EXISTENCE RESULTS FOR A CLASS OF ABSTRACT IMPULSIVE DIFFERENTIAL EQUATIONS

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Abstract

We study the existence of solutions for a class of abstract impulsive differential equations. Our technical framework allows us to study partial differential equations with impulsive conditions involving partial derivatives and nonlinear expressions of the solution. Some applications to impulsive partial differential equations are presented.

1. introduction

In this paper we study the existence of solutions for abstract impulsive differential equations of the form

$$u'(t) = Au(t) + f(t, u(t)), \quad t \in (0, a], t \neq t_i, i = 1, \ldots, N, \tag{1.1}$$

$$u(0) = x_0 \in X, \tag{1.2}$$

$$\triangle u(t_i) = I_i(u(t_i)), \quad i = 1, \ldots, N, \tag{1.3}$$

where $A : D(A) \subset X \to X$ is the infinitesimal generator of an analytic semigroup of bounded linear operators $(T(t))_{t \geq 0}$ defined on a Banach space $(X, \| \cdot \|)$, $x_0 \in X$, $0 < t_1 < \cdots < t_N < a$ are fixed numbers, $I_i \in C(X_\alpha, X)$ for all $i = 1, \ldots, N$, X_α denotes the domain of the α-fractional power of $-A$ endowed with the graph norm $\|(-A)^\alpha x\|$, $f : [0, a] \times X_\alpha \to X$ is a suitable function and $\triangle u(t_i)$ represents the jump of $u(\cdot)$ at t_i which is defined by $\triangle u(t_i) = u(t_i^+) - u(t_i^-)$.

The literature on abstract impulsive differential equations similar to (1.1)–(1.3) is very extensive and considers different topics on the existence and qualitative properties of solutions. Concerning general motivations, relevant developments and the current status of the theory we refer the reader to [1–19, 21, 22] and the references therein.

In comparison to the literature on abstract impulsive differential equations, the main contribution of our paper is the type of function I_i considered in (1.3). In this work we assume that the maps I_i are defined from X_α into X, which permit the study of partial differential equations with impulsive conditions involving partial derivatives and nonlinear expressions of the solution (see the examples in the final section). It is important to note that in [2, 5, 6, 9, 15, 21] some results are established on the existence of X_α-valued mild solutions for different classes of impulsive differential equations. However, in these papers the maps I_i are defined from X_α into X_α, and do not permit the applications mentioned.

This paper has three sections. In Section 2 we study the existence of mild and classical solutions for (1.1)–(1.3). To prove our results on the existence of a mild solution (see Definition 2.2), we need to look at the singularity of the function $t \mapsto \|T(t - t_i)I_i(u(t_i))\|_\alpha$ at $t = t_{t_i}$. To this end, we introduce a class of Banach spaces with weight (denoted $\mathcal{PC}_\alpha(X_\alpha)$) formed by piecewise continuous functions for which $\sup_{t \in (t_i, t_{i+1}]}(t - t_i)^\alpha \|u(t)\|_\alpha$ is finite. The existence and uniqueness of a mild solution via the contraction mapping principle is established in Theorem 2.6. In the main result of this paper (Theorem 2.13) we prove the existence of a mild solution via Schauder's fixed point criteria without assuming that the functions I_i are compact. To prove this theorem we use the approximation scheme of Fan and Li [8], but in the context of the space $\mathcal{PC}_\alpha(X_\alpha)$. First, we study the solvability of a family of integral equations (see equation (2.6)) which are a 'good approximation' of the integral equation used to define the concept of mild solution of (1.1)–(1.3). In Theorem 2.9, we prove the existence of a solution u_n, $n \in \mathbb{N}$, for (2.6) and in Lemmas 2.11 and 2.12 we establish some compactness properties of the set $\{u_n : n \in \mathbb{N}\}$. Finally, by proving the existence of a convergent subsequence of $(u_n)_{n \in \mathbb{N}}$, in Theorem 2.13 we show the existence of a mild solution for (1.1)–(1.3). We finish Section 2 by studying the existence of a classical solution for (1.1)–(1.3). In the final section, some applications to partial differential equations with impulsive terms involving partial derivatives and nonlinear expressions of the solution are considered.

Next, we introduce some definitions, notation and results used in this work. Let $(Z, \|\cdot\|_Z)$ and $(W, \|\cdot\|_W)$ be Banach spaces. We denote by $\mathcal{L}(Z, W)$ the space of bounded linear operators from Z into W endowed with the norm of operators denoted by $\|\cdot\|_{\mathcal{L}(Z,W)}$ and we write $\mathcal{L}(Z)$ and $\|\cdot\|_{\mathcal{L}(Z)}$ when $Z = W$. In addition, $B_l(z, Z)$ denotes the closed ball with centre at $z \in Z$ and radius l in Z and for $J \subset \mathbb{R}$ we use the notation $C(J; Z)$ for the space formed by all the bounded continuous functions from J into Z endowed with the uniform norm denoted by $\|\cdot\|_{C(J;X)}$. In this work, for a function $u : J \to X$ and $I \subset J$, the symbol $u_{|_I}$ represents the restriction of $u(\cdot)$ to the set I.

