Let $\mathcal{F} \subset \binom{[n]}{4}$ be a saturated intersecting family. Let $\mathcal{F}^*$ be the family of 4-complete sets in $\mathcal{F}$.

Claim 12. *If there are $F_1, F_2 \in \mathcal{F}^*$ with $|F_1 \cap F_2| = 1$, then $|\mathcal{F}^*| \leq 35$.*

Proof. Without loss of generality, assume that $F_1 = (x_1, x_2, x_3, z)$ and $F_2 = (y_1, y_2, y_3, z)$. Using that (x_1, x_2, x_3) is 4-fold covered in $\mathcal{F}$ which is intersecting, the only possible extra 4-sets are $(x_1, x_2, x_3, y_i) \in \mathcal{F}$, $1 \leq i \leq 3$. Similarly for (y_1, y_2, y_3) we infer $(y_1, y_2, y_3, x_j) \in \mathcal{F}$, $1 \leq j \leq 3$. Now consider (x_1, x_2, z), it is disjoint to (y_1, y_2, y_3, x_3). Hence its covering sets are (x_1, x_2, x_3, z) and $(x_1, x_2, z, y_i) \in \mathcal{F}$, $1 \leq i \leq 3$. Arguing in the same way with $(x_1, x_3, z), (x_2, x_3, z), (y_1, y_2, z)$, etc, we infer that all sets $R \in \binom{F_1 \cup F_2}{4}$ with $z \in R$ are in $\mathcal{F}$. If $\mathcal{F} \subset \binom{F_1 \cup F_2}{4}$ then $|\mathcal{F}^*| \leq |\mathcal{F}| \leq \binom{7}{4} = 35$, we are done. Otherwise we infer $z \in F$ for all $F \in \mathcal{F} \setminus \binom{F_1 \cup F_2}{4}$. Using $F \cap (x_1, x_2, x_3, y_i) \neq \emptyset$ and $F \cap (y_1, y_2, y_3, x_j) \neq \emptyset$, we infer that such F are of form (x_j, y_i, z, w_t) with w_t in the outside of $F_1 \cup F_2$. Now the intersecting property implies that (x_j, y_i, w_t) cannot be 4-fold covered by $\mathcal{F}$. Thus $(x_j, y_i, z, w_t) \notin \mathcal{F}^*$. Hence $|\mathcal{F}^*| \leq \binom{7}{4} = 35$. □

By Claim 12 and $n \geq 38$, we may assume that $\mathcal{F}^*$ is 2-intersecting.

Claim 13. *$\mathcal{F}^*$ contains no sunflower with 3 petals and a centre of size 2.*

Proof. Assume that $(1, 2, 3, 4), (1, 2, 5, 6), (1, 2, 7, 8)$ form such a sunflower in $\mathcal{F}^*$. Since $(2, 3, 4), (1, 5, 6)$ are 4-fold covered in $\mathcal{F}$, we may assume that

$$(2, 3, 4, a), (2, 3, 4, b), (2, 3, 4, c), (1, 5, 6, p), (1, 5, 6, q), (1, 5, 6, r) \in \mathcal{F}.$$

The intersecting property of $\mathcal{F}$ implies that (by symmetry) $(a, b) = (5, 6)$, $(p, q) = (3, 4)$, $c = r$. Since $(1, 7, 8)$ is 4-fold covered in $\mathcal{F}$, let $(1, 7, 8, u), (1, 7, 8, v), (1, 7, 8, w) \in \mathcal{F}$. Then one of u, v, w is not in $\{5, 6\}$, by symmetry assume $w \notin \{5, 6\}$, it implies that $(2, 3, 4, a), (1, 7, 8, w)$ are disjoint, a contradiction. □

Let $F \in \mathcal{F}^*$. For any $P \in \binom{F}{2}$, define

$$\mathcal{G}(P) = \left\{F' \setminus P \colon P \subset F' \in \mathcal{F}^*\right\}.$$

By Claim 13, $\nu(\mathcal{G}(P)) \leq 2$. By Lemma 6.1, we may assume that $\mathcal{F}^*$ contains no sunflower with 5 petals and a centre of size 3. It follows that the maximum degree of $\mathcal{G}(P)$ is at most 4. Then (25)

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implies $|\mathcal{G}(P)| \le 2 * 4 + 2 = 10$. Since $\mathcal{F}^*$ is 2-intersecting, we conclude that for $n \ge 63$,

$$|\mathcal{F}^*| \le \sum_{P \in \binom{F}{2}} |\mathcal{G}(P)| \le \binom{4}{2} * 10 = 60 \le n - 3$$

and the proposition is proven. □

Theorem 1.8 follows from Theorem 3.1 and Propositions 5.4, 5.5, 6.2, 6.3, 6.5.

