Proceedings of the Edinburgh Mathematical Society (2015) **58**, 739–767
DOI:10.1017/S0013091514000315

CLASSIFICATION OF SYMMETRIC SPECIAL BISERIAL ALGEBRAS WITH AT MOST ONE NON-UNISERIAL INDECOMPOSABLE PROJECTIVE

NICOLE SNASHALL[1] AND RACHEL TAILLEFER[2]

[1]*Department of Mathematics, University of Leicester, University Road, Leicester LE1 7RH, UK* (njs5@le.ac.uk)
[2]*Laboratoire de Mathématiques, UMR 6620, Université Blaise Pascal, Complexe Universitaire des Cézeaux, 63171 Aubière, France* (rachel.taillefer@math.univ-bpclermont.fr)

(Received 15 April 2013)

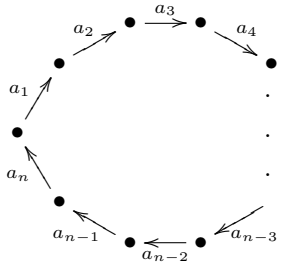
Abstract We consider a natural generalization of symmetric Nakayama algebras, namely, symmetric special biserial algebras with at most one non-uniserial indecomposable projective module. We describe the basic algebras explicitly by quiver and relations, then classify them up to derived equivalence and up to stable equivalence of Morita type. This includes the weakly symmetric algebras of Euclidean type $\tilde{\mathbb{A}}_n$, as studied by Bocian *et al.*, as well as some algebras of dihedral type.

Keywords: special biserial algebra; derived equivalence; stable equivalence of Morita type; generalized Brauer tree algebra

2010 *Mathematics subject classification:* Primary 16D50; 16E35
Secondary 16E40

Introduction

Let K be an algebraically closed field. In this paper we consider a generalization of symmetric Nakayama K-algebras. A symmetric Nakayama K-algebra is a symmetric K-algebra A such that all indecomposable projective modules are uniserial. These algebras are well known and have been classified up to Morita equivalence: every symmetric Nakayama algebra is Morita equivalent to exactly one algebra N_m^n defined by the quiver

$\Delta_n =$ 

and the ideal of relations L_m in $K\Delta_n$ generated by all paths of length $nm+1$. Note that in particular, the basic algebra associated with A is special biserial.

Our aim is to describe the basic indecomposable finite-dimensional K-algebras A that are symmetric special biserial algebras with at most one non-uniserial indecomposable projective module. These algebras include the symmetric Nakayama algebras, certain algebras in [**6**] that occur in the classification, up to derived equivalence, of all weakly symmetric algebras of Euclidean type, as well as some algebras of dihedral type (see [**7**]). In this paper we also distinguish, up to derived equivalence and up to stable equivalence of Morita type, the basic indecomposable finite-dimensional symmetric special biserial algebras that have at most one non-uniserial indecomposable projective module. It is well known that all special biserial algebras are tame [**27**]. Moreover, it was proved by Al-Nofayee [**1**] (and by Rickard [**23**] for the symmetric case) that if A and B are derived equivalent algebras, then A is self-injective if and only if B is self-injective. It was also proved by Pogorzały [**21**] that if A is a self-injective special biserial algebra that is not a Nakayama algebra and if A and B are stably equivalent of Morita type, then B is also a self-injective special biserial algebra. The algebras given in [**6**] are Brauer graph algebras and we recall that Brauer tree algebras play an important role in the Morita equivalence classification of blocks of group algebras of finite type (see [**2**, **4**]). We use the theory of generalized Brauer tree algebras as part of the classification of our algebras up to derived equivalence. We also refer the reader to [**25**], where Skowroński discusses the extensive programme to determine the derived equivalence classes of all tame self-injective algebras.

We begin this paper with some background and recall some properties of basic symmetric algebras so that, in § 2, we can describe by quiver and relations all basic indecomposable finite-dimensional algebras that are symmetric special biserial algebras with at most one non-uniserial indecomposable projective module. In order to distinguish our algebras up to derived equivalence and up to stable equivalence of Morita type, we use several invariants including Hochschild cohomology, which we discuss in § 3. The full classification of our algebras up to derived equivalence is contained in § 4 and, in addition to the dimensions of the Hochschild cohomology groups, we use Cartan invariants (see [**5**, Proposition 1.5] for a proof of derived invariance) and Külshammer invariants (or generalized Reynolds ideals, whose derived invariance was proved in [**29**]). The final section gives the full classification of our algebras up to stable equivalence of Morita type and the proof is based on the classification up to derived equivalence of § 4 and uses similar invariants.

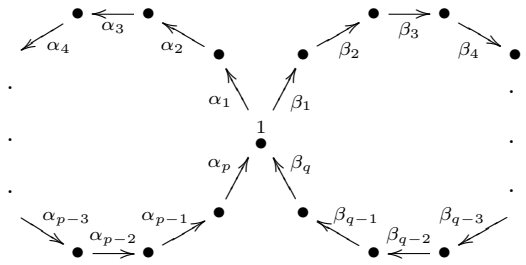
We assume throughout that A is a basic indecomposable finite-dimensional algebra over the algebraically closed field K so that A is isomorphic to $K\mathcal{Q}/I$ for some unique connected quiver $\mathcal{Q}$ and admissible ideal I of $K\mathcal{Q}$. We let $\mathrm{rad}(A)$ denote the Jacobson radical of A.

For any two positive integers p and q with $p \leqslant q$ we define the quiver $\mathcal{Q}_{(p,q)}$ to be the quiver formed of two oriented cycles, of lengths p and q, respectively, joined at one vertex

https://doi.org/10.1017/S0013091514000315 Published online by Cambridge University Press

labelled 1, as follows:

$Q_{(p,q)}$:
$1 \leqslant p \leqslant q$



We denote the trivial path at the vertex i by e_i. Paths are written from left to right. We write $\mathfrak{o}(\alpha)$ for the trivial path corresponding to the origin of the arrow α and write $\mathfrak{t}(\alpha)$ for the trivial path corresponding to the terminus of the arrow α. The vertices of the quiver $\mathcal{Q}_{(p,q)}$ are labelled by $1, \ldots, p+q-1$ in such a way that $\mathfrak{o}(\alpha_i) = i$ for $i = 1, \ldots, p$ and $\mathfrak{t}(\beta_j) = p + j$ for $j = 1, \ldots, q-1$. Thus, $\mathfrak{t}(\alpha_i) = i+1$ for $i = 1, \ldots, p-1$, $\mathfrak{t}(\alpha_p) = 1$, $\mathfrak{o}(\beta_1) = 1$ and $\mathfrak{o}(\beta_j) = p + j - 1$ for $j = 2, \ldots, q$.

Set $\gamma = \alpha_1\alpha_2\cdots\alpha_p$ and $\delta = \beta_1\beta_2\cdots\beta_q$. We define the following two admissible ideals in $K\mathcal{Q}_{(p,q)}$.

(a) For a positive integer r, let I_r be the ideal generated by

$$\begin{aligned} &\alpha_p\alpha_1, \qquad \beta_q\beta_1, \qquad (\gamma\delta)^r - (\delta\gamma)^r, \\ &\alpha_i\cdots\alpha_p(\delta\gamma)^{r-1}\delta\alpha_1\cdots\alpha_i \quad \text{for all } 2 \leqslant i \leqslant p-1, \\ &\beta_j\cdots\beta_q(\gamma\delta)^{r-1}\gamma\beta_1\cdots\beta_j \quad \text{for all } 2 \leqslant j \leqslant q-1. \end{aligned}$$

(b) For a pair of positive integers (s,t), let $J_{(s,t)}$ be the ideal generated by

$$\begin{aligned} &\alpha_p\beta_1, \qquad \beta_q\alpha_1, \qquad \gamma^s - \delta^t, \\ &\alpha_i\cdots\alpha_p\gamma^{s-1}\alpha_1\cdots\alpha_i \quad \text{for all } 2 \leqslant i \leqslant p-1, \\ &\beta_j\cdots\beta_q\delta^{t-1}\beta_1\cdots\beta_j \quad \text{for all } 2 \leqslant j \leqslant q-1, \end{aligned}$$

where if $p = 1$, then $s \geqslant 2$ and if $q = 1$, then $p = 1$, $s \geqslant 2$ and $t \geqslant 2$.

The algebras of type $\tilde{\mathbb{A}}_n$ considered in [**6**] are special cases of these algebras. Specifically, $A(p,q) = K\mathcal{Q}_{(p,q)}/I_1$ so that $r = 1$ and $\Lambda(n) = K\mathcal{Q}_{(1,n)}/J_{(2,2)}$ so that $p = 1$, $q = n$ and $s = 2 = t$. Moreover, some of these algebras are derived equivalent to algebras of dihedral type (see [**7**]) in the classification of Holm [**12**]: $K\mathcal{Q}_{(1,1)}/I_r$ is equal to $D(1\mathcal{A})^r_1$, $K\mathcal{Q}_{(1,2)}/I_r$ is derived equivalent to $D(2\mathcal{B})^{1,r}(0)$, and $K\mathcal{Q}_{(2,2)}/I_r$ is derived equivalent to $D(3\mathcal{K})^{r,1,1}$, all three of which come from tame blocks of finite groups when $\mathrm{char}(K) = 2$ and r is a power of 2, as well as $K\mathcal{Q}_{(2,2)}/J_{(s,t)}$ that is derived equivalent to $D(2\mathcal{R})^{1,s,t,1}$ and which does not come from blocks (see [**11**, **12**, **17**]).

https://doi.org/10.1017/S0013091514000315 Published online by Cambridge University Press

1. Background

The following result and especially its consequences will be used repeatedly. They are given in [**3**], but we include the proofs here for completeness.

Proposition 1.1. *Let $\mathcal{Q}$ be a quiver and let I be an admissible ideal in $K\mathcal{Q}$ such that $A = K\mathcal{Q}/I$ is a symmetric algebra. Let ρ be a path in $\mathcal{Q}$ with $\rho \neq 0$ in A. There then exists a cycle $\rho\rho_1$ in $\mathcal{Q}$ with $\rho\rho_1$ and $\rho_1\rho$ non-zero in A.*

Proof. Since A is a symmetric algebra, there exists a symmetric form $f\colon A \to K$ on A whose kernel contains no non-zero left or right ideals of A. Then ρA is not contained in $\operatorname{Ker} f$ so there exists a path ρ_1 such that $f(\rho\rho_1) \neq 0$. In particular, $\rho\rho_1 \neq 0$ and $\mathfrak{t}(\rho) = \mathfrak{o}(\rho_1)$. Moreover, since f is symmetric, $f(\rho_1\rho) = f(\rho\rho_1) \neq 0$ so $\rho_1\rho \neq 0$. Therefore, $\mathfrak{t}(\rho_1) = \mathfrak{o}(\rho)$. Hence, $\rho\rho_1$ and $\rho_1\rho$ are cycles in $\mathcal{Q}$ that are non-zero in A. □

Corollary 1.2. *Let $A = K\mathcal{Q}/I$ be an indecomposable finite-dimensional symmetric special biserial algebra that is not isomorphic to $N_1^1 \cong K[X]/(X^2)$. The following then hold.*

(1) *For any arrow α there is a unique arrow α' such that $\alpha'\alpha \neq 0$, and a unique arrow α'' such that $\alpha\alpha'' \neq 0$.*

(2) *For any vertex v in $\mathcal{Q}$ the number of arrows that start at v is equal to the number of arrows that end at v and this number is either* 1 *or* 2.

Proof. The second statement follows easily from the first and the definition of a special biserial algebra. Here we prove the first statement.

First suppose that α is an arrow that is not a loop. Then α is a non-zero path and so, by Proposition 1.1, there exists a path $\rho \neq \alpha$ such that $\alpha\rho$ and $\rho\alpha$ are non-zero cycles. Therefore, we can take α' to be the last arrow in ρ and α'' the first arrow in ρ. The uniqueness of these arrows follows from the definition of a special biserial algebra.

Now suppose that α is a loop. Assume for contradiction that $\alpha\beta = 0$ for every arrow β in $\mathcal{Q}$. Then α is in the socle of the indecomposable projective module $\mathfrak{o}(\alpha)A$. If no other arrow starts at $\mathfrak{o}(\alpha)$, then, since A is indecomposable and $\alpha^2 = 0$, we get $A \cong K[X]/(X^2)$, a contradiction. Therefore, there is another arrow ρ with $\mathfrak{o}(\alpha) = \mathfrak{o}(\rho)$. Choose a path σ that is maximal with the property $\rho\sigma \neq 0$ so that $\rho\sigma$ is in the socle of $\mathfrak{o}(\alpha)A$. Since A is a self-injective algebra, $\operatorname{soc}(\mathfrak{o}(\alpha)A)$ is one dimensional so that there exists a non-zero $c \in K$ such that $\alpha = c\rho\sigma$, which is a contradiction since the ideal I is admissible. Therefore, there exists an arrow α' with $\alpha'\alpha \neq 0$. The proof of the existence of the other arrow α'' is similar. □

2. Classification theorem

Our main theorem in this section is Theorem 2.2, in which we classify, by quiver and relations, all basic indecomposable finite-dimensional symmetric special biserial algebras with at most one non-uniserial indecomposable projective module.

https://doi.org/10.1017/S0013091514000315 Published online by Cambridge University Press

Proposition 2.1. *The algebras $K\mathcal{Q}_{(p,q)}/I_r$ and $K\mathcal{Q}_{(p,q)}/J_{(s,t)}$ are symmetric special biserial algebras with at most one non-uniserial indecomposable projective module.*

Proof. It is easy to see that these algebras are special biserial and that all but one of each of their indecomposable projective modules are uniserial. Moreover, the algebras are weakly symmetric, that is, the top and the socle of each indecomposable projective module are isomorphic. It remains to prove that the algebras are symmetric.

For $A = K\mathcal{Q}_{(p,q)}/I_r$, the socle of A has a K-basis consisting of

$$(\delta\gamma)^r, \quad \alpha_i \cdots \alpha_p (\delta\gamma)^{r-1} \alpha_1 \cdots \alpha_{i-1} \quad \text{and} \quad \beta_j \cdots \beta_q (\gamma\delta)^{r-1} \beta_1 \cdots \beta_{j-1}$$

for $2 \leqslant i \leqslant p$ and $2 \leqslant j \leqslant q$, that is, all the paths obtained from cyclic permutations of $(\gamma\delta)^r$, where we recall that $\gamma = \alpha_1\alpha_2\cdots\alpha_p$ and $\delta = \beta_1\beta_2\cdots\beta_q$. Complete this K-basis of $\operatorname{soc}(A)$ with paths in $K\mathcal{Q}_{(p,q)}$ to obtain a basis of A and define $f\colon A \to K$ on this basis by sending the elements in the socle to 1 and the others to 0. It then follows from [**13**, Proposition 3.1] that $\operatorname{Ker} f$ contains no non-zero left or right ideals of A. Moreover, f is clearly symmetric, since the paths on which it is non-zero are all the cyclic permutations of a single path. Thus, A is a symmetric algebra.