In this paper, $A : D(A) \subset X \to X$ is the generator of an analytic semigroup $(T(t))_{t \geq 0}$ on $(X, \|\cdot\|)$ and we assume that $0 \in \rho(A)$. From [20] we note that X_γ is a Banach space, $X_\gamma \hookrightarrow X_\beta$ for $0 < \beta < \gamma < \infty$, $(-A)^\beta T(\cdot) \in C((0, \infty), \mathcal{L}(X))$ and there is $C_\beta > 0$ such that $\|(-A)^\beta T(t)\|_{\mathcal{L}(X)} \leq C_\beta t^{-\beta} \|x\|$ for all $t \in (0, a]$.

In the remainder of this paper, $\alpha \in (0, 1)$, $t_0 = 0$, $t_{N+1} = a$ and $\delta_i = (t_{i+1} - t_i)$ for all $i = 0, \ldots, N$. To treat the impulsive conditions, we consider the space $\mathcal{PC}(X)$ formed

by all the functions $u : [0, a] \to X$ such that $u(\cdot)$ is continuous at $t \neq t_i$, $u(t_i^-) = u(t_i)$ and $u(t_i^+)$ exists for all $i = 1, \ldots, N$, endowed with the uniform norm denoted by $\| \cdot \|_{\mathcal{PC}(X)}$. The notation $\mathcal{PC}_\alpha(X_\alpha)$ is used for the space formed by all the functions $u : (0, a] \to X_\alpha$ such that $u_{|_{(t_i, t_{i+1}]}} \in C((t_i, t_{i+1}]; X_\alpha)$ and $\|u\|_{\alpha,i} = \sup_{t \in (t_i, t_{i+1}]} (t - t_i)^\alpha \|(-A)^\alpha u(t)\| < \infty$ for all $i = 0, \ldots, N$, endowed with the norm $\|u\|_\alpha = \max_{i=0,1,\ldots,N} \|u\|_{\alpha,i}$. It is easy to see that $\mathcal{PC}_\alpha(X_\alpha)$ is a Banach space.

For a function $u \in \mathcal{PC}(X)$ and $i \in \{0, 1, \ldots, N\}$, we use the notation $\tilde{u}_i$ for the function $\tilde{u}_i \in C([t_i, t_{i+1}]; X)$ given by

$$\widetilde{u_i}(t) = \begin{cases} u(t) & \text{for } t \in (t_i, t_{i+1}], \\ u(t_i^+) & \text{for } t = t_i. \end{cases} \tag{1.4}$$

If $B \subseteq \mathcal{PC}(X)$, we employ the notation $\widetilde{B}_i$ for the set $\widetilde{B}_i = \{\tilde{u}_i : u \in B\}$. We state without proof the following Ascoli–Arzelà type criteria.

LEMMA 1.1. *A set $B \subseteq \mathcal{PC}(X)$ is relatively compact in $\mathcal{PC}(X)$ if and only if $\widetilde{B}_i$ is relatively compact in $C([t_i, t_{i+1}], X)$ for all $i \in \{1, \ldots, N\}$.*

LEMMA 1.2. *Assume that $K \subset \mathcal{PC}_\alpha(X_\alpha)$, the set $\{u_{|_{[b,t_{i+1}]}} : u \in K\}$ is relatively compact in $C([b, t_{i+1}], X_\alpha)$ for all $b \in (t_i, t_{i+1})$ and each $i = 0, \ldots, N$, and $\lim_{t \downarrow t_i} \sup_{u \in K} (t - t_i)^\alpha \|(-A)^\alpha u(t)\| = 0$ for all $i = 1, \ldots, N$. Then K is relatively compact in $\mathcal{PC}_\alpha(X_\alpha)$.*

2. Existence of solutions

In this section we discuss the existence of solutions for the impulsive problem (1.1)–(1.3). To begin, we introduce the following concepts of solution.

DEFINITION 2.1. *A function $u \in \mathcal{PC}(X)$ is called a classical solution of the problem (1.1)–(1.3) if $u_{|_{(t_j, t_{j+1}]}} \in C((t_j, t_{j+1}], X_1)$ for all $j = 0, \ldots, N$ and $u(\cdot)$ satisfy (1.1)–(1.3).*

DEFINITION 2.2. *A function $u \in \mathcal{PC}(X)$ is called a mild solution of (1.1)–(1.3) if*

$$u(t) = T(t)x_0 + \int_0^t T(t - s)f(s, u(s)) \, ds + \sum_{t_i < t} T(t - t_i)I_i(u(t_i)), \quad \forall t \in [0, a].$$

To prove our results, we introduce the following conditions. Let q' denote the conjugate of a number $q > 1$ (that is, $1/q + 1/q' = 1$) and take $q' = \infty$ for $q = 1$.

H_1 The functions I_i belong to $C(X_\alpha; X)$ and there are nondecreasing functions $\mathcal{K}_{I_i} : [0, \infty) \to \mathbb{R}^+$ such that $\|I_i(x) - I_i(y)\| \leq \mathcal{K}_{I_i}(l)\|(-A)^\alpha(x - y)\|$ for all $x, y \in B_l(0, X_\alpha)$ and each $l > 0$.

H_2 For all $x \in X$, the function $f(\cdot, x)$ is strongly measurable on $[0, a]$ and $f(t, \cdot) \in C(X_\alpha; X)$ for each $t \in [0, a]$. There are $q \in [1, 1/\alpha)$, $m_f \in L^{q'}([0, a]; \mathbb{R}^+)$ and a nondecreasing function $W_f \in C([0, \infty); \mathbb{R}^+)$ such that $\|f(t, x)\| \leq m_f(t)W_f(\|(-A)^\alpha x\|)$ for all $(t, x) \in [0, a] \times X_\alpha$.