7. Concluding remarks

In this paper we considered intersecting k-graphs $\mathcal{F} \subset \binom{[n]}{k}$. For a positive parameter r we called an edge $F \in \mathcal{F}$ r-complete if all $G \in \binom{F}{k-1}$ were contained in at least r members of $\mathcal{F}$, including F. For $r = 1$ this condition is automatically satisfied. For $r \ge 2$ we defined $f(n, k, r)$ as the maximum of $|\mathcal{F}|$ for families with all edges being r-complete and $f^*(n, k, r)$ as the maximum number of r-complete edges in $\mathcal{F}$. The inequality $f(n, k, r) \le f^*(n, k, r)$ is obvious. For $2 \le r \le k$ all edges of the family

$$\mathcal{L}(n, k, r) = \left\{F \in \binom{[n]}{k} : |F \cap [2r-1]| \ge r\right\}$$

are r-complete. We showed that for $n \ge n_0(k, r)$, $f(n, k, r) = |\mathcal{L}(n, k, r)|$ and for $r = 2$ or 3 even $f^*(n, k, r)$ shares this value. However, for $r \ge 4$, $f^*(n, k, r)$ is much larger:

$$f^*(n, k, r) = (1 + o(1)) \binom{n-3}{k-3}.$$

In the case $r = 2$ we exploited some connections with the Erdős Matching Conjecture and succeeded in proving the statements with a linear constraint, $n \ge 28k$. However for $r \ge 3$ our proof requires $n_0(k, r) > k^{2(r+1)k}$.

Problem 7.1. *Does $f(n, k, r) = |\mathcal{L}(n, k, r)|$ hold for $3 \le r \le k$ and $n > ck$ with an absolute constant c?*

Another open problem is to determine the exact value of $f^*(n, k, r)$ for $4 \le r \le k - 1$ and $n > n_0(k, r)$.

As the analogous problems for t-intersecting families, we can define two more functions.

$$f(n, k, t, r) = \max\left\{|\mathcal{F}| : \mathcal{F} \subset \binom{[n]}{k} \text{ is } t\text{-intersecting and } r\text{-complete}\right\},$$

$$f^*(n, k, t, r) = \max\left\{|\mathcal{F}^*| : \begin{array}{l} \exists \mathcal{F} \subset \binom{[n]}{k} \text{ is } t\text{-intersecting}, \ \mathcal{F}^* \subset \mathcal{F}, \\ \mathcal{F}^* \text{ is relatively } r\text{-complete in } \mathcal{F} \end{array}\right\}.$$

Example 7.2. For $n \ge k \ge t \ge 1$ and $1 \le r \le k - t + 1$ define

$$\mathcal{A}(n, k, t, r) = \left\{A \in \binom{[n]}{k} : \ |F \cap [t + 2r - 2]| \ge t + r - 1\right\}.$$

By essentially the same proof as in Sections 3 and 4, one can obtain the following two results:

Theorem 7.3. *For $k \ge 3$, $r \ge 2$ and $n \ge n_0(k, r)$,*

$$f(n, k, t, r) = \begin{cases} |\mathcal{A}(n, k, t, r)|, & 2 \le r \le k - t + 1; \\ 0. & r \ge k - t + 2. \end{cases} \tag{26}$$

Theorem 7.4. *For $k \geq 3$, $r \geq 2$ and $n \geq n_0(k, r)$,*

$$f^*(n,k,t,r) = \begin{cases} |\mathcal{A}(n,k,t,r)|, & r = 2, 3; \\ \binom{n-t-2}{k-t-2} + O(n^{k-t-r+1}), & 4 \leq r \leq k-t+1; \\ \binom{n-t-2}{k-t-2}, & r \geq k-t+2. \end{cases} \tag{27}$$

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