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HALPHEN AND THE ELLIPTIC FUNCTIONS OF DU VAL

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Abstract

We show that the 'ternary' elliptic functions that were introduced and studied by Du Val in 1964 are the $n = 3$ instances of n-ary elliptic functions that are defined for arbitrary integers n greater than unity. We also trace the general n-ary elliptic function back to 1886 and the 'fort remarquable' function of Halphen.

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1. Introduction

One of the standard constructions of an elliptic function of Jacobian type starts from a Weierstrassian elliptic $\wp$-function p: subtracting from p its value $p(\omega)$ at a half-period yields an elliptic function whose poles and zeros all have order 2. A meromorphic square root of $p - p(\omega)$ is then elliptic, of Jacobian type. This construction of these elliptic functions is especially favoured by Neville in his masterly treatise [9].

Du Val takes this construction a step further: subtracting from the derivative p' a suitable affine expression in the Weierstrass function p itself, he fashions a coperiodic elliptic function whose poles and zeros all have order 3. Extraction of a meromorphic cube root then produces an elliptic function to which Du Val gives the name 'ternary'. Du Val presents a concise account of his ternary elliptic functions and their properties in [3] and expands upon this account in a chapter of his noteworthy book [4] on elliptic functions and elliptic curves.

It is entirely reasonable to ask whether the approach of Du Val may be pursued to higher orders. Here, we answer this question in the affirmative: when n is any integer greater than unity, we show that a suitable modification of the $(n - 2)$th derivative of the $\wp$-function p yields an elliptic function with not only nth-order poles but also nth-order zeros. An n-ary elliptic function then arises upon the extraction of a meromorphic nth root. Our construction of the appropriate elliptic function with nth-order poles and nth-order zeros relies on a classical theorem of Kiepert [7] which is recovered by Frobenius and Stickelberger in [5]. Having seen this classical theorem serve in the construction of n-ary elliptic functions, we turned for further inspiration to the first volume of Halphen [6] on elliptic functions and their applications. There, we were naturally drawn first to the calculations on page 223 and then to the very next page where, up to sign, the 'fort remarquable' function that Halphen displays is none other than an n-ary elliptic function, and in a very elegant formulation.

Our purpose here is thus twofold, and in part historical: to show that the Jacobian elliptic functions and the ternary elliptic functions of Du Val admit natural generalisations to n-ary elliptic functions; and to note that these elliptic functions were all anticipated by Halphen. In Section 2 we define and study the n-ary elliptic functions, both as meromorphic nth roots of suitably modified $(n-2)$th derivatives of $\wp$-functions and as affine combinations of shifted coperiodic zeta functions. Our approach in this section is deliberately patterned after that of Du Val, but is further supported by the classical theorem of Kiepert, Frobenius and Stickelberger. In Section 3 we develop the 'very remarkable' function of Halphen, offering an alternative route into the theory. In Section 4 we illustrate this alternative route in the $n = 3$ case by connecting the ternary elliptic function of Du Val to the elliptic functions sm and cm of Dixon [2].

Throughout our account, we assume a familiarity with the elements of elliptic function theory; for such elements, Chapter XX of the monumental treatise [11] by Whittaker and Watson is rather more than sufficient; alternative more recent sources are [4, 8]. Chapter 1 of the still more recent text [1] by Cooper offers a somewhat different perspective on such topics.

2. The n-ary elliptic function of Du Val

Let p be a Weierstrass elliptic $\wp$-function and write $\Lambda_p \subseteq \mathbb{C}$ for its period lattice. Let $(2\omega, 2\omega')$ be an integral basis for Λ_p. Thus, $(2\omega, 2\omega')$ is a fundamental pair of periods for p and we may write $p = \wp(-;\omega,\omega')$. As is often the custom, we shall suppose that the necessarily nonreal ratio ω'/ω lies in the upper halfplane $\mathcal{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. Associated to p are the corresponding zeta function $\zeta = \zeta(-;\omega,\omega')$ and sigma function $\sigma = \sigma(-;\omega,\omega')$: the former may be defined by

$$\zeta' = -p \quad \text{and} \quad \lim_{z\to 0}\left\{\zeta(z) - \frac{1}{z}\right\} = 0;$$

the latter by

$$\frac{\sigma'}{\sigma} = \zeta \quad \text{and} \quad \lim_{z\to 0}\frac{\sigma(z)}{z} = 1.$$