H_3 The functions I_i belong to $C(X_\alpha; X)$ and there are nondecreasing functions $c_i : [0, \infty) \to \mathbb{R}^+$ and constants $d_i > 0$ such that $\|I_i(x)\| \leq c_i(l)\|(-A)^\alpha x\| + d_i$ for all $x \in B_l(0, X_\alpha)$ and every $l > 0$.

H_4 The function f belongs to $C([0, a] \times X_\alpha; X)$ and there are $q \in [1, 1/\alpha)$ and a function $\mathcal{K}_f \in L^{q'}([0, a]; \mathbb{R}^+)$ such that $\|f(t, x) - f(t, y)\| \leq \mathcal{K}_f(t)\|(-A)^\alpha x - (-A)^\alpha y\|$ for all $x, y \in X_\alpha$ and every $t \in [0, a]$.

To shorten the proof of our existence results, we consider some lemmas.

LEMMA 2.3. *Assume that $\xi \in C((0, \infty); \mathbb{R}^+)$ is nondecreasing, $\vartheta > 0$, $q > 1$, $\eta \in L^{q'}([0, a]; \mathbb{R})$ and the function $\tau \to \xi^q(1/\tau^\alpha)$ is integrable on $[0, a]$. Then*

$$\int_c^d \frac{\eta(\tau)\xi\left(\frac{\vartheta}{(\tau-c)^\alpha}\right)}{(d-\tau)^\alpha}\, d\tau \leq \frac{2^{\alpha+1/q}}{(1-q\alpha)^{1/q}} \frac{\|\eta\|_{L^{q'}([c,d])}}{(d-c)^\alpha}\left(\int_0^{(d-c)/2} \xi^q\left(\frac{2^\alpha\vartheta}{\tau^\alpha}\right)d\tau\right)^{1/q}$$

for all $0 \leq c < d \leq a$.

PROOF. From the Hölder inequality, we see that

$$\int_c^d \frac{\eta(\tau)\xi\left(\frac{\vartheta}{(\tau-c)^\alpha}\right)}{(d-\tau)^\alpha}\, d\tau \leq \|\eta\|_{L^{q'}([c,d])}\left(\int_0^{d-c} \frac{\xi^q(\frac{\vartheta}{\tau^\alpha})}{(d-c-\tau)^{q\alpha}}\, d\tau\right)^{1/q}. \tag{2.1}$$

On the other hand, for $s \in [0, a]$,

$$
\begin{aligned}
\int_0^s \frac{\xi^q(\frac{\vartheta}{\tau^\alpha})}{(s-\tau)^{q\alpha}}\, d\tau &\leq \left(\frac{s}{2}\right)^{-q\alpha}\int_0^{s/2} \xi^q\left(\frac{\vartheta}{\tau^\alpha}\right)d\tau + \xi^q\left(\frac{\vartheta}{(\frac{s}{2})^\alpha}\right)\int_{s/2}^s \frac{d\tau}{(s-\tau)^{q\alpha}} \\
&\leq \frac{2^{q\alpha}}{s^{q\alpha}}\int_0^{s/2} \xi^q\left(\frac{\vartheta}{\tau^\alpha}\right)d\tau + \xi^q\left(\frac{2^\alpha\vartheta}{s^\alpha}\right)\frac{(\frac{s}{2})^{1-q\alpha}}{(1-q\alpha)} \\
&\leq \frac{2^{q\alpha}}{s^{q\alpha}}\int_0^{s/2} \xi^q\left(\frac{2^\alpha\vartheta}{\tau^\alpha}\right)d\tau + \frac{(\frac{s}{2})^{-q\alpha}}{(1-q\alpha)}\int_0^{s/2} \xi^q\left(\frac{2^\alpha\vartheta}{s^\alpha}\right)d\tau,
\end{aligned}
$$

from which we obtain that

$$\int_0^s \frac{\xi^q\left(\frac{\vartheta}{\tau^\alpha}\right)}{(s-\tau)^{q\alpha}}\, d\tau \leq \frac{2^{1+q\alpha}}{s^{q\alpha}(1-q\alpha)}\int_0^{s/2} \xi^q\left(\frac{2^\alpha\vartheta}{\tau^\alpha}\right)d\tau. \tag{2.2}$$

Now, the assertion follows by combining (2.1) and (2.2). The proof is complete.   □

LEMMA 2.4. *Assume that $\xi \in C((0, \infty); \mathbb{R}^+)$ is nondecreasing, $\eta \in L^{q'}([0, a]; \mathbb{R})$ for some $q > 1$, the function $\tau \to \xi^q(1/\tau^\alpha)$ is integrable on $[0, a]$, $u \in \mathcal{PC}_\alpha(X_\alpha)$ and $t \in (t_j, t_{j+1}]$ for some $j \in \{1, \ldots, N\}$. Then*

$$\int_0^t \frac{\eta(s)\xi(\|(-A)^\alpha u(s)\|)}{(t-s)^\alpha}\, ds \leq \frac{\|\eta\|_{L^{q'}([0,t])}}{(t-t_j)^\alpha}\left(N + \frac{2^{\alpha+1/q}}{(1-q\alpha)^{1/q}}\right)\left(\int_0^a \xi^q\left(\frac{2^\alpha\|u\|_\alpha}{s^\alpha}\right)ds\right)^{1/q}.$$