For $A = K\mathcal{Q}_{(p,q)}/I_{(s,t)}$ the argument is similar but this time the socle of A is generated by all the cyclic permutations of the two paths γ^s and δ^t. □

We now have the following result.

Theorem 2.2. *Let A be a basic indecomposable finite-dimensional symmetric special biserial algebra with at most one non-uniserial indecomposable projective module. Then A is isomorphic to a Nakayama algebra N_m^n for some positive integers m and n, or to $K\mathcal{Q}_{(p,q)}/I_r$ for some positive integers p, q and r, or to $K\mathcal{Q}_{(p,q)}/J_{(s,t)}$ for some positive integers p, q, s and t.*

Proof. Set $A = K\mathcal{Q}/I$ for some quiver $\mathcal{Q}$ and some admissible ideal I. It is already known that a symmetric (special biserial) algebra with no non-uniserial indecomposable projective module is isomorphic to a Nakayama algebra N_m^n. We may therefore assume that all except one of the indecomposable projective A-modules are uniserial. Consequently, using Corollary 1.2 (2), we must have one vertex that is the end point of exactly two arrows and the starting point of exactly two arrows, which we label 1, and the other vertices are the end point of exactly one arrow and the starting point of exactly one arrow. Therefore, the quiver of A must be $\mathcal{Q}_{(p,q)}$ for some positive integers p and q. Without loss of generality we may assume that $p \leqslant q$.

Now consider the composition $\alpha_p\alpha_1$. There are two cases: $\alpha_p\alpha_1 = 0$ and $\alpha_p\alpha_1 \neq 0$.

Case 1 ($\boldsymbol{\alpha_p\alpha_1 = 0}$). First assume that $\alpha_p\alpha_1 = 0$. Then it follows from Corollary 1.2 (1) that $\alpha_p\beta_1 \neq 0$, $\beta_q\alpha_1 \neq 0$ and $\beta_q\beta_1 = 0$. Now, for each vertex k with $k \neq 1$, e_kA is uniserial and, since A is symmetric, $\operatorname{top}(e_kA) \cong \operatorname{soc}(e_kA) \cong S_k$, the simple module at k. Therefore, for $i \neq 1$, we get a relation $\alpha_i \cdots \alpha_p (\delta\gamma)^{u_i} \delta\alpha_1 \cdots \alpha_i = 0$ with $\alpha_i \cdots \alpha_p (\delta\gamma)^{u_i} \delta\alpha_1 \cdots \alpha_{i-1} \neq 0$ for some integer u_i, and, for $j \neq 1$, we get a relation $\beta_j \cdots \beta_p (\gamma\delta)^{v_j} \gamma\beta_1 \cdots \beta_j = 0$ with $\beta_j \cdots \beta_p (\gamma\delta)^{v_j} \gamma\beta_1 \cdots \beta_{j-1} \neq 0$ for some integer v_j.

https://doi.org/10.1017/S0013091514000315 Published online by Cambridge University Press

Now consider e_1A. Since $\mathrm{rad}(e_1A) = \alpha_1 A + \beta_1 A$ and $\mathrm{soc}(e_1A) \cong S_1$, there is an element in $\mathrm{soc}(e_1A)$ of the form $(\gamma\delta)^r$ or $(\gamma\delta)^r\gamma$ for some integer r and there is an element of $\mathrm{soc}(e_1A)$ of the form $(\delta\gamma)^s$ or $(\delta\gamma)^s\delta$ for some integer s. But $\mathrm{soc}(e_1A)$ is simple so we must have a relation of one of the following forms.

(i) $(\gamma\delta)^r\gamma = c(\delta\gamma)^s \neq 0$ for some non-zero $c \in K$. Note that since I is admissible, we must have $s \geqslant 1$. But then, if $r \geqslant s$, we would have $(\gamma\delta)^r\gamma = \gamma(\delta\gamma)^s(\delta\gamma)^{r-s} = c^{-1}\gamma(\gamma\delta)^r\gamma(\delta\gamma)^{r-s} = 0$, which is a contradiction, and if $r < s$, we would have $(\delta\gamma)^s = \delta(\gamma\delta)^{s-1-r}(\gamma\delta)^r\gamma = c\delta(\gamma\delta)^{s-1-r}(\delta\gamma)^s = 0$, which is also a contradiction. Therefore, we cannot have this type of relation.

(ii) $(\gamma\delta)^r = c(\delta\gamma)^s\delta$ for some non-zero $c \in K$. As in the previous case, this relation cannot occur.

(iii) $(\gamma\delta)^r\gamma = c(\delta\gamma)^s\delta \neq 0$ for some non-zero $c \in K$. Then, if $r > s$, we have $(\gamma\delta)^r\gamma = \gamma(\delta\gamma)^r = \gamma(\delta\gamma)^s\delta(\gamma\delta)^{r-1-s}\gamma = c^{-1}\gamma(\gamma\delta)^r\gamma(\gamma\delta)^{r-1-s}\gamma = 0$, which is a contradiction, and if $s > r$, we have a similar contradiction. Therefore, $s = r$. Now, we also have $\alpha_2\cdots\alpha_p(\delta\gamma)^{u_2}\delta\alpha_1\alpha_2 = 0$ so, multiplying on the left by α_1 and on the right by $\alpha_3\cdots\alpha_p$, we get $\gamma(\delta\gamma)^{u_2+1} = 0$ so that $u_2 + 1 > r$. But $\alpha_2\cdots\alpha_p(\delta\gamma)^r\delta\alpha_1 = c^{-1}\alpha_2\cdots\alpha_p(\gamma\delta)^r\gamma\alpha_1 = 0$ so $u_2 < r$, which is impossible. Therefore, a relation of this form cannot occur either.

(iv) $(\gamma\delta)^r = c(\delta\gamma)^s$ for some non-zero $c \in K$ and this is the only possible type of relation. Here, again, if $r > s$, then $(\gamma\delta)^r = \gamma(\delta\gamma)^s(\delta\gamma)^{r-s-1}\delta = c^{-1}\gamma(\gamma\delta)^r(\delta\gamma)^{r-s-1}\delta = 0$, which is a contradiction and if $s > r$ we get a similar contradiction. Therefore, $r = s$ so that the relation is $(\gamma\delta)^r = c(\delta\gamma)^r$ for some $r \geqslant 1$ and $c \in K^*$.

Given this relation, we are now able to determine the u_i and the v_j. Since $\alpha_i\cdots\alpha_p(\delta\gamma)^r\delta\alpha_1\cdots\alpha_{i-1} = c^{-1}\alpha_i\cdots\alpha_p(\gamma\delta)^r\delta\alpha_1\cdots\alpha_{i-1} = 0$, we must have $r > u_i$. Moreover,

$$(\gamma\delta)^{u_i+2} = \alpha_1\cdots\alpha_{i-1}(\alpha_i\cdots\alpha_p(\delta\gamma)^{u_i}\delta\alpha_1\cdots\alpha_i)\alpha_{i+1}\cdots\alpha_p\delta = 0$$

so that $u_i + 2 > r$. Hence, $u_i = r - 1$ for all $i = 2, \ldots, p$. Similarly, $v_j = r - 1$ for all $j = 2, \ldots, q$. Moreover, we note that the relations $\alpha_p(\delta\gamma)^{r-1}\delta\alpha_1\cdots\alpha_p = 0$ (when $i = p$) and $\beta_p(\gamma\delta)^{r-1}\gamma\beta_1\cdots\beta_p = 0$ (when $j = q$) are superfluous and so are not required to give a minimal generating set of the ideal I_r.

Finally, we show that c must be equal to 1. Since A is symmetric, there exists a symmetric linear map $f\colon A \to K$ whose kernel does not contain any non-zero left or right ideal. In particular, the socle of A is not contained in $\mathrm{Ker}\, f$. But, from the relations obtained above, we see that the socle is generated as a K-vector space by all the paths obtained by cyclic permutations of $(\gamma\delta)^r$. Since f is symmetric, it follows that $f((\gamma\delta)^r) \neq 0$. But we have

$$f((\gamma\delta)^r) = f((\delta\gamma)^r) = f(c(\gamma\delta)^r) = cf((\gamma\delta)^r)$$

so that $c = 1$.

https://doi.org/10.1017/S0013091514000315 Published online by Cambridge University Press

We have hence shown that $A \cong K\mathcal{Q}_{(p,q)}/I_r$.

Case 2 ($\boldsymbol{\alpha_p\alpha_1 \neq 0}$). Now assume that $\alpha_p\alpha_1 \neq 0$. Then it follows from Corollary 1.2 (1) that $\alpha_p\beta_1 = 0$, $\beta_q\alpha_1 = 0$ and $\beta_q\beta_1 \neq 0$. The same methods as in the previous case show that we must have a relation of the form $\gamma^s = c\delta^t$ for some non-zero $c \in K$ and some positive integers s and t (by considering the structure of e_1A), and relations $\alpha_i \cdots \alpha_p\gamma^{s-1}\alpha_1 \cdots \alpha_i$ for all $2 \leqslant i \leqslant p$ and $\beta_j \cdots \beta_q\delta^{t-1}\beta_1 \cdots \beta_j$ for all $2 \leqslant j \leqslant q$ (from the structure of the other indecomposable projectives and using the relation $\gamma^s = c\delta^t$). Moreover, since K is algebraically closed, we may replace α_1 by $c'\alpha_1$, where c' is a tth root of c, and thus we may replace the relation $\gamma^s = c\delta^t$ by $\gamma^s = \delta^t$. Again, we may find a minimal set of relations and so conclude that $A \cong K\mathcal{Q}_{(p,q)}/J_{(s,t)}$. □

3. Hochschild cohomology groups

Our aim now is to investigate the derived equivalences among these algebras. It is well known that Hochschild cohomology is an invariant under derived equivalence and this section determines some of the Hochschild cohomology groups of the algebras $K\mathcal{Q}_{(p,q)}/I_r$ and $K\mathcal{Q}_{(p,q)}/J_{(s,t)}$ so that a full classification up to derived equivalence can be given in § 4.

Let $\Gamma(p,q;r) = K\mathcal{Q}_{(p,q)}/I_r$ and $\Lambda(p,q;s,t) = K\mathcal{Q}_{(p,q)}/J_{(s,t)}$. The special cases of the algebras $\Gamma(p,q;1)$ and $\Lambda(1,n;2,2)$ were considered in [**6**], where, in their notation, we have $A(p,q) = K\mathcal{Q}_{(p,q)}/I_1 = \Gamma(p,q;1)$ and $\Lambda(n) = K\mathcal{Q}_{(1,n)}/J_{(2,2)} = \Lambda(1,n;2,2)$.

We begin by describing $\mathrm{HH}^0(A)$ and $\mathrm{HH}^1(A)$ for the algebras $A = \Lambda(p,q;s,t)$ and $A = \Gamma(p,q;r)$. Recall that $\mathrm{HH}^0(A) = Z(A)$, the centre of the algebra A.

3.1. $\mathrm{HH}^0(A)$ and $\mathrm{HH}^1(A)$ for the algebra $A = \Lambda(p,q;s,t)$

We begin with the algebra $\Lambda(p,q;s,t) = K\mathcal{Q}_{(p,q)}/J_{(s,t)}$, where $1 \leqslant p \leqslant q$. Recall that $\gamma = \alpha_1\alpha_2 \cdots \alpha_p$ and $\delta = \beta_1\beta_2 \cdots \beta_q$. Let $\gamma_i = \alpha_i \cdots \alpha_p\alpha_1 \cdots \alpha_{i-1}$ for $i = 1, \ldots, p$ and let $\delta_j = \beta_j \cdots \beta_q\beta_1 \cdots \beta_{j-1}$ for $j = 1, \ldots, q$ so that $\gamma = \gamma_1$ and $\delta = \delta_1$.

Proposition 3.1. *Consider the algebra $\Lambda(p,q;s,t)$ and let $1 \leqslant p \leqslant q$. Let $x = \sum_{i=1}^{p}\gamma_i$ and $y = \sum_{j=1}^{q}\delta_j$. Then*

$$\dim \mathrm{HH}^0(\Lambda(p,q;s,t)) = p + q + s + t - 2$$

and the set

$$\{1, x, \ldots, x^{s-1}, y, \ldots, y^{t-1}, \gamma_i^s, \delta_j^t \text{ for } i = 1, \ldots, p \text{ and } j = 2, \ldots, q\}$$

is a K-basis of $\mathrm{HH}^0(\Lambda(p,q;s,t))$.

Proof. We note that $\gamma_i\beta_j = 0 = \beta_j\gamma_i$ and $\delta_j\alpha_i = 0 = \alpha_i\delta_j$ for all $i = 1, \ldots, p$, $j = 1, \ldots, q$ so $x, y \in Z(\Lambda(p,q;s,t))$. Moreover, $\Lambda(p,q;s,t)$ is weakly symmetric so all socle elements are central, namely, γ_i^s and δ_j^t are central for $i = 1, \ldots, p$, $j = 2, \ldots, q$. The result now follows. □

https://doi.org/10.1017/S0013091514000315 Published online by Cambridge University Press

We remark that Proposition 3.1 may be simplified if $p = q = 1$ since then $\Lambda(1,1;s,t)$ is the commutative algebra $K[\alpha,\beta]/(\alpha\beta, \alpha^s - \beta^t)$ with $s \geqslant 2$, $t \geqslant 2$. Thus, $\mathrm{HH}^0(\Lambda(1,1;s,t)) = \Lambda(1,1;s,t)$, which has a K-basis of $\{1, \alpha, \ldots, \alpha^s, \beta, \ldots, \beta^{t-1}\}$ and dimension $s+t$.

In order to compute the first Hochschild cohomology group, we now fix a minimal set of generators of the ideal $J_{(s,t)}$ and denote this set by g^2.