Having chosen the fundamental pair $(2\omega, 2\omega')$ of periods for the $\wp$-function p, we now fix an integer n greater than unity and focus on the nth period (or nth division point)

$$\alpha := 2\omega/n.$$

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Further, let $P = \wp(-;\omega, n\omega')$ be the Weierstrass $\wp$-function with $(2\omega, 2n\omega')$ as a fundamental pair of periods, so that its period lattice Λ_P comprises all linear combinations of 2ω and $2n\omega'$ with integer coefficients. Associated to P are the corresponding zeta function $Z = \zeta(-;\omega, n\omega')$ and sigma function $\Sigma = \sigma(-;\omega, n\omega')$. When convenient, we may instead write the chosen half-periods of P as $\Omega = \omega$ and $\Omega' = n\omega'$.

Now, to each choice of constants $a_1, \dots, a_{n-1}$ we associate the function

$$\phi := \frac{(-1)^n}{(n-1)!}\, p_{n-2} + a_1 p_{n-3} + \cdots + a_{n-2} p_0 + a_{n-1} \tag{2.1}$$

where we abbreviate to p_k the kth derivative of p. Note that for each such choice of constants, ϕ is an elliptic function with Λ_p as both its period lattice and its pole lattice, each pole having order n; note also that the coefficient of p_{n-2} in the expansion (2.1) secures the behaviour $z^n\phi(z) \to 1$ as $z \to 0$.

In the $n = 3$ case the following result is presented by Du Val in [3, page 141] and again in [4, page 129] as the basis for one of his constructions (the second) of a ternary elliptic function.

THEOREM 2.1. *There is a unique choice of constants $a_1, \dots, a_{n-1}$ for which the elliptic function ϕ defined at (2.1) has an nth-order zero at $\alpha = 2\omega/n$.*

PROOF. For convenience, let ϕ_k denote the kth derivative of ϕ and let us write $\lambda = (-1)^n/(n-1)!$. The requirement that ϕ have α as a zero is

$$a_{n-1} + p_0(\alpha)a_{n-2} + \cdots + p_{n-3}(\alpha)a_1 = -\lambda p_{n-2}(\alpha); \tag{2.2}$$

the requirement that the zero α of ϕ have order at least equal to n augments (2.2) by

$$p_k(\alpha)a_{n-2} + \cdots + p_{n-3+k}(\alpha)a_1 = -\lambda p_{n-2+k}(\alpha) \tag{2.3}$$

for $1 \leqslant k < n$. For k any positive integer, let $\mathbb{P}_k$ be the (Hankel) matrix-valued function

$$\mathbb{P}_k = \begin{bmatrix} p_1 & p_2 & \cdots & p_k \\ p_2 & p_3 & \cdots & p_{k+1} \\ \vdots & \vdots & \vdots & \vdots \\ p_k & p_{k+1} & \cdots & p_{2k-1} \end{bmatrix}.$$

The system comprising Equations (2.3) for $1 \leqslant k < n-1$ then assumes matrix form

$$\mathbb{P}_{n-2}(\alpha)\,[a_{n-2}, \dots, a_1]^T = -\lambda[p_{n-1}(\alpha), \dots, p_{2n-4}(\alpha)]^T. \tag{2.4}$$

On the one hand, $(n-1)\alpha$ is not a period of p, so a classical theorem of Kiepert (which we recall after our proof) implies that the coefficient matrix $\mathbb{P}_{n-2}(\alpha)$ here is nonsingular; in consequence, there is a unique vector $[a_{n-2}, \dots, a_1]^T$ satisfying the system (2.4). On the other hand, as $n\alpha$ is a period of p, the same theorem of Kiepert implies that the matrix $\mathbb{P}_{n-1}(\alpha)$ is singular; the one-dimensionality of its null space ensures that the very same constants $a_1, \dots, a_{n-2}$ further satisfy the system

$$\mathbb{P}_{n-1}(\alpha)\,[a_{n-2}, \dots, a_1, \lambda]^T = 0$$

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so that

$$\phi_1(\alpha) = \cdots = \phi_{n-1}(\alpha) = 0.$$

Precisely one value of the constant a_{n-1} now guarantees that $\phi(\alpha) = \phi_0(\alpha) = 0$ also, whence α is a zero of ϕ having order at least n. Finally, as ϕ is elliptic of order n, the zeros of ϕ have order at most n. □

The classical theorem of Kiepert [7] to which we refer in our proof of Theorem 2.1 asserts the following:

$$\mathrm{Det}\,\mathbb{P}_{k-1}(z) = (-1)^{k-1}\,[1!\;2!\;\cdots\;(k-1)!\,]^2\,\frac{\sigma(kz)}{\sigma(z)^{k^2}}, \tag{2.5}$$

with the notation established in our proof. Our application of this classical theorem rests on the fact that the zeros of σ are precisely the periods of p.