Proof. By noting that $\|(-A)^\alpha u(\tau)\| \le \|u\|_\alpha/(\tau - t_i)^\alpha$ and $\|(-A)^\alpha u(t_i)\| \le \|u\|_\alpha/\delta_{i-1}^\alpha$ for $i \in \{1, \ldots, N\}$ and $\tau \in (t_i, t_{i+1}]$, from Lemma 2.3 we have that

$$
\int_0^t \frac{\eta(s)\xi(\|(-A)^\alpha u(s)\|)}{(t-s)^\alpha}\,ds
$$

$$
\le \sum_{i=0}^{j-1} \int_{t_i}^{t_{i+1}} \frac{\eta(s)}{(t-s)^\alpha}\xi\left(\frac{\|u\|_\alpha}{(s-t_i)^\alpha}\right)ds + \int_{t_j}^t \frac{\eta(s)}{(t-s)^\alpha}\xi\left(\frac{\|u\|_\alpha}{(s-t_j)^\alpha}\right)ds
$$

$$
\le \frac{1}{(t-t_j)^\alpha} \sum_{i=0}^{j-1} \|\eta\|_{L^{q'}([t_i,t_{i+1}])}\left(\int_0^{t_{i+1}-t_i} \xi^q\left(\frac{\|u\|_\alpha}{s^\alpha}\right)ds\right)^{1/q}
$$

$$
+ \frac{2^{\alpha+1/q}\|\eta\|_{L^{q'}([t_j,t])}}{(t-t_j)^\alpha(1-q\alpha)^{1/q}}\left(\int_0^{(t-t_j)/2} \xi\left(\frac{2^\alpha\|u\|_\alpha}{s^\alpha}\right)ds\right)^{1/q}
$$

$$
\le \frac{\|\eta\|_{L^{q'}([0,t])}}{(t-t_j)^\alpha}\left(N + \frac{2^{\alpha+1/q}}{(1-q\alpha)^{1/q}}\right)\left(\int_0^a \xi\left(\frac{2^\alpha\|u\|_\alpha}{s^\alpha}\right)ds\right)^{1/q},
$$

which completes the proof.   □

Remark 2.5. To abbreviate some formulas, in the remainder of this paper Q is the constant $Q := N + 2^{\alpha+1/q}/|1 - q\alpha|^{1/q}$.

We can now establish our first result on the existence of a mild solution.

Theorem 2.6. *Assume that H_1, H_4 are satisfied and there are $r > 0$ and $\Theta \in (0, 1)$ such that*

$$
\Lambda = C_\alpha\left(\frac{a}{1-\alpha}\|f(\cdot, 0)\|_{C([0,a];X)} + \sum_{i=1}^N \|I_i(0)\|\right) < (1-\Theta)r, \tag{2.3}
$$

$$
C_\alpha\left(\|\mathcal{K}_f\|_{L^{q'}([0,a])}Q\frac{2^\alpha a^{1/q-\alpha}}{|1-q\alpha|^{1/q}}r + \sum_{i=1}^N \frac{\mathcal{K}_{I_i}(\frac{r}{\delta_{i-1}^\alpha})}{\delta_{i-1}^\alpha}\right) \le \Theta r. \tag{2.4}
$$

If $\|x_0\| < (r(1-\Theta) - \Lambda)/C_\alpha$, then there exists a unique mild solution $u \in \mathcal{PC}_\alpha(X_\alpha)$ of (1.1)–(1.3).

Proof. Let $\Gamma : \mathcal{PC}_\alpha(X_\alpha) \to \mathcal{PC}_\alpha(X_\alpha)$ be the map defined by

$$
\Gamma u(t) = T(t)x_0 + \int_0^t T(t-s)f(s, u(s))\,ds + \sum_{t_i < t} T(t-t_i)I_i(u(t_i)), \quad t \in (0, a].
$$

Next, we prove that the map Γ is a contraction on $B_r(0, \mathcal{PC}_\alpha(X_\alpha))$. To begin, we show that Γ has values in $B_r(0, \mathcal{PC}_\alpha(X_\alpha))$. Let $u \in B_r(0, \mathcal{PC}_\alpha(X_\alpha))$. By using Lemma 2.4,

for $t \in (t_j, t_{j+1}]$ we have that

$$
\begin{aligned}
\|(-A)^\alpha \Gamma u(t)\| &\leq \frac{C_\alpha}{t^\alpha} \|x_0\| + C_\alpha \int_0^t \frac{\mathcal{K}_f(s)\|u\|_\alpha}{(t-s)^\alpha (s-t_i)^\alpha} \, ds + C_\alpha \int_0^t \frac{\|f(s,0)\|}{(t-s)^\alpha} \, ds \\
&\quad + C_\alpha \sum_{i=1}^j \Big(\frac{\mathcal{K}_{I_i}(\frac{r}{\delta_{i-1}^\alpha})}{(t-t_i)^\alpha \delta_{i-1}^\alpha} \|u\|_\alpha^q + \frac{\|I_i(0)\|}{(t-t_i)^\alpha} \Big) \\
&\leq \frac{C_\alpha \|x_0\|}{(t-t_j)^\alpha} + \frac{C_\alpha \|\mathcal{K}_f\|_{L^{q'}([0,t_j])}}{(t-t_j)^\alpha} Q \Big(\int_0^a 2^{q\alpha} \frac{\|u\|_\alpha}{s^{q\alpha}} \, ds \Big)^{1/q} \\
&\quad + \frac{C_\alpha (t-t_j)^\alpha t^{1-\alpha}}{(t-t_j)^\alpha (1-\alpha)} \|f(\cdot,0)\|_{C([0,a];X)} \\
&\quad + \frac{C_\alpha}{(t-t_j)^\alpha} \sum_{i=1}^j \Big(\frac{\mathcal{K}_{I_i}(\frac{r}{\delta_{i-1}^\alpha})}{\delta_{i-1}^\alpha} \|u\|_\alpha + \|I_i(0)\| \Big),
\end{aligned}
$$