Proposition 3.2. *Consider the algebra $\Lambda(p,q;s,t)$ with $1 \leqslant p \leqslant q$.*

If $p \geqslant 2$, then the following elements form a minimal set of generators for the ideal $J_{(s,t)}$:

$$\begin{aligned} g_1^2 &= \gamma^s - \delta^t; \\ g_i^2 &= \alpha_i \cdots \alpha_p \gamma^{s-1} \alpha_1 \cdots \alpha_i \quad \text{for all } 2 \leqslant i \leqslant p-1; \\ g_p^2 &= \alpha_p \beta_1; \\ g_{p+1}^2 &= \beta_q \alpha_1; \\ g_{p+j}^2 &= \beta_j \cdots \beta_q \delta^{t-1} \beta_1 \cdots \beta_j \quad \text{for all } 2 \leqslant j \leqslant q-1. \end{aligned}$$

If $p = 1$, then the following elements form a minimal set of generators for the ideal $J_{(s,t)}$:

$$\begin{aligned} g_0^2 &= \alpha^s - \delta^t; \\ g_1^2 &= \alpha\beta_1; \\ g_2^2 &= \beta_q \alpha; \\ g_{j+1}^2 &= \beta_j \cdots \beta_q \delta^{t-1} \beta_1 \cdots \beta_j \quad \text{for all } 2 \leqslant j \leqslant q-1. \end{aligned}$$

We now compute the first Hochschild cohomology group $\mathrm{HH}^1(\Lambda(p,q;s,t))$. We use the explicit description of the start of a minimal projective bimodule resolution (P^*, d^*) for $\Lambda(p,q;s,t)$ as given in [**9**]. All tensors are over K so we write $\otimes$ for $\otimes_K$. For ease of notation, we write Λ for $\Lambda(p,q;s,t)$. Let

$$\begin{aligned} P^2 &= \bigoplus_k \Lambda \mathfrak{o}(g_k^2) \otimes \mathfrak{t}(g_k^2) \Lambda, \\ P^1 &= \bigoplus_{a \text{ arrow}} \Lambda \mathfrak{o}(a) \otimes \mathfrak{t}(a) \Lambda \end{aligned}$$

and

$$P^0 = \bigoplus_{v=1}^{p+q-1} \Lambda e_v \otimes e_v \Lambda.$$

Then the minimal projective bimodule resolution of Λ begins

$$\cdots \to P^2 \xrightarrow{d^2} P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} \Lambda \to 0$$

with the following maps. The map d^0 is the usual multiplication map. The map $d^1 \colon P^1 \to P^0$ is given by

$$d^1 \colon \mathfrak{o}(a) \otimes \mathfrak{t}(a) \mapsto \mathfrak{o}(a) \otimes a - a \otimes \mathfrak{t}(a),$$

https://doi.org/10.1017/S0013091514000315 Published online by Cambridge University Press

where the first term $\mathfrak{o}(a) \otimes a$ lies in the summand $\Lambda\mathfrak{o}(a) \otimes \mathfrak{o}(a)\Lambda$ and the second term $a \otimes \mathfrak{t}(a)$ lies in the summand $\Lambda\mathfrak{t}(a) \otimes \mathfrak{t}(a)\Lambda$. Now, each element of g^2 is a linear combination of paths in $K\mathcal{Q}_{(p,q)}$ so, for $x \in g^2$, we may write

$$x = \sum_{j=1}^{r} c_j a_{1j} \cdots a_{kj} \cdots a_{s_j j},$$

where $c_j \in K$ and the a_{kj} are arrows in $\mathcal{Q}_{(p,q)}$. With this notation for $x \in g^2$, the map $d^2 \colon P^2 \to P^1$ is given by

$$d^2 \colon \mathfrak{o}(x) \otimes \mathfrak{t}(x) \mapsto \sum_{j=1}^{r} c_j \sum_{k=1}^{s_j} a_{1j} \cdots a_{k-1j} \otimes a_{k+1j} \cdots a_{s_j j},$$

where the term $a_{1j} \cdots a_{k-1j} \otimes a_{k+1j} \cdots a_{s_j j}$ lies in the summand $\Lambda\mathfrak{o}(a_{kj}) \otimes \mathfrak{t}(a_{kj})\Lambda$ of P^1.

We now apply $\mathrm{Hom}_{\Lambda^e}(-,-)$ to this resolution, where $\Lambda^e = \Lambda \otimes \Lambda^{op}$ is the enveloping algebra of Λ. Let $\partial^1 \colon \mathrm{Hom}_{\Lambda^e}(P^1, \Lambda) \to \mathrm{Hom}_{\Lambda^e}(P^2, \Lambda)$ be the map induced by d^2 and let $\partial^0 \colon \mathrm{Hom}_{\Lambda^e}(P^0, \Lambda) \to \mathrm{Hom}_{\Lambda^e}(P^1, \Lambda)$ be the map induced by d^1. Then $\mathrm{HH}^1(\Lambda) = \mathrm{Ker}\,\partial^1 / \mathrm{Im}\,\partial^0$.

Proposition 3.3. *If $q \geqslant 2$, then*

$$\dim \mathrm{HH}^1(\Lambda(p,q;s,t)) = \begin{cases} s+t & \text{if } \mathrm{char}\, K \mid \gcd(s,t), \\ s+t-1 & \text{otherwise.} \end{cases}$$

If $q = 1$, then $p = 1$ and

$$\dim \mathrm{HH}^1(\Lambda(1,1;s,t)) = \begin{cases} s+t+1 & \text{if } \mathrm{char}\, K \mid \gcd(s,t), \\ s+t & \text{otherwise.} \end{cases}$$

Proof. There are three cases to consider.

Case 1 ($p \geqslant 2$). We start by calculating $\mathrm{Im}\,\partial^0$. Let $\varphi \in \mathrm{Hom}(P^0, \Lambda(p,q;s,t))$ so that $\partial^0(\varphi) = \varphi d^1$. Suppose that φ is given by

$$\begin{aligned} \varphi \colon \quad e_1 \otimes e_1 &\mapsto c_{1,0}e_1 + c_{1,1}\gamma_1 + \cdots + c_{1,s}\gamma_1^s + d_{1,1}\delta_1 + \cdots + d_{1,t-1}\delta_1^{t-1}, \\ e_i \otimes e_i &\mapsto c_{i,0}e_i + c_{i,1}\gamma_i + \cdots + c_{i,s}\gamma_i^s \qquad \text{for } i = 2, \ldots, p, \\ e_{p-1+i} \otimes e_{p-1+i} &\mapsto d_{i,0}e_{p-1+i} + d_{i,1}\delta_i + \cdots + d_{i,t}\delta_i^t \qquad \text{for } i = 2, \ldots, q, \end{aligned}$$

where $c_{i,j}, d_{i,j} \in K$.

We have

$$\begin{aligned} \varphi d^1(\mathfrak{o}(\alpha_1) \otimes \mathfrak{t}(\alpha_1)) &= \varphi(e_1 \otimes \alpha_1 - \alpha_1 \otimes e_2) \\ &= \varphi(e_1 \otimes e_1)\alpha_1 - \alpha_1\varphi(e_2 \otimes e_2) \\ &= (c_{1,0} - c_{2,0})\alpha_1 + (c_{1,1} - c_{2,1})\gamma_1\alpha_1 + \cdots + (c_{1,s-1} - c_{2,s-1})\gamma_1^{s-1}\alpha_1. \end{aligned}$$

https://doi.org/10.1017/S0013091514000315 Published online by Cambridge University Press

In a similar way we get

$$\begin{aligned}
\varphi d^1(\mathfrak{o}(\alpha_2)\otimes\mathfrak{t}(\alpha_2)) &= \varphi(e_2\otimes\alpha_2-\alpha_2\otimes e_3)\\
&= (c_{2,0}-c_{3,0})\alpha_2+(c_{2,1}-c_{3,1})\gamma_2\alpha_2+\cdots+(c_{2,s-1}-c_{3,s-1})\gamma_2^{s-1}\alpha_2,\\
&\vdots\\
\varphi d^1(\mathfrak{o}(\alpha_p)\otimes\mathfrak{t}(\alpha_p)) &= \varphi(e_p\otimes\alpha_p-\alpha_p\otimes e_1)\\
&= (c_{p,0}-c_{1,0})\alpha_p+(c_{p,1}-c_{1,1})\gamma_p\alpha_p+\cdots+(c_{p,s-1}-c_{1,s-1})\gamma_p^{s-1}\alpha_p,\\
\varphi d^1(\mathfrak{o}(\beta_1)\otimes\mathfrak{t}(\beta_1)) &= \varphi(e_1\otimes\beta_1-\beta_1\otimes e_{p+1})\\
&= (c_{1,0}-d_{2,0})\beta_1+(d_{1,1}-d_{2,1})\delta_1\beta_1+\cdots+(d_{1,t-1}-d_{2,t-1})\delta_1^{t-1}\beta_1,\\
&\vdots\\
\varphi d^1(\mathfrak{o}(\beta_q)\otimes\mathfrak{t}(\beta_q)) &= \varphi(e_{p+q-1}\otimes\beta_q-\beta_q\otimes e_1)\\
&= (d_{q,0}-c_{1,0})\beta_q+(d_{q,1}-d_{1,1})\delta_q\beta_q+\cdots+(d_{q,t-1}-d_{1,t-1})\delta_q^{t-1}\beta_q.
\end{aligned}$$

Thus, $\dim\operatorname{Im}\partial^0=(p-1)s+(q-1)t$.

Now let $\psi\in\operatorname{Ker}\partial^1$ so that $\psi d^2=0$ and suppose that $\psi\in\operatorname{Hom}(P^1,\Lambda(p,q;s,t))$ is given by

$$\begin{aligned}
\psi\colon \mathfrak{o}(\alpha_i)\otimes\mathfrak{t}(\alpha_i)&\mapsto c_{i,0}\alpha_i+c_{i,1}\gamma_i\alpha_i+\cdots+c_{i,s-1}\gamma_i^{s-1}\alpha_i,\\
\mathfrak{o}(\beta_j)\otimes\mathfrak{t}(\beta_j)&\mapsto d_{j,0}\beta_j+d_{j,1}\delta_j\beta_j+\cdots+d_{j,t-1}\delta_j^{t-1}\beta_j
\end{aligned}$$

for $i=1,\dots,p$, $j=1,\dots,q$ and where $c_{k,l},d_{k,l}\in K$.

From Proposition 3.2 with $p\geqslant 2$ and recalling that $\gamma=\gamma_1$ and $\delta=\delta_1$, it is easy to see that $\psi d^2(\mathfrak{o}(g_k^2)\otimes\mathfrak{t}(g_k^2))$ is immediately 0 for $k=2,\dots,p+q-1$ and so we do not get any restrictions on the constants in the cases where g_k^2 is a monomial. It remains to consider $g_1^2=\gamma^s-\delta^t$. The condition $\psi d^2(\mathfrak{o}(g_1^2)\otimes\mathfrak{t}(g_1^2))=0$ gives that

$$(s(c_{1,0}+c_{2,0}+\cdots+c_{p,0})-t(d_{1,0}+d_{2,0}+\cdots+d_{q,0}))\gamma^s=0$$

so that $s(c_{1,0}+c_{2,0}+\cdots+c_{p,0})-t(d_{1,0}+d_{2,0}+\cdots+d_{q,0})=0$. Hence,

$$\dim\operatorname{Ker}\partial^1=\begin{cases}ps+qt & \text{if } \operatorname{char}K\mid\gcd(s,t),\\ ps+qt-1 & \text{otherwise.}\end{cases}$$

Thus, for $p\geqslant 2$, we have

$$\dim\operatorname{HH}^1(\Lambda(p,q;s,t))=\begin{cases}s+t & \text{if } \operatorname{char}K\mid\gcd(s,t),\\ s+t-1 & \text{otherwise.}\end{cases}$$

https://doi.org/10.1017/S0013091514000315 Published online by Cambridge University Press

Case 2 ($p = 1$ and $q \geqslant 2$). To calculate $\operatorname{Im}\partial^0$, let $\varphi \in \operatorname{Hom}(P^0, \Lambda(1, q; s, t))$ so that $\partial^0(\varphi) = \varphi d^1$. Suppose that $\varphi \in \operatorname{Hom}(P^0, \Lambda(1, q; s, t))$ is given by

$$\begin{aligned}\varphi\colon e_1 \otimes e_1 &\mapsto c_{1,0}e_1 + c_{1,1}\alpha + \cdots + c_{1,s}\alpha^s + d_{1,1}\delta_1 + \cdots + d_{1,t-1}\delta_1^{t-1},\\ e_i \otimes e_i &\mapsto d_{i,0}e_i + d_{i,1}\delta_i + \cdots + d_{i,t}\delta_i^t\end{aligned}$$

for $i = 2, \ldots, q$ and where $c_{1,j}, d_{1,j}, d_{i,j} \in K$.

We have

$$\varphi d^1(\mathfrak{o}(\alpha) \otimes \mathfrak{t}(\alpha)) = \varphi(e_1 \otimes \alpha - \alpha \otimes e_1) = \varphi(e_1 \otimes e_1)\alpha - \alpha\varphi(e_1 \otimes e_1) = 0,$$

since $\alpha \in Z(\Lambda(1, q; s, t))$. Now,

$$\begin{aligned}\varphi d^1(\mathfrak{o}(\beta_1) \otimes \mathfrak{t}(\beta_1)) &= \varphi(e_1 \otimes \beta_1 - \beta_1 \otimes e_2)\\ &= (c_{1,0} - d_{2,0})\beta_1 + (d_{1,1} - d_{2,1})\delta_1\beta_1 + \cdots + (d_{1,t-1} - d_{2,t-1})\delta_1^{t-1}\beta_1,\\ &\vdots\\ \varphi d^1(\mathfrak{o}(\beta_q) \otimes \mathfrak{t}(\beta_q)) &= \varphi(e_q \otimes \beta_q - \beta_q \otimes e_1)\\ &= (d_{q,0} - c_{1,0})\beta_q + (d_{q,1} - d_{1,1})\delta_q\beta_q + \cdots + (d_{q,t-1} - d_{1,t-1})\delta_q^{t-1}\beta_q.\end{aligned}$$

Thus, $\dim \operatorname{Im}\partial^0 = (q-1)t$.

Now let $\psi \in \operatorname{Ker}\partial^1$ so that $\psi d^2 = 0$ and suppose that $\psi \in \operatorname{Hom}(P^1, \Lambda(1, q; s, t))$ is given by

$$\begin{aligned}\psi\colon \mathfrak{o}(\alpha) \otimes \mathfrak{t}(\alpha) &\mapsto c_{1,0}e_1 + c_{1,1}\alpha + \cdots + c_{1,s}\alpha^s + \tilde{c}_{1,1}\delta_1 + \cdots + \tilde{c}_{1,t-1}\delta_1^{t-1},\\ \mathfrak{o}(\beta_i) \otimes \mathfrak{t}(\beta_i) &\mapsto d_{i,0}\beta_i + d_{i,1}\delta_i\beta_i + \cdots + d_{i,t-1}\delta_i^{t-1}\beta_i\end{aligned}$$

for $i = 1, \ldots, q$ and where $c_{1,j}, \tilde{c}_{1,j}, d_{i,j} \in K$.