Henceforth, we choose the constants $a_1, \ldots, a_{n-1}$ that are singled out by Theorem 2.1. The Λ_p-periodic meromorphic function ϕ then has not only poles of order n but also zeros of order n. Accordingly, ϕ then has globally meromorphic nth roots; these nth roots are n in number and may be distinguished by their residues at the origin, these being the nth roots of unity since $z^n\phi(z) \to 1$ as $z \to 0$. We define the n-ary elliptic function ψ to be the meromorphic nth root of ϕ whose residue at the origin is unity itself; thus,

$$\psi^n = \phi \quad \text{and} \quad \mathrm{Res}_0\psi = 1.$$

Note that ϕ has Λ_p as its pole lattice, and its zeros are precisely the points congruent to α modulo this lattice. Naturally, the poles and zeros of ψ coincide with those of ϕ but are all simple. It is at once evident that the meromorphic function ψ is indeed elliptic as claimed: addition of 2ω or $2\omega'$ fixes $\psi^n = \phi$ and therefore multiplies ψ by an nth root of unity, whence ψ has $2n\omega$ and $2n\omega'$ as periods; however, these periods do not constitute a fundamental pair.

THEOREM 2.2. *The n-ary elliptic function ψ satisfies the identity*

$$\psi(z)\psi(z+\alpha)\cdots\psi(z+(n-1)\alpha) = \psi'(\alpha)\psi(2\alpha)\cdots\psi((n-1)\alpha).$$

PROOF. Consider the n-fold product on the left as a function of z. The poles and zeros of the n factors cancel one with another, in cyclic fashion, whence the product is constant. Evaluation of this constant may be effected by passage to the limit as $z \to 0$, since then

$$\psi(z)\psi(z+\alpha) = \{\psi(z)z\}\left\{\frac{\psi(z+\alpha)-\psi(\alpha)}{z}\right\} \to \psi'(\alpha).$$ □

Immediately, we see that 2ω is a period of ψ: simply observe the effect of replacing z by $z+\alpha$ in the identity of Theorem 2.2.

THEOREM 2.3. *The n-ary elliptic function ψ has $(2\omega, 2n\omega')$ as a fundamental pair of periods.*

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PROOF. Already, 2ω and $2n\omega'$ are periods of ψ. Let Π_n be the parallelogram with 0, 2ω, $2n\omega'$, $2\omega + 2n\omega'$ as its vertices (but translated slightly, so as to make 0 an interior point). In Π_n the elliptic function ψ has (simple) poles at the points

$$0,\ 2\omega',\ \ldots,\ 2(n-1)\omega'$$

and (simple) zeros at the points

$$\alpha,\ \alpha + 2\omega',\ \ldots,\ \alpha + 2(n-1)\omega'.$$

Now let m be a positive divisor of n. In the (similarly shifted) parallelogram Π_m with 0, 2ω, $2m\omega'$, $2\omega + 2m\omega'$ as its vertices, ψ has pole sum

$$0 + 2\omega' + \cdots + 2(m-1)\omega' = m(m-1)\omega'$$

and zero sum

$$\alpha + (\alpha + 2\omega') + \cdots + (\alpha + 2(m-1)\alpha) = m\alpha + m(m-1)\omega',$$

whence

$$(\text{zero sum}) - (\text{pole sum}) = m\alpha = \frac{m}{n} \cdot 2\omega.$$

If Π_m is a period parallelogram for ψ then this difference is a period of ψ and therefore of ϕ; thus, if $m < n$ then Π_m is not a period parallelogram and so the period pair $(2\omega, 2n\omega')$ is indeed fundamental. □