so that

$$
\begin{aligned}
\|\Gamma_n u\|_{\alpha,j} &\leq C_\alpha \Big(\|x_0\| + \|\mathcal{K}_f\|_{L^{q'}([0,t_j])} Q \frac{2^\alpha a^{1/q-\alpha}}{|1-q\alpha|^{1/q}} \|u\|_\alpha \Big) \\
&\quad + \frac{C_\alpha a}{1-\alpha} \|f(\cdot,0)\|_{C([0,a];X)} + C_\alpha \sum_{i=1}^j \Big(\frac{\mathcal{K}_{I_i}(\frac{r}{\delta_{i-1}^\alpha})}{\delta_{i-1}^\alpha} \|u\|_\alpha + \|I_i(0)\| \Big),
\end{aligned}
$$

from which

$$
\|\Gamma u\|_\alpha \leq C_\alpha \|x_0\| + \Theta r + C_\alpha \Big(\frac{a}{1-\alpha} \|f(\cdot,0)\|_{C([0,a];X)} + \sum_{i=1}^N \|I_i(0)\| \Big) \leq r
$$

and $\Gamma u \in B_r(0, \mathcal{PC}_\alpha(X_\alpha))$. Moreover, arguing as above, for $u, v \in B_r(0, \mathcal{PC}_\alpha(X_\alpha))$,

$$
\|\Gamma u - \Gamma v\|_{\alpha,j} \leq C_\alpha \Big(\|\mathcal{K}_f\|_{L^{q'}([0,a])} Q \frac{2^\alpha a^{1/q-\alpha}}{|1-q\alpha|^{1/q}} + \sum_{i=1}^N \frac{\mathcal{K}_{I_i}(\frac{r}{\delta_{i-1}^\alpha})}{\delta_{i-1}^\alpha} \Big) \|u-v\|_\alpha,
$$

which implies that $\|\Gamma u - \Gamma v\|_\alpha \leq \Theta \|u-v\|_\alpha$. Thus, Γ is a contraction and there exists a unique fixed point $v \in B_r(0, \mathcal{PC}_\alpha(X_\alpha))$ of $\Gamma(\cdot)$.

Let $u : [0,a] \to X$ be defined by $u(t) = v(t)$ for $t \neq 0$ and $u(0) = x_0$. To complete the proof, next we prove that $u(\cdot)$ is a mild solution of (1.1)–(1.3). It is easy to see that $u(\cdot)$ is left continuous on $[0,a]$ and that all the limits $u(t_i^+)$ exist. Thus, it remains to show that $\|\Gamma u\|_{\mathcal{PC}(X)}$ is finite. To prove this, we note that for $s \in (t_i, t_{i+1}]$,

$$
\begin{aligned}
\int_{t_i}^s \|f(\tau, u(\tau))\| d\tau &\leq \int_{t_i}^s \frac{\mathcal{K}_f(\tau)\|u\|_\alpha}{(\tau-t_i)^\alpha} \, d\tau + \int_{t_i}^s \|f(\tau,0)\| \, d\tau \\
&\leq \Upsilon := \|\mathcal{K}_f\|_{L^{q'}([0,a])} \frac{ra^{1/q-\alpha}}{(1-q\alpha)^{1/q}} + \|f(\cdot,0)\|_{L^1([0,a])}.
\end{aligned}
\tag{2.5}
$$

By using condition (2.4) and inequality (2.5), for $t \in (t_j, t_{j+1}]$ we see that

$$\|u(t)\| \leq C_0 \|x_0\| + C_0 \sum_{i=0}^{j-1} \int_{t_i}^{t_{i+1}} \|f(\tau, u(\tau))\| \, d\tau + C_0 \int_{t_j}^{t} \|f(\tau, u(\tau))\| \, d\tau$$

$$+ C_0 \sum_{i=1}^{j} \frac{\mathcal{K}_{I_i}(\frac{r}{\delta_{i-1}^\alpha}) r}{\delta_{i-1}^\alpha} + C_0 \sum_{i=1}^{j} \|I_i(0)\|$$

$$\leq C_0 \|x_0\| + C_0 N \Upsilon + \frac{C_0}{C_\alpha} r + C_0 \sum_{i=1}^{j} \|I_i(0)\|,$$

which implies that $\|u\|_{\mathcal{PC}(X)} < \infty$. The proof is complete. $\qquad \square$

Next, we establish some consequences of Theorem 2.6.

COROLLARY 2.7. *If the conditions H_1, H_4 hold and*

$$C_\alpha \left(Q \frac{2^\alpha a^{1/q-\alpha}}{|1 - q\alpha|^{1/q}} \|\mathcal{K}_f\|_{L^{q'}([0,a])} + \limsup_{l \to \infty} \frac{1}{l} \sum_{i=1}^{N} \frac{\mathcal{K}_{I_i}(\frac{l}{\delta_{i-1}^\alpha})}{\delta_{i-1}^\alpha} \right) < 1,$$

then there exists a unique mild solution $u \in \mathcal{PC}_\alpha(X_\alpha)$ of (1.1)–(1.3). In particular, if the functions $\mathcal{K}_f(\cdot)$, $\mathcal{K}_{I_i}(\cdot)$ are bounded and $C_\alpha Q(2^\alpha a^{1-\alpha}/|1 - q\alpha|^{1/q}) \|\mathcal{K}_f\|_{L^\infty([0,a])} < 1$, then there exists a unique mild solution $u \in \mathcal{PC}_\alpha(X_\alpha)$ for (1.1)–(1.3).