From Proposition 3.2, the minimal generating set for $J_{(s,t)}$ is

$$\{g_0^2 = \alpha^s - \delta^t, g_1^2 = \alpha\beta_1, g_2^2 = \beta_q\alpha,\ g_{j+1}^2 = \beta_j \cdots \beta_q\delta^{t-1}\beta_1 \cdots \beta_j \mid 2 \leqslant j \leqslant q-1\},$$

where we recall that $\delta = \delta_1$. Starting with g_1^2, the equation $\psi d^2(\mathfrak{o}(g_1^2) \otimes \mathfrak{t}(g_1^2)) = 0$ gives that

$$\begin{aligned}0 &= \psi(\mathfrak{o}(\alpha) \otimes \mathfrak{t}(\alpha))\beta_1 + \alpha\psi(\mathfrak{o}(\beta_1) \otimes \mathfrak{t}(\beta_1))\\ &= (c_{1,0}e_1 + \tilde{c}_{1,1}\delta_1 + \cdots + \tilde{c}_{1,t-1}\delta_1^{t-1})\beta_1.\end{aligned}$$

Hence, $c_{1,0} = \tilde{c}_{1,1} = \cdots = \tilde{c}_{1,t-1} = 0$. So we may immediately simplify our expression for ψ to

$$\begin{aligned}\psi\colon \mathfrak{o}(\alpha) \otimes \mathfrak{t}(\alpha) &\mapsto c_{1,1}\alpha + \cdots + c_{1,s}\alpha^s,\\ \mathfrak{o}(\beta_i) \otimes \mathfrak{t}(\beta_i) &\mapsto d_{i,0}\beta_i + d_{i,1}\delta_i\beta_i + \cdots + d_{i,t-1}\delta_i^{t-1}\beta_i\end{aligned}$$

for $i = 1, \ldots, q$. It then follows that $\psi d^2(\mathfrak{o}(g_j^2) \otimes \mathfrak{t}(g_j^2))$ is 0 for $j = 2, \ldots, q$ and so we do not get any restrictions on the constants here. Finally,

$$0 = \psi d^2(\mathfrak{o}(g_0^2) \otimes \mathfrak{t}(g_0^2)) = (sc_{1,1} - t(d_{1,0} + d_{2,0} + \cdots + d_{q,0}))\alpha^s$$

https://doi.org/10.1017/S0013091514000315 Published online by Cambridge University Press

so that $sc_{1,1} - t(d_{1,0} + d_{2,0} + \cdots + d_{q,0}) = 0$. Hence,

$$\dim \operatorname{Ker} \partial^1 = \begin{cases} s + qt & \text{if } \operatorname{char} K \mid \gcd(s,t), \\ s + qt - 1 & \text{otherwise.} \end{cases}$$

Thus, for $q \geqslant 2$, we have

$$\dim \mathrm{HH}^1(\Lambda(1,q;s,t)) = \begin{cases} s + t & \text{if } \operatorname{char} K \mid \gcd(s,t), \\ s + t - 1 & \text{otherwise.} \end{cases}$$

Case 3 ($p = 1 = q$). To calculate $\operatorname{Im} \partial^0$, let $\varphi \in \operatorname{Hom}(P^0, \Lambda(1,1;s,t))$ so that $\partial^0(\varphi) = \varphi d^1$. We have

$$\varphi d^1(\mathfrak{o}(\alpha) \otimes \mathfrak{t}(\alpha)) = \varphi(e_1 \otimes \alpha - \alpha \otimes e_1) = \varphi(e_1 \otimes e_1)\alpha - \alpha\varphi(e_1 \otimes e_1) = 0$$

since $\Lambda(1,1;s,t)$ is commutative. Similarly, $\varphi d^1(\mathfrak{o}(\beta) \otimes \mathfrak{t}(\beta)) = 0$. Thus, $\operatorname{Im} \partial^0 = (0)$. Hence, $\mathrm{HH}^1(\Lambda(1,1;s,t)) = \operatorname{Ker} \partial^1$. Let $\psi \in \operatorname{Ker} \partial^1$ so that $\psi d^2 = 0$ and suppose that $\psi \in \operatorname{Hom}(P^1, \Lambda(1,1;s,t))$ is given by

$$\begin{aligned} \psi\colon \mathfrak{o}(\alpha) \otimes \mathfrak{t}(\alpha) &\mapsto c_{1,0}e_1 + c_{1,1}\alpha + \cdots + c_{1,s}\alpha^s + d_{1,1}\beta + \cdots + d_{1,t-1}\beta^{t-1}, \\ \mathfrak{o}(\beta) \otimes \mathfrak{t}(\beta) &\mapsto c_{2,0}e_1 + c_{2,1}\alpha + \cdots + c_{2,s}\alpha^s + d_{2,1}\beta + \cdots + d_{2,t-1}\beta^{t-1}, \end{aligned}$$

where $c_{i,j}, d_{i,j} \in K$.

From Proposition 3.2, the minimal generating set for $J_{(s,t)}$ is $\{g_0^2 = \alpha^s - \beta^t, g_1^2 = \alpha\beta, g_2^2 = \beta\alpha\}$. Starting with g_1^2, the equation $\psi d^2(\mathfrak{o}(g_1^2) \otimes \mathfrak{t}(g_1^2)) = 0$ gives that

$$c_{1,0}\beta + d_{1,1}\beta^2 + \cdots + d_{1,t-2}\beta^{t-1} + c_{2,0}\alpha + c_{2,1}\alpha^2 + \cdots + c_{2,s-2}\alpha^{s-1} + (d_{1,t-1} + c_{2,s-1})\alpha^s = 0.$$

Hence, $c_{1,0} = d_{1,1} = \cdots = d_{1,t-2} = c_{2,0} = c_{2,1} = \cdots = c_{2,s-2} = 0$ and $d_{1,t-1} + c_{2,s-1} = 0$. So we may simplify our expression for ψ to

$$\begin{aligned} \psi\colon \mathfrak{o}(\alpha) \otimes \mathfrak{t}(\alpha) &\mapsto c_{1,1}\alpha + \cdots + c_{1,s}\alpha^s + d_{1,t-1}\beta^{t-1}, \\ \psi\colon \mathfrak{o}(\beta) \otimes \mathfrak{t}(\beta) &\mapsto -d_{1,t-1}\alpha^{s-1} + c_{2,s}\alpha^s + d_{2,1}\beta + \cdots + d_{2,t-1}\beta^{t-1}. \end{aligned}$$

We then have that

$$0 = \psi d^2(\mathfrak{o}(g_0^2) \otimes \mathfrak{t}(g_0^2)) = sc_{1,1}\alpha^s - td_{2,1}\beta^t = (sc_{1,1} - td_{2,1})\alpha^s$$

and hence $sc_{1,1} - td_{2,1} = 0$. The final equation $\psi d^2(\mathfrak{o}(g_0^2) \otimes \mathfrak{t}(g_0^2)) = 0$ gives no new information. Hence, $\psi \in \operatorname{Ker} \partial^1$ is given by

$$\begin{aligned} \psi\colon \mathfrak{o}(\alpha) \otimes \mathfrak{t}(\alpha) &\mapsto c_{1,1}\alpha + \cdots + c_{1,s}\alpha^s + d_{1,t-1}\beta^{t-1}, \\ \psi\colon \mathfrak{o}(\beta) \otimes \mathfrak{t}(\beta) &\mapsto -d_{1,t-1}\alpha^{s-1} + c_{2,s}\alpha^s + d_{2,1}\beta + \cdots + d_{2,t-1}\beta^{t-1} \end{aligned}$$

with the additional linear dependency that $sc_{1,1} - td_{2,1} = 0$. Therefore,

$$\dim \mathrm{HH}^1(\Lambda(1,1;s,t)) = \dim \operatorname{Ker} \partial^1 = \begin{cases} s + t + 1 & \text{if } \operatorname{char} K \mid \gcd(s,t), \\ s + t & \text{otherwise.} \end{cases}$$

This completes the proof. □

https://doi.org/10.1017/S0013091514000315 Published online by Cambridge University Press

3.2. $\mathbf{HH^0}(A)$ and $\mathbf{HH^1}(A)$ for the algebras $A = \Gamma(p, q; r)$

We now turn to the algebras $\Gamma(p, q; r) = K\mathcal{Q}_{(p,q)}/I_r$. Set $\eta_i = \alpha_i \cdots \alpha_p \delta \alpha_1 \cdots \alpha_{i-1}$ for $1 \leqslant i \leqslant p$ (so that $\eta_1 = \gamma\delta$) and set $\theta_j = \beta_j \cdots \beta_q \gamma \beta_1 \cdots \beta_{j-1}$ for $1 \leqslant j \leqslant q$ (so that $\theta_1 = \delta\gamma$).

Proposition 3.4. *Let $p \leqslant q$ be positive integers and consider the algebra $\Gamma(p, q; r)$. Set*

$$z = \sum_{i=1}^{p} \eta_i + \sum_{j=1}^{q} \theta_j.$$

Then

$$\dim \mathrm{HH}^0(\Gamma(p, q; r)) = \begin{cases} p + q + r - 1 & \text{if } p > 1, \\ q + r + 1 & \text{if } p = 1 \text{ and } q > 1, \\ r + 3 & \text{if } p = 1 = q \end{cases}$$

and a basis for $\mathrm{HH}^0(\Gamma(p, q; r))$ is given by

$$\begin{aligned} &\{1; (\gamma\delta)^r; z^k,\ 1 \leqslant k \leqslant r-1; \eta_i^r; \theta_j^r \text{ for } 2 \leqslant i \leqslant p \text{ and } 2 \leqslant j \leqslant q\} && \text{if } p > 1, \\ &\{1; (\gamma\delta)^r; z^k,\ 1 \leqslant k \leqslant r-1; (\delta\gamma)^{r-1}\delta; \theta_j^r \text{ for } 2 \leqslant j \leqslant q\} && \text{if } p = 1 \text{ and } q > 1, \\ &\{1; (\gamma\delta)^r; z^k,\ 1 \leqslant k \leqslant r-1; (\gamma\delta)^{r-1}\gamma; (\delta\gamma)^{r-1}\delta\} && \text{if } p = 1 = q. \end{aligned}$$

Proof. It is clear that $(\gamma\delta)^r$, η_i^r and θ_j^r are in the centre of $\Gamma(p, q; r)$ since they are socle elements and it is easy to check that $(\gamma\delta)^{r-1}\gamma$ and $(\delta\gamma)^{r-1}\delta$ are in the centre in the appropriate cases.

Conversely, a central element ζ must be in $\bigoplus_{v=1}^{p+q-1} e_v \Gamma(p, q; r) e_v$ and therefore is a linear combination of $(\gamma\delta)^k$ for $0 \leqslant k \leqslant r$, $(\delta\gamma)^k$ for $1 \leqslant k \leqslant r-1$, $(\gamma\delta)^k\gamma$ and $(\delta\gamma)^k\delta$ for $0 \leqslant k \leqslant r-1$, and η_i^k and θ_j^k for $2 \leqslant i \leqslant p$, $2 \leqslant j \leqslant q$ and $1 \leqslant k \leqslant r$. Writing the equations $\alpha_i\zeta = \zeta\alpha_i$ and $\beta_j\zeta = \zeta\beta_j$ gives the result (noting that $\eta_i^k = \alpha_i \cdots \alpha_p (\delta\gamma)^{k-1} \delta \alpha_1 \cdots \alpha_{i-1}$ and $\theta_j^k = \beta_j \cdots \beta_q (\gamma\delta)^{k-1} \gamma \beta_1 \cdots \beta_{j-1}$). □

We now use the same method as for $\Lambda(p, q; s, t)$ to compute $\mathrm{HH}^1(\Gamma(p, q; r))$, starting with a minimal set g^2 of generators of I_r.

Proposition 3.5. *Consider the algebra $\Gamma(p, q; r)$ with $1 \leqslant p \leqslant q$.*
If $p \geqslant 2$, then the following elements form a minimal set of generators for the ideal I_r:

$$\begin{aligned} g_1^2 &= (\gamma\delta)^r - (\delta\gamma)^r; \\ g_i^2 &= \eta_i^r \alpha_i \quad \text{for all } 2 \leqslant i \leqslant p-1; \\ g_p^2 &= \alpha_p \alpha_1; \\ g_{p+1}^2 &= \beta_q \beta_1; \\ g_{p+j}^2 &= \theta_j^r \beta_j \quad \text{for all } 2 \leqslant j \leqslant q-1. \end{aligned}$$

https://doi.org/10.1017/S0013091514000315 Published online by Cambridge University Press

If $p = 1$, then the following elements form a minimal set of generators for the ideal I_r:

$$\begin{aligned} g_0^2 &= (\gamma\delta)^r - (\delta\gamma)^r; \\ g_1^2 &= \alpha^2; \\ g_2^2 &= \beta_q\beta_1; \\ g_{j+1}^2 &= \theta_j^r\beta_j \quad \text{for all } \ 2 \leqslant j \leqslant q-1. \end{aligned}$$

The proof of the next result giving the first Hochschild cohomology group is a similar calculation to that of Proposition 3.3 and so is left to the reader.

Proposition 3.6. *If $p \geqslant 2$, then*

$$\dim \mathrm{HH}^1(\Gamma(p,q;r)) = r+1.$$

If $p = 1$ and $q \geqslant 2$, then

$$\dim \mathrm{HH}^1(\Gamma(1,q;r)) = \begin{cases} r+4 & \text{if } \operatorname{char} K = 2, \\ r+2 & \text{if } \operatorname{char} K \neq 2. \end{cases}$$

If $q = 1$, then $p = 1$ and

$$\dim \mathrm{HH}^1(\Gamma(1,1;r)) = \begin{cases} 2r+6 & \text{if } \operatorname{char} K = 2, \\ 2r+2 & \text{if } \operatorname{char} K \neq 2. \end{cases}$$

3.3. Higher Hochschild cohomology groups for $\Gamma(p,q;r)$

In order to distinguish the algebras of the form $\Gamma(p,q;r)$ up to derived equivalence we need the dimensions of the Hochschild cohomology groups up to $\mathrm{HH}^{2p-2}(\Gamma(p,q;r))$. If $p = 1$, this is just $\mathrm{HH}^0(\Gamma(1,q;r))$, which we already know, so we shall assume that $p > 1$ in the remainder of this section. We begin by giving the start of a projective bimodule resolution of $\Gamma(p,q;r)$ to enable us to find these groups. For ease of notation, set $\Gamma = \Gamma(p,q;r)$.

The projective bimodules P^n in a minimal projective bimodule resolution of Γ are known from [**10**]; specifically, the multiplicity of $\Gamma e_i \otimes e_j\Gamma$ as a direct summand of P^n is equal to the dimension of $\mathrm{Ext}^n_\Gamma(S_i, S_j)$, where S_k is the simple module at the vertex k. We thus define projective Γ–Γ-bimodules (equivalently Γ^e-modules) $P^0, P^1, \ldots, P^{2p}$ that will be the projectives in our minimal projective bimodule resolution for Γ.