From this, it follows that addition of $2\omega'$ to its argument multiplies ψ by a primitive nth root of unity. In fact, we can identify this nth root of unity explicitly: it is precisely the 'first' nth root of unity, namely

$$\varepsilon := \exp\{2\pi i/n\}.$$

THEOREM 2.4. *The n-ary elliptic function ψ satisfies the identity*

$$\psi(z + 2\omega') = \varepsilon \cdot \psi(z).$$

PROOF. Almost as in the proof of Theorem 2.3, we take the following full sets of Λ_P-incongruent zeros and poles of ψ, all of them simple: for the zeros

$$a_0 = \alpha,\ a_1 = \alpha + 2\omega',\ \ldots,\ a_{n-1} = \alpha + 2(n-1)\omega'$$

and for the poles

$$b_0 = 2\omega,\ b_1 = 2\omega',\ \ldots,\ b_{n-1} = 2(n-1)\omega'.$$

As the zero sum and pole sum agree, an elliptic function f may be defined by

$$f(z) = \prod_{j=0}^{n-1} \frac{\Sigma(z - a_j)}{\Sigma(z - b_j)}$$

where Σ is the sigma function associated to the lattice Λ_P. As ψ and f share a period lattice and have precisely the same poles and zeros to matching order, there exists $\mu = \operatorname{Res}_0 f \in \mathbb{C}$ such that

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$$\mu\psi = f.$$

The precise value of μ is immaterial for our purposes: its mere existence is sufficient, as it implies that identically

$$\frac{\psi(z+2\omega')}{\psi(z)} = \frac{f(z+2\omega')}{f(z)}.$$

The last-mentioned ratio is readily calculated using appropriate care and the quasiperiodicity of Σ, according to which

$$\Sigma(z+2\Omega) = -\exp\{2Z(\Omega)[z+\Omega]\}\Sigma(z),$$
$$\Sigma(z+2\Omega') = -\exp\{2Z(\Omega')[z+\Omega']\}\Sigma(z),$$

and is found to have the value

$$\exp\left\{\frac{4}{n}[Z(\Omega)\Omega' - Z(\Omega')\Omega]\right\}$$

where we recall that $\Omega = \omega$ and $\Omega' = n\omega'$. Finally, as Ω'/Ω lies in the upper halfplane, the Legendre relation

$$Z(\Omega)\Omega' - Z(\Omega')\Omega = \tfrac{1}{2}\pi i$$

ends the proof. □

Note that this is a consequence of our requiring that the ratio ω'/ω be in the upper halfplane; were we to assume instead that ω'/ω has negative imaginary part, then ε would be replaced by its reciprocal.

The ternary elliptic functions of Du Val enjoy a variety of interesting properties, many of which are explored in [3, 4]. The same is true for the n-ary elliptic functions; here we offer but a couple of examples. Thus, ψ satisfies the reflection identity

$$\psi(\alpha - z)\psi(z) = -\psi'(\alpha).$$

Viewing the left-hand side as a function of z, the poles and zeros of the one factor cancel the zeros and poles of the other, so the elliptic product is constant; its value appears in the limit as $z \to 0$ since ψ has unit residue at the origin. Also, ψ satisfies the identity

$$\psi(z)\psi(-z) = p(\alpha) - p(z).$$

The elliptic functions of z on the two sides of this identity have the same poles (double, at points of the lattice Λ_p) and the same zeros (simple, at points congruent to $\pm\alpha$ modulo Λ_p); their ratio is therefore a constant, whose value is unity since $\text{Res}_0\psi = 1$ and $z^2 p(z) \to 1$ as $z \to 0$.

Our account thus far has already involved the fact that, up to a multiplicative constant, an elliptic function with given periods is determined by its poles and zeros, via a sigma function expansion. Our further discussion will make use of the fact that, up to an additive constant, an elliptic function with given periods is determined by its principal parts at its poles, via a zeta function expansion.

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Note that $P = \wp(-;\omega, n\omega')$ is the $\wp$-function coperiodic with the n-ary elliptic function ψ, which has Λ_P as its period lattice. Recalling that ψ has its poles (all of them simple) at the points of Λ_p, we may take $0, 2\omega', \ldots, 2(n-1)\omega'$ as a full set of poles, incongruent modulo periods. As ψ has unit residue at the origin, Theorem 2.4 implies that ψ has residue ε^j at each pole $2j\omega'$, these residues having zero sum. Now consider the function ψ_0 defined by

$$\psi_0(z) = \sum_{j=0}^{n-1} \varepsilon^j Z(z - 2j\omega')$$

where Z is the zeta function associated to the lattice Λ_P. The elliptic functions ψ and ψ_0 share the period lattice Λ_P and have the very same poles, at which they have the very same principal parts; as remarked above, it follows that ψ and ψ_0 differ by an additive constant, which we identify to produce the following expression for ψ.