PROOF. Let Λ be the number introduced in (2.3). From the assumption, we can select numbers $r > 0$ and $\Theta \in (0, 1)$ such that

$$C_\alpha \left(Q \frac{2^\alpha a^{1/q-\alpha}}{|1 - q\alpha|^{1/q}} \|\mathcal{K}_f\|_{L^{q'}([0,a])} r + \sum_{i=1}^{N} \frac{\mathcal{K}_{I_i}(\frac{r}{\delta_{i-1}^\alpha})}{\delta_{i-1}^\alpha} \right) < \Theta r,$$

$\Lambda < (1 - \Theta) r$ and $\|x_0\| < (r(1 - \Theta) - \Lambda)/C_\alpha$. Now, the assertion follows from Theorem 2.6. $\qquad \square$

REMARK 2.8. It is convenient to include some comments on Theorem 2.6. The assumption on the existence of r is a restriction, since under this condition the result is only applicable for a initial condition in a neighbourhood of zero. However, this restriction is natural and predicable for two reasons.

(a) Our abstract approach is designed to study partial differential equations with impulsive conditions involving nonlinear expressions of the solution. Note that if some of the functions I_i are nonlinear and $\lim_{s \to \infty} \mathcal{K}_{I_i}(s) s^{-\gamma} = \infty$ for some $\gamma > 1$ (respectively, $\lim_{s \to 0} \mathcal{K}_{I_i}(s) s^{-\gamma} = \infty$ for some $\gamma < 1$), then (2.4) is not satisfied for r large (respectively, small).

(b) To preserve the basic characteristic of the problem (the presence of impulses), we need to establish the existence of a mild solution on an interval containing the times t_i. In problems without impulses, the dependence on r can be eliminated by studying the existence of solutions local in time.

To prove our main result (Theorem 2.13), we consider an approximation scheme based on the solvability of the associated abstract integral equations

$$u(t) = T(t)x_0 + \int_0^t T(t-s)f(s, u(s))\, ds + T\Big(\frac{1}{n}\Big) \sum_{t_i < t} T(t-t_i)I_i(u(t_i)), \quad n \in \mathbb{N}. \quad (2.6)$$

To begin, we study the existence of solutions for the integral equation (2.6).

THEOREM 2.9. *Assume that H_2 and H_3 are satisfied, $(T(t))_{t \geq 0}$ is compact, the function $\tau \to W_f^q(1/\tau^\alpha)$ is integrable on $[0, a]$,*

$$\lim_{h \downarrow 0} \|m_f\|_{L^{q'}([c,c+h])} \frac{1}{h^\alpha} \left(\int_0^h W_f^q\Big(\frac{1}{s^\alpha}\Big)\, ds \right)^{1/q} = 0$$

for all $c \in [0, a)$, $\lim_{h \to 0} \|m_f\|_{L^{q'}([d-h,d])}(1/h^\alpha)(\int_0^h W_f^q(1/s^\alpha)\, ds)^{1/q} = 0$ uniformly for d in compact subsets of $(0, a]$, and there is $r > 0$ such that

$$C_\alpha\Big(\|x_0\| + Q\|m_f\|_{L^{q'}([0,a])}\Big(\int_0^a W_f^q\Big(\frac{2^\alpha r}{s^\alpha}\Big)\, ds\Big)^{1/q} + C_0 \sum_{i=1}^N \Big(\frac{c_i(\frac{r}{\delta_{i-1}^\alpha})r}{\delta_{i-1}^\alpha} + d_i\Big)\Big) \leq r. \quad (2.7)$$

Then, for all $n \in \mathbb{N}$, there exists a solution $u_n \in B_r(0, \mathcal{PC}_\alpha(X_\alpha)) \bigcap \mathcal{PC}(X)$ of (2.6).

PROOF. For $n \in \mathbb{N}$, we define the map $\Gamma_n : \mathcal{PC}_\alpha(X_\alpha) \to \mathcal{PC}_\alpha(X_\alpha)$ by

$$\Gamma_n u(t) = T(t)x_0 + \int_0^t T(t-s)f(s, u(s))\, ds + T\Big(\frac{1}{n}\Big) \sum_{t_i < t} T(t-t_i)I_i(u(t_i)).$$

Next, we prove that Γ_n is a condensing map on $B_r(0, \mathcal{PC}_\alpha(X_\alpha))$. To begin, we show that Γ has values in $B_r(0, \mathcal{PC}_\alpha(X_\alpha))$. Let $u \in B_r(0, \mathcal{PC}_\alpha(X_\alpha))$. From Lemma 2.4, for $t \in (t_j, t_{j+1}]$,

$$\|(-A)^\alpha \Gamma_n u(t)\| \leq \frac{C_\alpha}{t^\alpha}\|x_0\| + C_\alpha \int_0^t \frac{m_f(s)}{(t-s)^\alpha} W_f(\|(-A)^\alpha u(s)\|)\, ds$$

$$+ \sum_{i=1}^j \frac{C_\alpha C_0}{(t-t_i)^\alpha}\Big(\frac{c_i(\frac{r}{\delta_{i-1}^\alpha})r}{\delta_{i-1}^\alpha} + d_i\Big)$$