Definition 3.7. Let $\Gamma = \Gamma(p,q;r)$ with $p > 1$. We define projective Γ–Γ-bimodules $P^0, P^1, \ldots, P^{2p}$ as follows:

$$\begin{aligned} P^0 &= \bigoplus_{i=1}^{p+q-1} \Gamma e_i \otimes e_i\Gamma; \\ P^1 &= \bigoplus_{i=1}^{p-1} \Gamma e_i \otimes e_{i+1}\Gamma \oplus \Gamma e_p \otimes e_1\Gamma \\ &\quad \oplus \bigoplus_{j=2}^{q-1} \Gamma e_{p+j-1} \otimes e_{p+j}\Gamma \oplus \Gamma e_{p+q-1} \otimes e_1\Gamma \oplus \Gamma e_1 \otimes e_{p+1}\Gamma; \end{aligned}$$

https://doi.org/10.1017/S0013091514000315 Published online by Cambridge University Press

$$P^{2n} = \bigoplus_{i=2}^{p-n} \Gamma e_i \otimes e_{i+n}\Gamma \oplus \bigoplus_{i=p-n+1}^{p-1} \Gamma e_i \otimes e_{i+n+1-p}\Gamma \oplus \Gamma e_p \otimes e_{n+1}\Gamma$$
$$\oplus \bigoplus_{j=2}^{q-n} \Gamma e_{p+j-1} \otimes e_{p+j+n-1}\Gamma$$
$$\oplus \bigoplus_{j=q-n+1}^{q-1} \Gamma e_{p+j-1} \otimes e_{p+j+n-q}\Gamma \oplus \Gamma e_{p+q-1} \otimes e_{p+n}\Gamma \oplus \Gamma e_1 \otimes e_1\Gamma$$
$$\text{for } 1 \leqslant n < p;$$

$$P^{2n-1} = \bigoplus_{i=2}^{p-n} \Gamma e_i \otimes e_{i+n}\Gamma \oplus \bigoplus_{i=p-n+1}^{p-1} \Gamma e_i \otimes e_{i+n-p}\Gamma \oplus \Gamma e_p \otimes e_n\Gamma \oplus \Gamma e_1 \otimes e_{n+1}\Gamma$$
$$\oplus \bigoplus_{j=2}^{q-n} \Gamma e_{p+j-1} \otimes e_{p+j+n-1}\Gamma$$
$$\oplus \bigoplus_{j=q-n+2}^{q-1} \Gamma e_{p+j-1} \otimes e_{p+j+n-q-1}\Gamma \oplus \Gamma e_{p+q-1}$$
$$\otimes e_{p+n-1}\Gamma \oplus \Gamma e_1 \otimes e_{p+n}\Gamma \oplus \Gamma e_{p+q-n} \otimes e_1\Gamma \quad \text{for } 2 \leqslant n < p;$$

$$P^{2p-1} = \bigoplus_{i=1}^{p} \Gamma e_i \otimes e_i\Gamma \oplus \bigoplus_{j=2}^{q-p} \Gamma e_{p+j-1} \otimes e_{2p+j-1}\Gamma$$
$$\oplus \bigoplus_{j=q-p+2}^{q} \Gamma e_{p+j-1} \otimes e_{2p+j-q-1}\Gamma \oplus \Gamma e_q \otimes e_1\Gamma \oplus \Gamma e_1 \otimes e_{2p}\Gamma;$$

$$P^{2p} = \bigoplus_{i=1}^{p-1} \Gamma e_i \otimes e_{i+1}\Gamma \oplus \bigoplus_{j=2}^{q-p} \Gamma e_{p+j-1} \otimes e_{2p+j-1}\Gamma$$
$$\oplus \bigoplus_{j=q-p+1}^{q} \Gamma e_{p+j-1} \otimes e_{2p+j-q}\Gamma \oplus \Gamma e_p \otimes e_1\Gamma \oplus \Gamma e_1 \otimes e_1\Gamma.$$

The first maps

$$d^i \colon P^i \to P^{i-1} \quad \text{for } i = 1, 2, 3$$

of a minimal projective bimodule resolution are given in [**9**]. For our algebra Γ, we extend the resolution in [**9**] in a similar way to [**26**] to make the following definition of maps

$$d^i \colon P^i \to P^{i-1} \quad \text{for } i = 1, \ldots, 2p.$$

Definition 3.8. Let $\Gamma = \Gamma(p, q; r)$ with $p > 1$. We define Γ–Γ-bimodule homomorphisms $d^i \colon P^i \to P^{i-1}$ for $i = 1, \ldots, 2p$ as follows.

$\triangleright$ $d^{2n-1} \colon P^{2n-1} \to P^{2n-2}$ for $1 \leqslant n < p$:

- $e_i \otimes e_{i+n} \mapsto e_i \otimes \alpha_{i+n-1} - \alpha_i \otimes e_{i+n}$ for $2 \leqslant i \leqslant p - n$,

https://doi.org/10.1017/S0013091514000315 Published online by Cambridge University Press

- $e_i \otimes e_{i+n-p} \mapsto$
$$(-1)^{i+p+1}\Big[e_i \otimes \eta^r_{i+n-p} + (-1)^{(p-i)(n-1)}\alpha_i \cdots \alpha_p \otimes \alpha_1 \cdots \alpha_{n-p+i-1} - \eta^r_i \otimes e_{i+n-p}$$
$$- (-1)^n(-1)^{(p-i)(n-1)} \sum_{m=1}^{p-i} (-1)^{m(n-1)} \alpha_i \cdots \alpha_{p-m} \otimes \alpha_{n-m+1} \cdots \alpha_p \delta(\gamma\delta)^{r-1}\alpha_1 \cdots \alpha_{n-p+i-1}$$
$$+ (-1)^n(-1)^{(p-i)(n-1)} \sum_{m=p-i+2}^{n-1} (-1)^{m(n-1)} \alpha_i \cdots \alpha_p (\delta\gamma)^{r-1}\delta\alpha_1 \cdots \alpha_{p-m} \otimes \alpha_{n-m+1} \cdots \alpha_{n-p+i-1}\Big] \quad \text{for } p-n+2 \leqslant i \leqslant p,$$

- $e_{p-n+1} \otimes e_1 \mapsto e_{p-n+1} \otimes \alpha_p$
$$+ (-1)^n \sum_{m=1}^{n-1} (-1)^{m(n-1)} \alpha_{p-n+1} \cdots \alpha_{p-m} \otimes \alpha_{n-m+1} \cdots \alpha_p \delta(\gamma\delta)^{r-1} - \alpha_{p-n+1} \cdots \alpha_p \otimes e_1,$$

- $e_1 \otimes e_{n+1} \mapsto (-1)^{n-1}\Big[e_1 \otimes \alpha_1 \cdots \alpha_n + (-1)^n \sum_{m=1}^{n-1} (-1)^{m(n-1)} \delta(\gamma\delta)^{r-1}\alpha_1 \cdots \alpha_{p-m} \otimes \alpha_{n-m+1} \cdots \alpha_n + (-1)^n \alpha_1 \otimes e_{n+1}\Big],$

- $e_{p+j-1} \otimes e_{p+j+n-1} \mapsto -e_{p+j-1} \otimes \beta_{j+n-1} + \beta_j \otimes e_{p+j+n-1} \quad \text{for } 2 \leqslant j \leqslant q-n,$

- $e_{p+j-1} \otimes e_{p+j+n-q-1} \mapsto$
$$(-1)^{j+q}\Big[e_{p+j-1} \otimes \theta^r_{j+n-q} + (-1)^{(q-j)(n-1)}\beta_j \cdots \beta_q \otimes \beta_1 \cdots \beta_{n+j-q-1} - \theta^r_j \otimes e_{p+j+n-q-1}$$
$$- (-1)^n(-1)^{(q-j)(n-1)} \sum_{m=1}^{q-j} (-1)^{m(n-1)} \beta_j \cdots \beta_{q-m} \otimes \beta_{n-m+1} \cdots \beta_q (\gamma\delta)^{r-1}\gamma\beta_1 \cdots \beta_{n-q+j-1}$$
$$+ (-1)^n(-1)^{(q-j)(n-1)} \sum_{m=q-j+2}^{n-1} (-1)^{m(n-1)} \beta_j \cdots \beta_q \gamma(\delta\gamma)^{r-1}\beta_1 \cdots \beta_{q-m} \otimes \beta_{n-m+1} \cdots \beta_{n-q+j-1}\Big] \quad \text{for } q-n+2 \leqslant j \leqslant q,$$

- $e_{p+q-n} \otimes e_1 \mapsto -\Big[e_{p+q-n} \otimes \beta_q + (-1)^n \sum_{m=1}^{n-1} (-1)^{m(n-1)} \beta_{q-n+1} \cdots \beta_{q-m} \otimes \beta_{n-m+1} \cdots \beta_q (\gamma\delta)^{r-1}\gamma - \beta_{q-n+1} \cdots \beta_q \otimes e_1\Big],$

- $e_1 \otimes e_{p+n} \mapsto (-1)^n\Big[e_1 \otimes \beta_1 \cdots \beta_n + (-1)^n \sum_{m=1}^{n-1} (-1)^{m(n-1)} (\gamma\delta)^{r-1}\gamma\beta_1 \cdots \beta_{q-m} \otimes \beta_{n-m+1} \cdots \beta_n + (-1)^n \beta_1 \otimes e_{p+n}\Big].$

$\triangleright\ d^{2n} \colon P^{2n} \to P^{2n-1}$ for $1 \leqslant n < p$:

- $e_i \otimes e_{i+n} \mapsto \sum_{k=0}^{r} \eta^k_i \otimes \eta^{r-k}_{i+n}$
$$+ \sum_{k=0}^{r-1}\Big[\sum_{m=0}^{p-i-n} \eta^k_i \alpha_i \cdots \alpha_{i+m} \otimes \alpha_{i+m+n+1} \cdots \alpha_p \delta\alpha_1 \cdots \alpha_{i+n-1}\eta^{r-k-1}_{i+n}$$

https://doi.org/10.1017/S0013091514000315 Published online by Cambridge University Press

$$+\sum_{m=0}^{i-2}\eta_i^k\alpha_i\cdots\alpha_p\delta\alpha_1\cdots\alpha_m\otimes\alpha_{m+n+1}\cdots\alpha_{i+n-1}\eta_{i+n}^{r-k-1}$$
$$+(-1)^n\sum_{j=0}^{q-n}\eta_i^k\alpha_i\cdots\alpha_p\beta_1\cdots\beta_j\otimes\beta_{j+n+1}\cdots\beta_q\alpha_1\cdots\alpha_{i+n-1}\eta_{i+n}^{r-k-1}\Big] \quad \text{for } 2\leqslant i\leqslant p-n,$$

- $e_i\otimes e_{i+n+1-p}\mapsto e_i\otimes\alpha_{i+n-p}-(-1)^n\alpha_i\otimes e_{i+n+1-p}$ for $p+1-n\leqslant i\leqslant p$,
- $$e_1\otimes e_1\mapsto\sum_{k=0}^{r-1}\Big[\sum_{i=0}^{p-n}(\delta(\gamma\delta)^k\alpha_1\cdots\alpha_i\otimes\alpha_{i+n+1}\cdots\alpha_p(\delta\gamma)^{r-k-1}$$
$$+(-1)^n(\gamma\delta)^k\alpha_1\cdots\alpha_i\otimes\alpha_{i+n+1}\cdots\alpha_p(\delta\gamma)^{r-k-1}\delta)$$
$$+\sum_{j=0}^{q-n}(\gamma(\delta\gamma)^k\beta_1\cdots\beta_j\otimes\beta_{j+n+1}\cdots\beta_q(\gamma\delta)^{r-k-1}$$
$$+(-1)^n(\delta\gamma)^k\beta_1\cdots\beta_j\otimes\beta_{j+n+1}\cdots\beta_q(\gamma\delta)^{r-k-1}\gamma)\Big],$$
- $e_{p+j-1}\otimes e_{p+j+n-q}\mapsto e_{p+j-1}\otimes\beta_{j+n-q}-(-1)^n\beta_j\otimes e_{p+j+n-q}$ for $q+1-n\leqslant j\leqslant q$,
- $$e_{p+j-1}\otimes e_{p+j+n-1}\mapsto\sum_{k=0}^{r}\theta_j^k\otimes\theta_{j+n}^{r-k}$$
$$+\sum_{k=0}^{r-1}\Big[\sum_{m=0}^{q-j-n}\theta_j^k\beta_j\cdots\beta_{j+m}\otimes\beta_{j+m+n+1}\cdots\beta_q\gamma\beta_1\cdots\beta_{j+n-1}\theta_{j+n}^{r-k-1}$$
$$+\sum_{m=0}^{j-2}\theta_j^k\beta_j\cdots\beta_q\gamma\beta_1\cdots\beta_m\otimes\beta_{m+n+1}\cdots\beta_{j+n-1}\theta_{j+n}^{r-k-1}$$
$$+(-1)^n\sum_{i=0}^{p-n}\theta_j^k\beta_j\cdots\beta_q\alpha_1\cdots\alpha_i\otimes\alpha_{i+n+1}\cdots\alpha_p\beta_1\cdots\beta_{j+n-1}\theta_{j+n}^{r-k-1}\Big] \quad \text{for } 2\leqslant j\leqslant q-n.$$

▷ $d^{2p-1}\colon P^{2p-1}\to P^{2p-2}$:

- $$e_i\otimes e_i\mapsto$$
$$(-1)^i\Big[e_i\otimes\eta_i^r+(-1)^{(p-i)(p-1)}\alpha_i\cdots\alpha_p\otimes\alpha_1\cdots\alpha_{i-1}-\eta_i^r\otimes e_i$$
$$-(-1)^p(-1)^{(p-i)(p-1)}\sum_{m=1}^{p-i}(-1)^{m(p-1)}\alpha_i\cdots\alpha_{p-m}\otimes\alpha_{p-m+1}\cdots\alpha_p\delta(\gamma\delta)^{r-1}\alpha_1\cdots\alpha_{i-1}$$
$$+(-1)^p(-1)^{(p-i)(p-1)}\sum_{m=p-i+2}^{p-1}(-1)^{m(p-1)}\alpha_i\cdots\alpha_p(\delta\gamma)^{r-1}\delta\alpha_1\cdots\alpha_{p-m}\otimes\alpha_{p-m+1}\cdots\alpha_{i-1}\Big]$$
for $2\leqslant i\leqslant p$,
- $$e_1\otimes e_1\mapsto e_1\otimes\gamma-\sum_{m=1}^{p-1}(-1)^{m(p-1)}\alpha_1\cdots\alpha_{p-m}\otimes\alpha_{p-m+1}\cdots\alpha_p\delta(\gamma\delta)^{r-1}$$
$$+(-1)^p\sum_{m=1}^{p-1}(-1)^{m(p-1)}\delta(\gamma\delta)^{r-1}\alpha_1\cdots\alpha_{p-m}\otimes\alpha_{p-m+1}\cdots\alpha_p+(-1)^p\gamma\otimes e_1,$$
- $e_{p+j-1}\otimes e_{2p+j-1}\mapsto -e_{p+j-1}\otimes\beta_{p+j-1}+\beta_j\otimes e_{2p+j-1}$ for $2\leqslant j\leqslant q-p$,
- $$e_{p+j-1}\otimes e_{2p+j-q-1}\mapsto$$
$$(-1)^{j+q}\Big[e_{p+j-1}\otimes\theta_{p+j-q}^r+(-1)^{(q-j)(p-1)}\beta_j\cdots\beta_q\otimes\beta_1\cdots\beta_{p+j-q-1}-\theta_j^r\otimes e_{2p+j-q-1}$$

https://doi.org/10.1017/S0013091514000315 Published online by Cambridge University Press

$$- (-1)^p(-1)^{(q-j)(p-1)} \sum_{m=1}^{q-j} (-1)^{m(p-1)} \beta_j \cdots \beta_{q-m} \otimes \beta_{p-m+1} \cdots \beta_q (\gamma\delta)^{r-1} \gamma \beta_1 \cdots \beta_{p-q+j-1}$$
$$+ (-1)^p(-1)^{(q-j)(p-1)} \sum_{m=q-j+2}^{p-1} (-1)^{m(p-1)} \beta_j \cdots \beta_q \gamma (\delta\gamma)^{r-1} \beta_1 \cdots \beta_{q-m} \otimes \beta_{p-m+1} \cdots \beta_{p-q+j-1} \Big] \quad \text{for } q-p+2 \leqslant j \leqslant q,$$

- $e_q \otimes e_1 \mapsto -\Big[e_q \otimes \beta_q + (-1)^p \sum_{m=1}^{p-1} (-1)^{m(p-1)} \beta_{q-p+1} \cdots \beta_{q-m} \otimes \beta_{p-m+1} \cdots \beta_q (\gamma\delta)^{r-1} \gamma - \beta_{q-p+1} \cdots \beta_q \otimes e_1 \Big]$,
- $e_1 \otimes e_{2p} \mapsto (-1)^p \Big[e_1 \otimes \beta_1 \cdots \beta_p + (-1)^p \sum_{m=1}^{p-1} (-1)^{m(p-1)} (\gamma\delta)^{r-1} \gamma \beta_1 \cdots \beta_{q-m} \otimes \beta_{p-m+1} \cdots \beta_p + (-1)^p \beta_1 \otimes e_{2p} \Big]$.

$\triangleright$ $d^{2p}\colon P^{2p} \to P^{2p-1}$:

- $e_i \otimes e_{i+1} \mapsto e_i \otimes \alpha_i - (-1)^p \alpha_i \otimes e_{i+1}$ for $1 \leqslant i \leqslant p-1$,
- $e_p \otimes e_1 \mapsto e_p \otimes \alpha_p - (-1)^p \alpha_p \otimes e_1$,
- $e_1 \otimes e_1 \mapsto \sum_{k=0}^{r-1} \Big[(-1)^p \delta(\gamma\delta)^k \otimes (\delta\gamma)^{r-k-1} + (\gamma\delta)^k \otimes (\delta\gamma)^{r-k-1}\delta - \sum_{j=0}^{q-p} (\gamma(\delta\gamma)^k \beta_1 \cdots \beta_j \otimes \beta_{j+p+1} \cdots \beta_q (\gamma\delta)^{r-k-1} + (-1)^p (\delta\gamma)^k \beta_1 \cdots \beta_j \otimes \beta_{j+p+1} \cdots \beta_q (\gamma\delta)^{r-k-1}\gamma) \Big]$,
- $e_{p+j-1} \otimes e_{p+j+p-q} \mapsto e_{p+j-1} \otimes \beta_{j+p-q} - (-1)^p \beta_j \otimes e_{p+j+p-q}$ for $q+1-p \leqslant j \leqslant q$,
- $e_{p+j-1} \otimes e_{2p+j-1} \mapsto \sum_{k=0}^{r} \theta_j^k \otimes \theta_{j+p}^{r-k}$
$+ \sum_{k=0}^{r-1} \Big[\sum_{m=0}^{q-j-p} \theta_j^k \beta_j \cdots \beta_{j+m} \otimes \beta_{j+m+p+1} \cdots \beta_q \gamma \beta_1 \cdots \beta_{j+p-1} \theta_{j+p}^{r-k-1} + \sum_{m=0}^{j-2} \theta_j^k \beta_j \cdots \beta_q \gamma \beta_1 \cdots \beta_m \otimes \beta_{m+p+1} \cdots \beta_{j+p-1} \theta_{j+p}^{r-k-1} - \theta_j^k \beta_j \cdots \beta_q \otimes \beta_1 \cdots \beta_{j+p-1} \theta_{j+p}^{r-k-1} \Big]$ for $2 \leqslant j \leqslant q-p$.

It remains to show that the projective bimodules and homomorphisms that we have defined do indeed give the start of a minimal projective bimodule resolution of Γ.

Theorem 3.9. *With the above notation,*

$$\cdots \longrightarrow P^{2p} \xrightarrow{d^{2p}} P^{2p-1} \xrightarrow{d^{2p-1}} \cdots \xrightarrow{d^2} P^1 \xrightarrow{d^1} P^0 \longrightarrow \Gamma \longrightarrow 0$$

is the beginning of a minimal projective resolution of Γ as a Γ–Γ-bimodule (when $p > 1$).

https://doi.org/10.1017/S0013091514000315 Published online by Cambridge University Press

Proof. It may be verified directly from the definitions that $d^2 = 0$, and thus we have a complex. The strategy for proving exactness is identical to that of [**26**, Theorem 1.6] (see also [**9**, Proposition 2.8]), whereby we apply $(\Gamma/\mathrm{rad}(\Gamma) \otimes -)$ to the complex and show that the resulting sequence is a minimal projective resolution of $\Gamma/\mathrm{rad}(\Gamma)$ as a right Γ-module. Minimality is then immediate since we know that the projectives are those of a minimal projective resolution of Γ as a Γ–Γ-bimodule from [**10**]. □

We are now in a position to give the dimensions of the Hochschild cohomology groups up to $\mathrm{HH}^{2p-2}(\Gamma)$. We only give those in even degree since we shall not need the others. The details of the proof are left to the reader.

Theorem 3.10. *For $2 \leqslant n < p \leqslant q$ we have*

$$\dim \mathrm{HH}^{2n-2}(\Gamma) = \begin{cases} r & \text{if } n \text{ is odd and } \mathrm{char}\, K \nmid 2r, \\ r+1 & \text{if } n \text{ is odd and } \mathrm{char}\, K \mid 2r, \\ r & \text{if } n \text{ is even and } \mathrm{char}\, K \neq 2, \\ r+1 & \text{if } n \text{ is even and } \mathrm{char}\, K = 2 \end{cases}$$

and for $2 \leqslant p < q$ we have

$$\dim \mathrm{HH}^{2p-2}(\Gamma) = \begin{cases} r-1 & \text{if } p \text{ is odd and } \mathrm{char}\, K \nmid 2r, \\ r & \text{if } p \text{ is odd, } \mathrm{char}\, K \neq 2 \text{ and } \mathrm{char}\, K \mid 2r, \\ r+1 & \text{if } p \text{ is even and } \mathrm{char}\, K \neq 2, \\ r+2 & \text{if } \mathrm{char}\, K = 2. \end{cases}$$

4. Derived equivalence classes

It was shown in [**6**] that two algebras of the form $A(p,q) = \Gamma(p,q;1)$ or $\Lambda(n) = \Lambda(1,n;2,2)$ are derived equivalent if and only if they are isomorphic. The main result in this section is to extend this to all algebras of the form $\Gamma(p,q;r)$ and $\Lambda(p,q;s,t)$, and hence to all basic indecomposable finite-dimensional K-algebras A that are symmetric special biserial algebras with at most one non-uniserial indecomposable projective module.

We start with some properties of these algebras, all of which are invariants under derived equivalence.

Proposition 4.1. *Suppose that $1 \leqslant p \leqslant q$. The algebras $\Gamma(p,q;r)$ and $\Lambda(p,q;s,t)$ have the following properties.*

(1) *The number of simples of*

$$\begin{aligned} \Gamma(p,q;r) &\quad \text{is } p+q-1, \\ \Lambda(p,q;s,t) &\quad \text{is } p+q-1. \end{aligned}$$

https://doi.org/10.1017/S0013091514000315 Published online by Cambridge University Press

(2) *The Cartan invariants of*

$$\Gamma(p,q;r) \quad \text{are} \quad \begin{cases} \underbrace{1,1,\dots,1}_{p+q-2},4r & \text{if } r(p+q-2) \text{ is even,} \\ \underbrace{1,1,\dots,1}_{p+q-3},2,2r & \text{if } r(p+q-2) \text{ is odd,} \end{cases}$$

$$\Lambda(p,q;s,t) \quad \text{are} \quad \underbrace{1,1,\dots,1}_{p+q-2},s+t+(p+q-2)st.$$

(3) *The Cartan determinant of*

$$\begin{aligned} \Gamma(p,q;r) &\quad \text{is } 4r, \\ \Lambda(p,q;s,t) &\quad \text{is } s+t+(p+q-2)st. \end{aligned}$$

Proof. (1) This is immediate from the number of vertices of the quiver $\mathcal{Q}_{(p,q)}$.

(2) Let $\mathcal{I}_m$ be the $m \times m$ identity matrix, let $\mathcal{J}_n$ be the $n \times n$ matrix with all entries equal to 1 and set $u = p+q-2$. We start with the algebra $\Gamma(p,q;r)$. The Cartan matrix of $\Gamma(p,q;r)$ is the $(p+q-1) \times (p+q-1)$ matrix

$$C_\Gamma = \begin{bmatrix} 4r & 2r & \cdots & 2r \\ 2r & & & \\ \vdots & & \mathcal{I}_u + r\mathcal{J}_u & \\ 2r & & & \end{bmatrix}.$$

The Smith normal form for C_Γ is

$$\begin{bmatrix} \mathcal{I}_u & 0 \\ 0 & 4r \end{bmatrix} \quad \text{if } ru \text{ is even,}$$

$$\begin{bmatrix} \mathcal{I}_{u-1} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2r \end{bmatrix} \quad \text{if } ru \text{ is odd,}$$

and thus the Cartan invariants of $\Gamma(p,q;r)$ are

$$\begin{aligned} &\underbrace{1,1,\dots,1}_{p+q-2},4r \quad \text{if } ru \text{ is even,} \\ &\underbrace{1,1,\dots,1}_{p+q-3},2,2r \quad \text{if } ru \text{ is odd.} \end{aligned}$$

https://doi.org/10.1017/S0013091514000315 Published online by Cambridge University Press

Now consider the algebra $\Lambda(p,q;s,t)$. The Cartan matrix of $\Lambda(p,q;s,t)$ is the $(p+q-1)\times(p+q-1)$ matrix

$$C_\Lambda = \begin{bmatrix} s+t & t & \cdots & t & s & \cdots & s \\ t & & & & & & \\ \vdots & & \mathcal{I}_{p-1}+t\mathcal{J}_{p-1} & & & 0 & \\ t & & & & & & \\ s & & & & & & \\ \vdots & & 0 & & & \mathcal{I}_{q-1}+s\mathcal{J}_{q-1} & \\ s & & & & & & \end{bmatrix}.$$

The Smith normal form for C_Λ is

$$\begin{bmatrix} \mathcal{I}_u & 0 \\ 0 & s+t+ust \end{bmatrix}$$

so the Cartan invariants of $\Lambda(p,q;s,t)$ are $\underbrace{1,1,\ldots,1,}_{p+q-2} s+t+(p+q-2)st$.

(3) This is immediate from (2). □

We now consider isomorphism classes of algebras of the same form. It is clear that $\Lambda(q,q;s,t)=\Lambda(q,q;t,s)$ and the next result shows that, with this one exception, two algebras both of the form $\Gamma(p,q;r)$ or of the form $\Lambda(p,q;s,t)$ are pairwise non-isomorphic.

Theorem 4.2.

(1) *The algebras of the form $\Gamma(p,q;r)$ (with $1\leqslant p\leqslant q$) are pairwise non-isomorphic.*

(2) *The algebras of the form $\Lambda(p,q;s,t)$ (with $1\leqslant p\leqslant q$) are pairwise non-isomorphic with the exception that $\Lambda(q,q;s,t)=\Lambda(q,q;t,s)$.*

Proof. (1) First, suppose that the algebras $\Gamma(p,q;r)$ and $\Gamma(p',q';r')$ are isomorphic with $1\leqslant p\leqslant q$ and $1\leqslant p'\leqslant q'$. Since both algebras are basic, the quivers are uniquely determined and hence $\mathcal{Q}_{(p,q)}=\mathcal{Q}_{(p',q')}$. Thus, $p=p'$ and $q=q'$. From the Cartan determinant in Proposition 4.1 (3), we have that $r=r'$.

(2) Suppose that $\Lambda(p,q;s,t)\cong\Lambda(p',q';s',t')$ with $1\leqslant p\leqslant q$ and $1\leqslant p'\leqslant q'$. Since both algebras are basic, we again have that $p=p'$ and $q=q'$. Then, using the zeroth Hochschild cohomology group from Proposition 3.1, we have $s+t=s'+t'$. Equality of the Cartan invariants from Proposition 4.1 (2) gives that $st=s't'$. Hence s', t' are the two roots of the equation $x^2-(s+t)x+st=0$. Thus, we have either $s=s'$ and $t=t'$ (which gives us the algebra $\Lambda(p,q;s,s)$) or $s=t'$ and $t=s'$. In the latter case we have the algebras $\Lambda(p,q;s,t)$ and $\Lambda(p,q;t,s)$, and it remains to show $p=q$ when $s\neq t$. The K dimension of $\Lambda(p,q;s,t)$ is $tp^2+sq^2+p+q-2$ and that of $\Lambda(p,q;t,s)$ is $sp^2+tq^2+p+q-2$. Thus, $tp^2+sq^2=sp^2+tq^2$ so that $(t-s)(p+q)(p-q)=0$. Since $p+q>0$ and $s\neq t$ we have $p=q$, which is precisely the case $\Lambda(q,q;s,t)=\Lambda(q,q;t,s)$. □

https://doi.org/10.1017/S0013091514000315 Published online by Cambridge University Press

Our next theorem classifies up to derived equivalence all basic indecomposable finite-dimensional K-algebras A that are symmetric special biserial algebras with at most one non-uniserial indecomposable projective module.