THEOREM 2.5. *The n-ary elliptic function ψ is given by*

$$\psi(z) = \frac{2Z(n\omega')}{\varepsilon - 1} + \sum_{j=0}^{n-1} \varepsilon^j Z(z - 2j\omega').$$

PROOF. Say $\psi = c + \psi_0$ with $c \in \mathbb{C}$ constant. Recall that the zeta function Z is quasiperiodic: in particular, it satisfies the identity

$$Z(z + 2\Omega') = Z(z) + 2Z(\Omega').$$

It follows easily from this that ψ_0 satisfies the identity

$$\psi_0(z + 2\omega') - \varepsilon \cdot \psi_0(z) = 2Z(\Omega'),$$

whence from $\psi = c + \psi_0$ we deduce that

$$\psi(z + 2\omega') - \varepsilon \cdot \psi(z) = (1 - \varepsilon)c + 2Z(\Omega'),$$

and reference to Theorem 2.4 concludes the proof. □

In the $n = 3$ case this result is tantamount to the first construction of a ternary elliptic function by Du Val (see [3, page 136] and [4, page 119]).

Incidentally, we note that Theorem 2.5 implies a cyclotomic decomposition of the zeta function at a half-period, on account of the fact that $\psi(\alpha)$ vanishes:

$$2Z(\Omega') = (1 - \varepsilon) \sum_{j=0}^{n-1} \varepsilon^j Z(\alpha - 2j\omega').$$

We may reformulate this identity in more familiar terms by renaming Ω and Ω' as ω and ω': this results in

$$2\zeta(\omega') = (1 - \varepsilon) \sum_{j=0}^{n-1} \varepsilon^j \zeta([2\omega - 2j\omega']/n).$$

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3. The very remarkable function of Halphen

Here we offer an entirely different approach to the n-ary elliptic function, prompted by an inspection of [6] by Halphen. There, his 'very remarkable' function appears at Equation (24) on page 224, in a section devoted to 'Fonctions analogues à sn u, cn u, dn u', and emerges from his prior discussion involving the classical results of Kiepert, Frobenius and Stickelberger. Here, we do not propose to reproduce the genesis of this function in [6], whose perusal we recommend to the reader; instead, we present the 'very remarkable' function as an important object of study in its own right, by means of which to represent and investigate an n-ary elliptic function.

Recall that the n-ary elliptic function ψ has simple poles at the points of the lattice Λ_p and has simple zeros at the points congruent to α modulo Λ_p. As a function of z, the ratio $\sigma(z-\alpha)/\sigma(z)$ plainly has these poles and zeros, but it fails to be elliptic. Taking into account the fact that quasiperiodicity of the sigma function σ introduces exponentials when its argument is shifted by elements of Λ_p, we modify this ratio and consider (with Halphen) the function ψ_* defined by

$$\psi_*(z) = \frac{\sigma(z-\alpha)}{\sigma(z)} \exp\{\beta z\}$$

where $\beta \in \mathbb{C}$ is a constant whose value will be chosen shortly. The aforementioned quasiperiodicity of σ leads directly to the identities

$$\psi_*(z+2\omega) = \psi_*(z) \exp\{2[\beta\omega - \alpha\eta]\},$$
$$\psi_*(z+2\omega') = \psi_*(z) \exp\{2[\beta\omega' - \alpha\eta']\},$$

where $\eta = \zeta(\omega)$ and $\eta' = \zeta(\omega')$ as is customary. We choose

$$\beta = \frac{\alpha}{\omega}\eta = \frac{2\eta}{n}$$

so that ψ_* has 2ω as a period. It further follows that

$$\beta\omega' - \alpha\eta' = \frac{2\eta}{n}\omega' - \frac{2\omega}{n}\eta' = \frac{2}{n}[\eta\omega' - \omega\eta']$$

which the Legendre relation $\eta\omega' - \omega\eta' = \frac{1}{2}\pi i$ converts to

$$\beta\omega' - \alpha\eta' = \frac{\pi i}{n}$$

and results in

$$\psi_*(z+2\omega') = \psi_*(z) \exp\left\{\frac{2\pi i}{n}\right\} = \psi_*(z)\cdot\varepsilon$$

so that ψ_* has $2n\omega'$ as a period.

The following representation of the n-ary elliptic function ψ is now almost immediate.

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THEOREM 3.1. *The n-ary elliptic function ψ is given by*

$$\psi(z) = \frac{\sigma(\alpha - z)}{\sigma(\alpha)\sigma(z)} \exp\{\beta z\}$$

where $\alpha = 2\omega/n$ and $\beta = 2\zeta(\omega)/n$.