$$\leq \frac{C_\alpha\|x_0\|}{(t-t_j)^\alpha} + \frac{\|m_f\|_{L^{q'}([0,a])}}{(t-t_j)^\alpha} Q\Big(\int_0^a W_f\Big(\frac{2^\alpha r}{s^\alpha}\Big)\, ds\Big)^{1/q}$$

$$+ \frac{C_\alpha C_0}{(t-t_j)^\alpha} \sum_{i=1}^j \Big(\frac{c_i(\frac{r}{\delta_{i-1}^\alpha})r}{\delta_{i-1}^\alpha} + d_i\Big),$$

and hence,

$$\|\Gamma_n u\|_{\alpha,j} \leq C_\alpha\Big(\|x_0\| + \|m_f\|_{L^{q'}([0,a])} Q\Big(\int_0^a W_f\Big(\frac{2^\alpha r}{s^\alpha}\Big)\, ds\Big)^{1/q}\Big) \quad (2.8)$$

$$+ C_\alpha C_0 \sum_{i=1}^j \Big(\frac{c_i(\frac{r}{\delta_{i-1}^\alpha})r}{\delta_{i-1}^\alpha} + d_i\Big), \quad (2.9)$$

which implies that $\|\Gamma_n u\|_\alpha \leq r$ and $\Gamma_n B_r(0, \mathcal{PC}_\alpha(X_\alpha)) \subset B_r(0, \mathcal{PC}_\alpha(X_\alpha))$.

To show that the map Γ_n is completely continuous, we introduce the decomposition $\Gamma_n = \Gamma^1 + \Gamma^2 + \Gamma_n^3$ where $\Gamma^1 u(t) = T(t)x_0$, $\Gamma^2 u(t) = \int_0^t T(t-s)f(s, u(s))\, ds$ and $\Gamma_n^3 u(t) = T(1/n) \sum_{t_i < t} T(t-t_i)I_i(u(t_i))$. The continuity of the map $\Gamma_n(\cdot)$ is proved by a standard procedure. We divide the remainder of the proof into several steps. Next, we use the notation $\Gamma^i B_r(0, \mathcal{PC}_\alpha(X_\alpha))(s) = \{\Gamma^i u(s) : u \in B_r(0, \mathcal{PC}_\alpha(X_\alpha))\}$.

Step 1. For $t_j < c < d \leq t_{j+1}$, the set $\cup_{s \in [c,d]} \Gamma^2 B_r(0, \mathcal{PC}_\alpha(X_\alpha))(s)$ is relatively compact in X_α.

Let $t_j < \mu < c$. From Lemmas 2.3 and 2.4, for $t \in [c, d]$, $u \in B_r(0, \mathcal{PC}_\alpha(X_\alpha))$ and $0 < \varepsilon < (c - \mu)/2$,

$$
\|(-A)^\alpha \Gamma^2 u(t)\| \leq C_\alpha \int_0^t \frac{m_f(s)}{(t-s)^\alpha} W_f(\|(-A)^\alpha u(s)\|)\, ds
$$

$$
\leq r_1 := \frac{C_\alpha \|m_f\|_{L^{q'}([0,a])}}{(\mu - t_j)^\alpha} Q\left(\int_0^a W_f\left(\frac{2^\alpha r}{\tau^\alpha} \right) d\tau \right)^{1/q},
$$

$$
\left\| \int_{t-\varepsilon}^t (-A)^\alpha T(t-\tau)f(\tau, u(\tau))\, ds \right\|
$$

$$
\leq C_\alpha \int_0^t \frac{m_f(s)}{(t-s)^\alpha} W_f(\|(-A)^\alpha u(s)\|)\, ds
$$

$$
\leq \frac{C_\alpha 2^{\alpha+1/q}}{|1-q\alpha|^{1/q}} \|m_f\|_{L^{q'}([t-\varepsilon,t])} \frac{1}{\varepsilon^\alpha} \left(\int_0^{\varepsilon/2} W_f^q\left(\frac{2^\alpha r}{\tau^\alpha} \right) d\tau \right)^{1/q}
$$

$$
\leq r_\varepsilon := \frac{C_\alpha 2^{\alpha+1/q}}{|1-q\alpha|^{1/q}} \sup_{s \in [\mu,d]} \|m_f\|_{L^{q'}([s-\varepsilon,s])} \frac{1}{\varepsilon^\alpha} \left(\int_0^\varepsilon W_f^q\left(\frac{2^\alpha r}{\tau^\alpha} \right) d\tau \right)^{1/q}.
$$

By noting that the above estimates are independent of $t \in [c, d]$, from the decomposition

$$
(-A)^\alpha \Gamma^2 u(s) = T(\varepsilon) \int_0^{s-\varepsilon} (-A)^\alpha T(s-\varepsilon-\tau)f(\tau, u(\tau))\, d\tau
$$

$$
+ \int_{s-\varepsilon}^s (-A)^\alpha T(s-\tau)f(\tau, u(\tau))\, d\tau
$$

we infer that $\cup_{s \in [c,d]}(-A)^\alpha \Gamma^2 B_r(0, \mathcal{PC}_\alpha(X_\alpha))(s) \subset T(\varepsilon)B_{r_1}(0, X) + B_{2r_\varepsilon}(0, X)$, which proves the assertion since $T(\varepsilon)B_{r_1}(0, X)$ is compact in X and $r_\varepsilon \to 0$ as $\varepsilon \to 0$.