Theorem 4.3.

(1) *An algebra of the form $\Lambda(p,q;s,t)$ (with $1 \leqslant p \leqslant q$) is derived equivalent to exactly one algebra in the following list:*

   (a) *$\Lambda(1,p+q-1;s,t)$ with $2 \leqslant s \leqslant t$;*
   (b) *N_M^{p+q-1} with $p+q>2$ and $\min(s,t)=1$, $\max(s,t)=M$.*

(2) *An algebra of the form $\Gamma(p,q;r)$ (with $1 \leqslant p \leqslant q$) is derived equivalent to an algebra of the form $\Lambda(p,q;s,t)$ if and only if they are isomorphic. This is only the case for $\Gamma(1,1;1) \cong \Lambda(1,1;2,2)$ and $\operatorname{char} K \neq 2$.*

(3) *The algebras $\Gamma(p,q;r)$ and $\Gamma(p',q';r')$ (with $1 \leqslant p \leqslant q$ and $1 \leqslant p' \leqslant q'$) are derived equivalent if and only if $(p,q,r)=(p',q',r')$.*

Proof. (1) The algebra $\Lambda(p,q;s,t)$ is the generalized Brauer tree algebra, associated with the Brauer tree in Figure 1, in which the vertices a and b have multiplicities s and t, respectively (we refer the reader to [**4**, § 4.18] or [**20**] for the definition of a Brauer tree algebra and a generalized Brauer tree algebra).

It was proved in [**20**, Theorem 9.7] that generalized Brauer tree algebras up to derived equivalence depend only on the number of edges in the graph and the set of multiplicities. Therefore, $\Lambda(p,q;s,t)$ is derived equivalent to the generalized Brauer tree algebra associated with the Brauer tree in Figure 2, in which the vertices c and d have multiplicities $m=\min(s,t)$ and $M=\max(s,t)$, respectively, and $\{a,b\}=\{c,d\}$. This algebra is equal to

- either $K\Delta_{p+q-1}/L_M = N_M^{p+q-1}$ if $m=1$, that is, if $(s,t)=(M,1)$ or $(s,t)=(1,M)$, with $M \geqslant 1$,
- or $\Lambda(1,p+q-1;m,M)$ if $m>1$, that is, if $s \geqslant 2$ and $t \geqslant 2$.

Moreover, none of these algebras are derived equivalent, again by [**20**, Theorem 9.7].

(2) First, we show that the algebras $\Gamma(1,1;1)$ and $\Lambda(1,1;2,2)$ are isomorphic when $\operatorname{char} K \neq 2$. Since K is algebraically closed, let ε be a square root of -1 in K. Then the map

$$\varphi\colon \Gamma(1,1;1) \to \Lambda(1,1;2,2) \quad \text{given by} \quad \begin{cases} \alpha \mapsto \alpha + \varepsilon\beta, \\ \beta \mapsto \alpha - \varepsilon\beta \end{cases}$$

is an isomorphism of algebras.

Suppose that there is a derived equivalence between the algebras $\Gamma(p,q;r)$ and $\Lambda(p',q';s,t)$. Then the algebras have the same number of simple modules so, from Proposition 4.1, we have $p+q=p'+q'$. From (1), the algebra $\Lambda(p',q';s,t)$ is derived equivalent to exactly one algebra in the list above. Moreover, $s+t=m+M$, where $m=\min(s,t)$ and $M=\max(s,t)$.

https://doi.org/10.1017/S0013091514000315 Published online by Cambridge University Press

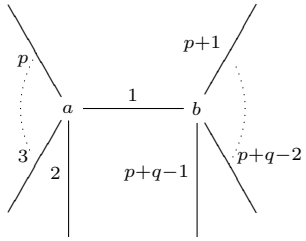


Figure 1. Brauer tree for $\Lambda(p, q; s, t)$.

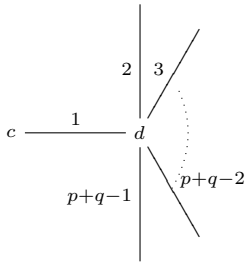


Figure 2. Brauer tree derived equivalent to that in Figure 1.

Case 1 ($m = \min(s, t) = 1$). Let $\Gamma = \Gamma(p, q; r)$ and $\Lambda = \Lambda(p', q'; s, t)$. From Propositions 3.1 and 3.3, $\dim \mathrm{HH}^0(\Lambda) = p + q + M - 1$ and $\dim \mathrm{HH}^1(\Lambda) = M$. We first assume that $p = 1$ so that $q > 1$. From Proposition 3.4, $\dim \mathrm{HH}^0(\Gamma) = q + r + 1$ so that $M = r + 1$. However, Proposition 3.6 gives

$$\dim \mathrm{HH}^1(\Gamma) = \begin{cases} r + 4 & \text{if } \operatorname{char} K = 2, \\ r + 2 & \text{if } \operatorname{char} K \neq 2, \end{cases}$$

which is a contradiction. So $\Gamma(1, q; r)$ is not derived equivalent to $\Lambda(p', q'; s, t)$.

On the other hand, if $p > 1$, then $\dim \mathrm{HH}^0(\Gamma) = p + q + r - 1$ so that $M = r$. But $\dim \mathrm{HH}^1(\Gamma) = r + 1$ so that $M = r + 1$, a contradiction. Again, $\Gamma(p, q; r)$ is not derived equivalent to $\Lambda(p', q'; s, t)$.

Case 2 ($m = \min(s, t) > 1$). We use Propositions 3.1, 3.3, 3.4 and 3.6 without further comment. We begin with the case $p = 1$. If $q = 1$, then $\dim \mathrm{HH}^0(\Lambda) = s + t$ and $\dim \mathrm{HH}^0(\Gamma) = r + 3$. From Proposition 4.1 (3), the Cartan determinant of Γ is $4r$ and that of Λ is $s + t$. Hence, $r + 3 = s + t = 4r$ so that $r = 1$ and $s = 2 = t$. If $\operatorname{char} K \neq 2$, then we have an isomorphism $\Gamma(1, 1; 1) \cong \Lambda(1, 1; 2, 2)$ from above. If $\operatorname{char} K = 2$, then $\dim \mathrm{HH}^1(\Lambda) = 5$ and $\dim \mathrm{HH}^1(\Gamma) = 8$, which is a contradiction and $\Gamma(1, 1; 1)$ is not derived equivalent to $\Lambda(1, 1; 2, 2)$. For the case in which $q > 1$ (keeping $p = 1$), a similar consideration of the zeroth and first Hochschild cohomology groups and the Cartan determinant shows that $\Gamma(1, q; r)$ is not derived equivalent to $\Lambda(p', q'; s, t)$.

Now suppose that $p > 1$. We have $\dim \mathrm{HH}^0(\Lambda) = p + q + s + t - 2$ and $\dim \mathrm{HH}^0(\Gamma) = p + q + r - 1$ so that $r = s + t - 1$. So $\dim \mathrm{HH}^1(\Gamma) = r + 1 = s + t$ and we must have

https://doi.org/10.1017/S0013091514000315 Published online by Cambridge University Press

$\operatorname{char} K = \ell | \gcd(s,t)$. From Proposition 4.1 (3), the Cartan determinant of Γ is $4r$ and the Cartan determinant of Λ is $s+t+(p+q-2)st$ so that $3(s+t) = (p+q-2)st-4$. Thus, $\ell|4$ so that $\ell = 2$ and s and t are both even. Thus, we are in the situation where $\Gamma(p,q;r)$ is derived equivalent to $\Lambda(p',q';s,t)$, $\operatorname{char} K = 2$, both s and t are even, $r = s+t-1$ and $p+q = p'+q'$. We shall use Külshammer invariants and the same arguments as in [**13**, § 4.5.2] for this case. Recall from Proposition 2.1 and its proof that the algebras $\Lambda(p',q';s,t)$ and $\Gamma(p,q;r)$ are symmetric; moreover, once we have fixed a K-basis of paths for the socle of a symmetric algebra A and completed it to a K-basis $\mathcal{B}_A$ of paths for A, the linear map $f\colon A \to K$ that is defined on $\mathcal{B}_A$ by sending socle elements to 1 and the rest to 0 defines a symmetric non-degenerate associative bilinear form on A. Orthogonals will be taken with respect to this bilinear form. Let $\kappa(A)$ be the commutator subspace of A and, for any non-negative integer n, define $T_n(A) = \{x \in A;\ x^{2^n} \in \kappa(A)\}$. It was proved in [**29**] that the generalized Reynolds ideals (or Külshammer invariants) $T_n(A)^\perp$ are derived invariant. Note that $\operatorname{soc}(A) \subseteq T_n(A)^\perp \subsetneq Z(A)$ for every n. It is well known that the centre $Z(A)$ is a derived invariant. Given a vector space V, let $\mathcal{B}_V$ denote a basis of V.

• We start with the algebra $\Gamma = \Gamma(p,q;r)$. A basis $\mathcal{B}_\Gamma$ of paths of Γ is given by the union over all i, j with $1 \leqslant i,\ j \leqslant p+q-1$ of all paths from e_i to e_j of length at most $(p+q)r$ except $(\delta\gamma)^r$. Recall from Proposition 2.1 the basis $\mathcal{B}_{\operatorname{soc}(\Gamma)} = \{(\gamma\delta)^r; \eta_i^r,\ 2 \leqslant i \leqslant p; \theta_i^r,\ 2 \leqslant j \leqslant q\} \subset \mathcal{B}_\Gamma$ of $\operatorname{soc}(\Gamma)$ and from Proposition 3.4 the basis

$$\mathcal{B}_{Z(\Gamma)} = \{1; (\gamma\delta)^r; z^k,\ 1 \leqslant k \leqslant r-1; \eta_i^r; \theta_j^r \text{ for } 2 \leqslant i \leqslant p \text{ and } 2 \leqslant j \leqslant q\}$$

of $Z(\Gamma)$, where

$$z = \sum_{i=1}^{p} \eta_i + \sum_{j=1}^{q} \theta_j.$$

Then $\dim \kappa(\Gamma) = r((p+q)^2-1)-1$ and a basis of $\kappa(\Gamma)$ is given by

$$\mathcal{B}_{\kappa(\Gamma)} = \{\gamma(\delta\gamma)^k; \delta(\gamma\delta)^k; (\gamma\delta)^{k+1} - (\delta\gamma)^{k+1}; \eta_i^{k+1} - \eta_1^{k+1},\ 1 \leqslant i \leqslant p; \\ \theta_j^{k+1} - \theta_1^{k+1}, 2 \leqslant j \leqslant q;\ \text{for } 0 \leqslant k \leqslant r-1\} \cup \{b \in \mathcal{B}_\Gamma; \mathfrak{o}(b) \neq \mathfrak{t}(b)\}.$$

Now, as in § 4.5.2 of [**13**], work in $\Gamma/\kappa(\Gamma)$ to find a basis $\mathcal{B}_{T_1(\Gamma)} = \mathcal{B}_{\kappa(\Gamma)} \cup \{(\gamma\delta)^k;\ (r+1)/2 \leqslant k \leqslant r\}$ of $T_1(\Gamma)$ (recall that r is odd), then work in $Z(\Gamma)/\operatorname{soc}(\Gamma)$ to find a basis $\mathcal{B}_{T_1(\Gamma)^\perp} = \mathcal{B}_{\operatorname{soc}(\Gamma)} \cup \{z^k;\ (r+1)/2 \leqslant k \leqslant r-1\}$ for $T_1(\Gamma)^\perp$ so that $\Gamma' := Z(\Gamma)/T_1(\Gamma)^\perp$ has a basis $\mathcal{B}_{\Gamma'} = \{1; z^k; 1 \leqslant k \leqslant (r-1)/2\}$. We now consider the Jacobson radical $\mathfrak{r}_{\Gamma'}$ of the algebra Γ', which is spanned by $\{z^k;\ 1 \leqslant k \leqslant (r-1)/2\}$, and its square $\mathfrak{r}_{\Gamma'}^2$, which is spanned by $\{z^k;\ 2 \leqslant k \leqslant (r-1)/2\}$, so that $\dim_K \mathfrak{r}_{\Gamma'}/\mathfrak{r}_{\Gamma'}^2 = 1$.

• We now turn to the algebra $\Lambda = \Lambda(p',q';s,t)$. Since they are derived equivalent, we may assume that $\Lambda = \Lambda(1,n;m,M)$ with $n = p+q-1$ (to simplify notation). Note that m and M are even. Set $\alpha = \alpha_1$. We follow the same method as for Γ, using Propositions 2.1

https://doi.org/10.1017/S0013091514000315 Published online by Cambridge University Press

and 3.1:

$$\begin{aligned}
\mathcal{B}_\Lambda &= \bigcup_{\substack{1\leqslant i,\\ j\leqslant p+q-1}} \{\text{all paths from } e_i \text{ to } e_j \text{ of length at most } nM \text{ that do not contain } \alpha\} \cup \{\alpha^k,\ 1\leqslant k\leqslant m-1\};\\
\mathcal{B}_{\mathrm{soc}(\Lambda)} &= \{\delta_i^m,\ 1\leqslant i\leqslant n\};\\
\mathcal{B}_{Z(\Lambda)} &= \{1;\alpha^k,\ 1\leqslant k\leqslant m; y^\ell,\ 1\leqslant \ell\leqslant t-1; \delta_j^M,\ 2\leqslant j\leqslant q\},\quad \text{where } y=\sum_{j=1}^q \delta_j;\\
\mathcal{B}_{\kappa(\Lambda)} &= \{\delta^\ell-\delta_j^\ell;\ 2\leqslant j\leqslant q,\ 1\leqslant \ell\leqslant M\}\cup\{b\in\mathcal{B}_\Lambda;\ \mathfrak{o}(b)\neq \mathfrak{t}(b)\};\\
\mathcal{B}_{T_1(\Lambda)} &= \mathcal{B}_{\kappa(\Lambda)}\cup\{\alpha^k,\ \tfrac{1}{2}m<k\leqslant m;\delta^\ell,\ \tfrac{1}{2}M<\ell\leqslant M;\ \alpha^{m/2}+\delta^{M/2}\};\\
\mathcal{B}_{T_1(\Lambda)^\perp} &= \mathcal{B}_{\mathrm{soc}(\Lambda)}\cup\{\alpha^k,\ \tfrac{1}{2}m<k\leqslant m-1;\ y^\ell,\ \tfrac{1}{2}M<\ell\leqslant M-1;\ \alpha^{m/2}+y^{M/2}\}.
\end{aligned}$$

So the Jacobson radical $\mathfrak{r}_{\Lambda'}$ of $\Lambda' := Z(\Lambda)/T_1(\Lambda)^\perp$ has a basis $\{\alpha^k,\ 1\leqslant k\leqslant \frac{1}{2}m; y^\ell,\ 1\leqslant \ell<\frac{1}{2}M\}$ and $\mathfrak{r}^2_{\Lambda'}$ has basis $\{\alpha^k,\ 2\leqslant k\leqslant \frac{1}{2}m; y^\ell,\ 2\leqslant \ell<\frac{1}{2}M\}$ so that

$$\dim_K \mathfrak{r}_{\Lambda'}/\mathfrak{r}^2_{\Lambda'} = \begin{cases} 2 & \text{if } M\geqslant 4,\\ 1 & \text{if } M=2 \text{ (and therefore } m=2).\end{cases}$$

Since we assumed that Λ and Γ are derived equivalent, the algebras Λ' and Γ' are isomorphic and hence we have $\dim_K \mathfrak{r}_{\Gamma'}/\mathfrak{r}^2_{\Gamma'} = \dim_K \mathfrak{r}_{\Lambda'}/\mathfrak{r}^2_{\Lambda'}$, which implies that $m=2=M$, that is, $s=2=t$. Therefore, $r=s+t-1=3$. We now use the Cartan determinants: $0=\det C_\Gamma - \det C_\Lambda = 4r-(s+t+(p+q-2)st) = 12-(4+(p+q-2)4) = 4(4-(p+q))$ so that $p+q=4$. Since $1<p\leqslant q$, we must have $p=2=q$. Therefore, $\Gamma=\Gamma(2,2;3)$ and Λ is derived equivalent to $\Lambda(2,2;2,2)$. However, it was shown in [**12**, § 3] that $\Gamma(2,2;3) = \mathcal{D}(3\mathcal{A})_1^3$ and $\Lambda(2,2;2,2)=\mathcal{D}(3\mathcal{A})_2^{2,2}$ are not derived equivalent. Therefore, Λ and Γ are not derived equivalent.