PROOF. The functions ψ_* and ψ have precisely the same poles and zeros by design; as we have just seen, they also share 2ω and $2n\omega'$ as periods. It follows that ψ_* is a constant multiple of ψ: namely, $\psi_* = (\mathrm{Res}_0\psi_*)\,\psi$ where

$$\mathrm{Res}_0\psi_* = \lim_{z\to 0}\left\{\frac{z}{\sigma(z)}\sigma(z-\alpha)\exp\{\beta z\}\right\} = \sigma(-\alpha) = -\sigma(\alpha)$$

whence the claim in the theorem. □

This elegant expression for the n-ary elliptic function ψ naturally facilitates alternative proofs of its various properties. Thus, the reflection property $\psi(\alpha - z)\psi(z) = -\psi'(\alpha)$ may be established as follows: on the one hand, simple cancellations yield

$$\psi(\alpha - z)\psi(z) = \frac{\exp\{\beta\alpha\}}{\sigma(\alpha)^2};$$

on the other hand,

$$\psi'(\alpha) = \lim_{z\to\alpha}\frac{\psi(z)}{z-\alpha} = \lim_{z\to\alpha}\left\{\frac{\sigma(\alpha - z)}{z-\alpha}\,\frac{\exp\{\beta z\}}{\sigma(\alpha)\sigma(z)}\right\} = -\frac{\exp\{\beta\alpha\}}{\sigma(\alpha)^2}.$$

Also, the property $\psi(z)\psi(-z) = p(\alpha) - p(z)$ amounts to a familiar classical relation between the $\wp$-function p and the sigma function σ: indeed,

$$\psi(z)\psi(-z) = \frac{\sigma(\alpha - z)}{\sigma(\alpha)\sigma(z)}\exp\{\beta z\}\frac{\sigma(\alpha + z)}{\sigma(\alpha)\sigma(-z)}\exp\{-\beta z\} = -\frac{\sigma(\alpha - z)\sigma(\alpha + z)}{\sigma(\alpha)^2\sigma(z)^2}.$$

As a further example, Theorem 3.1 leads to an evaluation of the (constant) product displayed in Theorem 2.2 as follows:

$$\psi(z+\alpha)\cdot\,\cdots\,\cdot\psi(z+n\alpha) = \frac{(-1)^{n-1}}{\sigma(\alpha)^n}\exp\left\{\frac{2}{n}\omega\,\zeta(\omega)\right\}.$$

In the $n = 2$ case the elliptic function of Theorem 3.1 is entirely classical, as Halphen notes in [6]. To make contact with the relevant notation, let the $\wp$-function p have ω_1 and ω_2 as a fundamental pair of half-periods and write $\omega_3 = -\omega_1 - \omega_2$ for symmetry. For $j = 1, 2, 3$, let $\eta_j = \zeta(\omega_j)$ and define the sigma function σ_j by

$$\sigma_j(z) = \exp\{-\eta_j z\}\,\frac{\sigma(z+\omega_j)}{\sigma(\omega_j)}$$

(see [11, page 448], [4, page 110] or [8, page 151]). The elliptic ratio σ_j/σ is then a square root of $p - p(\omega_j)$ and hence a Jacobian function (see [11, page 451], [4, page 110] or [8, page 158]). Quasiperiodicity of σ reveals this ratio to be the very function that appears in Theorem 3.1.

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4. The elliptic functions of Dixon

In [2] Dixon introduces and quite thoroughly analyses a family of third-order elliptic functions that effectively lay dormant for more than a century, until their connections with combinatorics, with geometry, and more recently with generalised trigonometry, were exposed. His elliptic functions $s = \mathrm{sm}(-, \kappa)$ and $c = \mathrm{cm}(-, \kappa)$ of complex modulus κ, not a cube root of -1, are the solutions to the initial value problem

$$s' = c^2 - \kappa\, s, \quad s(0) = 0,$$

$$c' = \kappa\, c - s^2, \quad c(0) = 1.$$

In [10] we show that the ternary elliptic functions of Du Val are intimately related to the classical elliptic functions sm and cm of Dixon: indeed, they are so related that either can be obtained from the other by elementary shifts. Our purpose in this section is to present a brief account of this relationship, taking advantage of the realisation that the ternary elliptic function is the very remarkable function of Halphen in the $n = 3$ case.

We begin by deriving a first-order differential equation that is satisfied by ψ. This differential equation appears first in [3, Section 20] and then as [10, Theorem 8] but derived by a very different line of argument.