(3) If the algebras $\Gamma(p,q;r)$ and $\Gamma(p',q';r')$ (with $p\leqslant q$ and $p'\leqslant q'$) are derived equivalent, then from the Cartan determinant and number of simples (see Proposition 4.1) we know that $r=r'$ and $p+q=p'+q'$. Assume for contradiction that $(p,q)\neq(p',q')$. We may suppose that $p<p'$. It follows that $p<q$ (for otherwise $p=q$ and hence $q'=(p'+q')-p'=(p+q)-p'=2p-p'<2p'-p'=p'$, a contradiction). Using Theorem 3.10 and Proposition 3.4, we then have

$$\mathrm{HH}^{2p-2}(\Gamma(p,q;r))\neq \mathrm{HH}^{2p-2}(\Gamma(p',q';r')),$$

which contradicts the fact that the algebras are derived equivalent. Thus, $(p,q)=(p',q')$. □

5. Stable equivalence of Morita type classes

Finally, we give a classification up to stable equivalence of Morita type of all algebras of the form $\Gamma(p,q;r)$ and $\Lambda(p,q;s,t)$, and hence of all basic indecomposable finite-dimensional K-algebras A that are symmetric special biserial algebras with at most

https://doi.org/10.1017/S0013091514000315 Published online by Cambridge University Press

one non-uniserial indecomposable projective module. This is based on the classification up to derived equivalence of Theorem 4.3.

Theorem 5.1.

(1) *An algebra of the form* $\Lambda(p,q;s,t)$ *(with* $1 \leqslant p \leqslant q$*) is stably equivalent of Morita type to exactly one algebra in the following list:*

   (a) $\Lambda(1,p+q-1;s,t)$ *with* $2 \leqslant s \leqslant t$*;*

   (b) N_M^{p+q-1} *with* $p+q>2$ *and* $\min(s,t)=1$, $\max(s,t)=M$.

(2) *An algebra of the form* $\Gamma(p,q;r)$ *(with* $1 \leqslant p \leqslant q$*) is stably equivalent of Morita type to an algebra of the form* $\Lambda(p,q;s,t)$ *if and only if they are isomorphic. This is only the case for* $\Gamma(1,1;1) \cong \Lambda(1,1;2,2)$ *and* $\operatorname{char} K \neq 2$.

(3) *The algebras* $\Gamma(p,q;r)$ *and* $\Gamma(p',q';r')$ *(with* $1 \leqslant p \leqslant q$ *and* $1 \leqslant p' \leqslant q'$*) are stably equivalent of Morita type if and only if* $(p,q,r)=(p',q',r')$.

Proof. It was proved by Rickard [**22**] and Keller and Vossieck [**15**] that if two self-injective K-algebras are derived equivalent, then they are stably equivalent of Morita type. Therefore, to prove the result, we need only prove that the algebras listed in Theorem 4.3 are not stably equivalent of Morita type since they are all self-injective.

We shall use the following invariants of stable equivalences of Morita type repeatedly:

- the dimension of $\mathrm{HH}^n(\Lambda)$ for $n \geqslant 1$ for Artin K-algebras (see [**28**, Theorem 4.2]);
- the number of simple Λ-modules, where Λ is an indecomposable self-injective special biserial algebra (see [**21**]);
- the dimension of the centre $Z(\Lambda) \cong \mathrm{HH}^0(\Lambda)$, where Λ is an indecomposable symmetric special biserial algebra (using a result of Liu *et al.* [**19**, Corollary 1.2]);
- the absolute value of the Cartan determinant of Λ (see [**28**, Proposition 5.1]).

All of our algebras are indecomposable symmetric special biserial algebras. We now prove the theorem.

(1) Assume first that $\Lambda(1,a;s,t)$ and $\Lambda(1,b;s',t')$, with $a \geqslant 1$, $b \geqslant 1$, $2 \leqslant s \leqslant t$ and $2 \leqslant s' \leqslant t'$, are stably equivalent of Morita type. Then $a=b$ since the numbers of simples are equal and, using the dimensions of the centres, we have $s+t=s'+t'$. Moreover, their Cartan determinants are equal (they are positive) so that $st=s't'$. Finally, we have $(s,t)=(s',t')$ so that $\Lambda(1,a;s,t)=\Lambda(1,b;s',t')$.

Now assume that N_M^a and $N_{M'}^b$, with $a>1$, $b>1$, $M \geqslant 1$ and $M' \geqslant 1$, are stably equivalent of Morita type. Then $a=b$ since the numbers of simples are equal and, using the Cartan determinant, we have $1+M+(a-1)M = 1+M'+(a-1)M'$ so that $M=M'$. Therefore, $N_M^a = N_{M'}^b$.

Finally, if $\Lambda(1,a;s,t)$ and N_M^b, with $a \geqslant 1$, $b>1$, $2 \leqslant s \leqslant t$ and $M \geqslant 1$, are stably equivalent of Morita type, then, since N_M^b is a symmetric Nakayama algebra,

https://doi.org/10.1017/S0013091514000315 Published online by Cambridge University Press

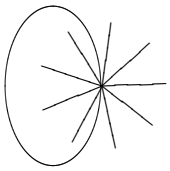


Figure 3. Brauer graph for $\Gamma(p,q;r)$

$\Lambda(1,a;s,t)$ must be a Brauer tree algebra by [**8**] and hence derived equivalent to a symmetric Nakayama algebra by [**20**], which we know is not the case. Therefore, $\Lambda(1,a;s,t)$ and N_M^b are not stably equivalent of Morita type.

This concludes the proof of (1).

(2) The proof is essentially the same as in Theorem 4.3 since almost all the derived invariants used there are invariants of stable equivalence of Morita type between indecomposable symmetric special biserial algebras, the exception being the Külshammer ideal $T_1^{\perp}(A)$ of an algebra A. However, for symmetric algebras, the algebra $Z(A)/T_1^{\perp}(A)$ is a stable invariant of Morita type. Indeed, let $Z^{\mathrm{st}}(A) = \underline{\mathrm{End}}_{A^e}(A)$ be the stable centre of A (the endomorphisms of A in the stable A^e-module category) and let $Z^{\mathrm{pr}}(A) = \mathrm{Ker}(\mathrm{End}_{A^e}(A) \to \underline{\mathrm{End}}_{A^e}(A))$ be the projective centre of A. Then $Z^{\mathrm{st}}(A)$ and $T_1^{\perp}(A)/Z^{\mathrm{pr}}(A)$ are invariants of stable equivalences of Morita type for symmetric algebras (see [**16**, **19**]) and moreover, $Z(A)/T_1^{\perp}(A) \cong Z^{\mathrm{st}}(A)/(T_1^{\perp}(A)/Z^{\mathrm{pr}}(A))$.

Therefore, the proof of Theorem 4.3 (2) still holds for stable equivalences of Morita type.

(3) The proof is the same as in Theorem 4.3 since the dimensions of the Hochschild cohomology groups in positive degrees are invariants of stable equivalences of Morita type. □

Remark 5.2. Recall from [**8**] that the Nakayama algebras are the distinct representatives of the stable equivalence classes of Brauer tree algebras. It follows from [**21**] and [**18**] that any algebra that is stably equivalent of Morita type to one of the $\Gamma(p,q;r)$ or $\Lambda(p,q;s,t)$ is a symmetric Brauer graph algebra. Moreover, the list of algebras given in Theorem 5.1 are 'normal forms' for derived equivalences of Brauer graph algebras (a specific type of generalized star as in [**14**] and [**24**, Theorem 5.7]), and hence, since Brauer graph algebras are self-injective, for stable equivalences of Morita type. Indeed, the algebra $\Gamma(p,q;r)$ is the Brauer graph algebra associated with the graph of Figure 3, which has $p-1$ edges inside the loop, $q-1$ edges outside the loop and with multiplicity r at the central vertex. However, it is still an open question in general as to whether two such normal forms are derived equivalent or not.

Acknowledgements. N.S. thanks the Université Jean Monnet (Saint-Etienne) and the Université Blaise Pascal (Clermont-Ferrand 2) for two months of invited professorships during which this collaboration took place.

References

1. S. Al-Nofayee, Equivalences of derived categories for self-injective algebras, *J. Alg.* **313**(2) (2007), 897–904.
2. J. L. Alperin, *Local representation theory: modular representations as an introduction to the local representation theory of finite groups*, Cambridge Studies in Advanced Mathematics, Volume 11 (Cambridge University Press, 1986).
3. M. A. Antipov and A. I. Generalov, The Yoneda algebras of symmetric special biserial algebras are finitely generated, *St Petersburg Math. J.* **17**(3) (2006), 377–392.
4. D. Benson, *Representations and cohomology, I: basic representation theory of finite groups and associative algebras*, Cambridge Studies in Advanced Mathematics, Volume 30 (Cambridge University Press, 1991).
5. R. Bocian and A. Skowroński, Weakly symmetric algebras of Euclidean type, *J. Reine Angew. Math.* **580** (2005), 157–199.
6. R. Bocian, T. Holm and A. Skowroński, Derived equivalence classification of weakly symmetric algebras of Euclidean type, *J. Pure Appl. Alg.* **191**(1–2) (2004), 43–74.
7. K. Erdmann, *Blocks of tame representation type and related algebras*, Springer Lecture Notes in Mathematics, Volume 1428 (Springer, 1990).
8. P. Gabriel and Ch. Riedtmann, Group representations without groups, *Comment. Math. Helv.* **54**(2) (1979), 240–287.
9. E. L. Green and N. Snashall, Projective bimodule resolutions of an algebra and vanishing of the second Hochschild cohomology group, *Forum Math.* **16**(1) (2004), 17–36.
10. D. Happel, Hochschild cohomology of finite-dimensional algebras, in *Séminaire d'Algèbre Paul Dubreil et Marie-Paul Malliavin*, Lecture Notes in Mathematics, Volume 1404, pp. 108–126 (Springer, 1989).
11. T. Holm, Derived equivalent tame blocks, *J. Alg.* **194** (1997), 178–200.
12. T. Holm, Derived equivalence classification of algebras of dihedral, semidihedral, and quaternion type, *J. Alg.* **211**(1) (1999), 159–205.
13. T. Holm and A. Zimmermann, Generalized Reynolds ideals and derived equivalences for algebras of dihedral and semidihedral type, *J. Alg.* **320**(9) (2008), 3425–3437.
14. M. Kauer, Derived equivalence of graph algebras, in *Trends in the representation theory of finite dimensional algebras*, Contemporary Mathematics, Volume 229, pp. 201–213 (American Mathematical Society, Providence, RI, 1998).
15. B. Keller and D. Vossieck, Sous les catégories dérivées, *C. R. Acad. Sci. Paris I* **305**(6) (1987), 225–228.
16. S. Koenig, Y. Liu and G. Zhou, Transfer maps in Hochschild (co)homology and applications to stable and derived invariants and to the Auslander–Reiten conjecture, *Trans. Am. Math. Soc.* **364**(1) (2012), 195–232.
17. M. Linckelmann, A derived equivalence for blocks with dihedral defect groups, *J. Alg.* **164** (1994), 244–255.
18. Y. Liu, Summands of stable equivalences of Morita type, *Commun. Alg.* **36** (2008), 3778–3782.
19. Y. Liu, G. Zhou and A. Zimmermann, Higman ideal, stable Hochschild homology and Auslander-Reiten conjecture, *Math. Z.* **270**(3–4) (2012), 759–781.
20. F. H. Membrillo-Hernández, Brauer tree algebras and derived equivalence, *J. Pure Appl. Alg.* **114**(3) (1997), 231–258.
21. Z. Pogorzały, Algebras stably equivalent to self-injective special biserial algebras, *Commun. Alg.* **22**(4) (1994), 1127–1160.
22. J. Rickard, Derived categories and stable equivalence, *J. Pure Appl. Alg.* **61**(3) (1989), 303–317.
23. J. Rickard, Derived equivalences as derived functors, *J. Lond. Math. Soc.* **43** (1991), 37–48.

24. K. W. Roggenkamp, Biserial algebras and graphs, in *Algebras and modules II*, Conference Proceedings, Canadian Mathematical Society, Volume 24, pp. 481–496 (American Mathematical Society, Providence, RI, 1998).
25. A. Skowroński, Selfinjective algebras: finite and tame type, in *Trends in representation theory of algebras and related topics*, Contemporary Mathematics, Volume 406, pp. 169–238 (American Mathematical Society, Providence, RI, 2006).
26. N. Snashall and R. Taillefer, The Hochschild cohomology ring of a class of special biserial algebras, *J. Alg. Appl.* **9**(1) (2010), 73–122.
27. B. Wald and J. Waschbüsch, Tame biserial algebras, *J. Alg.* **95**(2) (1985), 480–500.
28. C. Xi, Stable equivalences of adjoint type, *Forum Math.* **20**(1) (2008), 81–97.
29. A. Zimmermann, Invariance of generalized Reynolds ideals under derived equivalences, *Math. Proc. R. Irish Acad.* A **107**(1) (2007), 1–9.