THEOREM 4.1. *The ternary elliptic function ψ satisfies the differential equation*

$$\psi'(z) = A\,\psi(z) - \psi(-z)^2$$

where the constant A is given by

$$A = 2\zeta(\alpha) - \frac{4}{3}\zeta(\omega) = -\frac{1}{3}\frac{p''(\alpha)}{p'(\alpha)}.$$

PROOF. Logarithmic differentiation of the Halphen expression for ψ in Theorem 3.1 yields

$$\psi'(z) = \{\beta - \zeta(\alpha - z) - \zeta(z)\}\psi(z)$$

with $\beta = 2\eta/3$ and $\eta = \zeta(\omega)$ as usual. Further application of Theorem 3.1 then shows that

$$A(z) := \frac{\psi'(z) + \psi(-z)^2}{\psi(z)} = \{\beta - \zeta(\alpha - z) - \zeta(z)\} + \frac{\sigma(\alpha + z)^2}{\sigma(\alpha)\sigma(\alpha - z)\sigma(z)}\,\exp\{-3\beta z\}$$

and we claim that this function A is constant. Note that

$$\zeta(2\alpha) = \zeta(-\alpha + 2\omega) = \zeta(-\alpha) + 2\eta = 2\eta - \zeta(\alpha),$$

whence the vanishing of σ at 0 implies that

$$A(-\alpha) = \beta - \zeta(2\alpha) + \zeta(\alpha) = \beta + 2\zeta(\alpha) - 2\eta = 2\zeta(\alpha) - 4\zeta(\omega)/3.$$

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Accordingly, we claim that this is the constant value of A; equivalently, we claim that identically

$$\frac{\sigma(\alpha+z)^2}{\sigma(\alpha)\sigma(\alpha-z)\sigma(z)}\exp\{-2\eta z\} = 2\zeta(\alpha)+\zeta(\alpha-z)+\zeta(z)-2\eta.$$

As a function of z, each side of this equation is elliptic, with 2ω and $2\omega'$ as periods: on the left-hand side, this follows from (multiplicative) quasiperiodicity of σ; on the right-hand side, it follows from (additive) quasiperiodicity of ζ. Further, it is readily confirmed that each side has simple poles of residue $+1$ at the points of the lattice Λ_p and simple poles of residue -1 at points of the coset $\alpha+\Lambda_p$. Thus, the two sides share both periods and poles along with their principal parts, so the two sides differ by a constant, which vanishes by evaluation at $z=-\alpha$. Finally, the identification of $2\zeta(\alpha)-4\zeta(\omega)/3$ as $-p''(\alpha)/3p'(\alpha)$ is an easy consequence of the zeta function duplication law, according to which

$$2\zeta(2z)-4\zeta(z)=p''(z)/p'(z).$$

□

Before we proceed further, it is convenient to observe here the fact that

$$\psi(-\alpha)^3 = p'(\alpha).$$

To see this, note first that

$$\sigma(2\alpha)=\sigma(-\alpha+2\omega)=\sigma(\alpha)\exp\{\eta\alpha\}$$

by quasiperiodicity. Now

$$\psi(-\alpha)^3=\left\{\frac{\sigma(2\alpha)}{\sigma(\alpha)\sigma(-\alpha)}\exp\{-\beta\alpha\}\right\}^3=-\frac{\exp\{\eta\alpha\}}{\sigma(\alpha)^3}=-\frac{\sigma(2\alpha)}{\sigma(\alpha)^4}=p'(\alpha)$$

where the last equation instances the sigma function duplication law.

As a consequence of the preceding theorem we have the next, which appears first in [3, Section 19], then in [4, Section 59] and subsequently as [10, Theorem 6].

THEOREM 4.2. *The ternary elliptic function ψ satisfies the cubic identity*

$$\psi(z)^3+\psi(-z)^3-3A\psi(z)\psi(-z)=p'(\alpha).$$

PROOF. Denote by $g(z)$ the left-hand side of the claimed identity. Theorem 4.1 shows that $g'=0$; the constant value of g is $g(\alpha)=\psi(-\alpha)^3$ since $\psi(\alpha)=0$; and the observation made ahead of the theorem ends the proof. □

The following theorem is established in [10] but without the assistance of Halphen. Here, we use the convenient abbreviation

$$\mu := \psi(-\alpha).$$

THEOREM 4.3. *The functions s and c defined by*

$$s(z)=\frac{1}{\mu}\psi\left(\alpha-\frac{z}{\mu}\right)\quad \textit{and}\quad c(z)=\frac{1}{\mu}\psi\left(\frac{z}{\mu}-\alpha\right)$$

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are the elliptic functions of Dixon with modulus

$$\kappa = -\frac{1}{3\mu}\frac{p''(\alpha)}{p'(\alpha)} = -\frac{1}{3}\frac{p''(\alpha)}{\psi(-\alpha)^4}.$$

PROOF. Plainly, $s(0) = 0$ since $\psi(\alpha) = 0$ and $c(0) = 1$ by definition of μ. Further, direct application of Theorem 4.1 shows that

$$s' = c^2 - \kappa s \quad \text{and} \quad c' = \kappa c - s^2$$

with the modulus $\kappa = A/\mu$ as advertised. □

We remark here that the stated modulus κ is not a cube root of -1; we see this as follows. First, recall that the $\wp$-function p satisfies the 'eliminant' equation $(p')^2 = 4p^3 - g_2 p - g_3$ and the derived equation $p'' = 6p^2 - \frac{1}{2}g_2$ along with the resulting $p''' = 12pp'$; here, the constants g_2 and g_3 are the invariants of p. As α is a third period of p, the Kiepert formula (2.5) informs us that

$$0 = p'(\alpha)p'''(\alpha) - p''(\alpha)^2 = 12p(\alpha)p'(\alpha)^2 - p''(\alpha)^2.$$

If possible, suppose $\kappa^3 = -1$: that is, suppose $p''(\alpha)^3 = 27\psi(-\alpha)^{12} = 27p'(\alpha)^4$. By elimination of the derivatives between the equations here assembled, it may be deduced that $g_2 = 4p(\alpha)^2/3$ and $g_3 = 8p(\alpha)^3/27$ and therefore that $g_2^3 = 27g_3^2$; for a $\wp$-function, this is impossible.

We refer to [10] for a proof of the 'converse' fact that for any $-\kappa$ other than a cube root of unity, there exist a $\wp$-function p and third period α with associated ternary function ψ such that the formulas in Theorem 4.3 recover the elliptic functions $s = \mathrm{sm}(-, \kappa)$ and $c = \mathrm{cm}(-, \kappa)$ of Dixon; in [10] it is further shown that we may insist upon $\mu := \psi(-\alpha) = 1$ and so simplify the formulas in Theorem 4.3.

Incidentally, Theorems 4.2 and 4.3 combine to show that s and c give an elliptic parametrisation of the cubic to which Dixon refers in the title of [2].

Finally, we draw attention to the fact that Cooper presents an alternative, q-theoretic construction of the Dixonian functions sm and cm in Chapter 1 of his recent comprehensive text [1] on Ramanujan's theta functions. Of course, once his q-analogue of the sine function is expressed in terms of the Weierstrass sigma function, the echo of the Halphen function in his formulas for sm and cm is promoted to an exact agreement with the formulas for sm and cm that we present here.

References

[1] S. Cooper, *Ramanujan's Theta Functions* (Springer, Cham, 2017).

[2] A. C. Dixon, 'On the doubly periodic functions arising out of the curve $x^3 + y^3 - 3\alpha xy = 1$', *Quarterly J. Pure Appl. Math.* **24** (1890), 167–233.

[3] P. Du Val, 'On certain elliptic functions of order three', *Acta Arith.* **9** (1964), 133–147.

[4] P. Du Val, *Elliptic Functions and Elliptic Curves*, London Mathematical Society Lecture Note Series, 9 (Cambridge University Press, Cambridge, 1973).

[5] F. G. Frobenius and L. Stickelberger, 'Zur Theorie der elliptischen Functionen', *J. reine angew. Math.* **83** (1877), 175–179.

[6] G.-H. Halphen, *Traité des fonctions elliptiques et de leurs applications, Volume 1* (Gauthier-Villars, Paris, 1886).

[7] L. Kiepert, 'Wirkliche Ausführung der ganzzahligen Multiplication der elliptischen Functionen', *J. reine angew. Math.* **76** (1873), 21–33.

[8] D. F. Lawden, *Elliptic Functions and Applications*, Applied Mathematical Sciences, 80 (Springer-Verlag, New York, 1989).

[9] E. H. Neville, *Jacobian Elliptic Functions* (Oxford University Press, Oxford, 1944).

[10] P. L. Robinson, 'The elliptic functions of Dixon and Du Val', *J. Math. Anal. Appl.* **549**(1) (2025), Article no. 129496.

[11] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, 4th edn (Cambridge University Press, Cambridge, 1927).

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