Step 2. $\Gamma^2 B_r(0, \mathcal{PC}_\alpha(X_\alpha)) = \{\Gamma^2 u : u \in B_r(0, \mathcal{PC}_\alpha(X_\alpha))\}$ is an equicontinuous subset of $C([0, a]; X_\alpha)$.

Let $t \in (0, a]$. To begin, we assume that $t \in (t_j, t_{j+1})$. Since $T(\cdot) \in C((0, a], \mathcal{L}(X))$ and $\Gamma^2 B_r(0, \mathcal{PC}_\alpha(X_\alpha))(t)$ is relatively compact in X, for given $\varepsilon > 0$ we can select $0 < \delta \leq \min\{t_{j+1} - t\}$ such that $\|(T(\theta) - I)x\| \leq \varepsilon$ for every $x \in \Gamma^2 B_r(0, \mathcal{PC}_\alpha(X_\alpha))(t)$ and

all $0 < \theta \leq \delta$. Then, for $0 < h < \delta$ and $u \in B_r(0, \mathcal{PC}_\alpha(X_\alpha))$,

$$\|(-A)^\alpha \Gamma^2 u(t + h) - (-A)^\alpha \Gamma^2 u(t)\|$$

$$\leq \|(T(h) - I)(-A)^\alpha \Gamma^2 u(t)\| + \int_t^{t+h} \|(-A)^\alpha T(t + h - s) f(s, u(s))\| \, ds$$

$$\leq \sup\{\|(T(h) - I)(-A)^\alpha \Gamma^2 v(t)\| : v \in B_r(0, \mathcal{PC}_\alpha(X_\alpha))\}$$

$$+ C_\alpha \int_t^{t+h} \frac{m_f(s)}{(t + h - s)^\alpha} W_f\left(\frac{\|u\|_\alpha}{(s - t)^\alpha}\right) ds$$

$$\leq \varepsilon + \frac{C_\alpha 2^{\alpha + 1/q} \|m_f\|_{L^{q'}([t, t+h])}}{|1 - q\alpha|^{1/q}} \frac{1}{h^\alpha} \left(\int_0^{h/2} W_f^q\left(\frac{2^\alpha r}{s^\alpha}\right) ds\right)^{1/q},$$

which shows that $(-A)^\alpha \Gamma^2 B_r(0, \mathcal{PC}_\alpha(X_\alpha))$ is right equicontinuous at t.

We now show the left equicontinuity for the case where $t \in (t_j, t_{j+1}]$. Let $t_j < \mu < c < t$. By noting that the set $\cup_{s \in [\mu, t]} \Gamma^2 B_r(0, \mathcal{PC}_\alpha(X_\alpha))(s)$ is relatively compact in X_α (see Step 1), for $\varepsilon > 0$ there exists $0 < \delta < (c - \mu)/2$ such that $\|(T(\theta) - I)x\| \leq \varepsilon$ for every $x \in \cup_{s \in [\mu, t]} \Gamma^2 B_r(0, \mathcal{PC}_\alpha(X_\alpha))(s)$ and all $0 < \theta \leq \delta$. Then, for $0 < h < \delta$ and $u \in B_r(0, \mathcal{PC}_\alpha(X_\alpha))$,

$$\|(-A)^\alpha \Gamma^2 u(t - h) - (-A)^\alpha \Gamma^2 u(t)\|$$

$$\leq \|(I - T(h))(-A)^\alpha \Gamma^2 u(t - h)\| + \int_{t-h}^t \|(-A)^\alpha T(t - s) f(s, u(s))\| \, ds$$

$$\leq \sup\{\|(I - T(h))(-A)^\alpha \Gamma^2 v(s)\| : s \in [\mu, t], v \in B_r(0, \mathcal{PC}_\alpha(X_\alpha))\}$$

$$+ C_\alpha \int_{t-h}^t \frac{m_f(s)}{(t - s)^\alpha} W_f\left(\frac{\|u\|_\alpha}{(s - (t - h))^\alpha}\right) ds$$

$$\leq \varepsilon + \frac{C_\alpha 2^{\alpha + 1/q}}{(1 - q\alpha)^{1/q}} \|m_f\|_{L^{q'}([t-h, t]; \mathbb{R})} \frac{1}{h^\alpha} \left(\int_0^h W_f^q\left(\frac{2^\alpha r}{s^\alpha}\right) ds\right)^{1/q},$$

which shows the left equicontinuity at t.

Step 3. For $t_j < c < d \leq t_{j+1}$, the set $\cup_{s \in [c, d]} \Gamma_n^3 B_r(0, \mathcal{PC}_\alpha(X_\alpha))(s)$ is relatively compact in X_α.

The case where $j = 0$ is obvious. Assume that $j \geq 1$ and let $t_j < \mu < c$. For $u \in B_r(0, \mathcal{PC}_\alpha(X_\alpha))$ and $s \in [\mu, d]$ we see that

$$\|(-A)^\alpha \Gamma_n^3 u(s)\| \leq r^* := \frac{C_\alpha C_0}{(\mu - t_j)^\alpha} \sum_{i=1}^j \left(\frac{c_i(\frac{r}{\delta_{i-1}^\alpha})r}{\delta_{i-1}^\alpha} + d_i\right),$$

and hence, $\cup_{\tau \in [\mu, d]} (-A)^\alpha \Gamma_n^3 B_r(0, \mathcal{PC}_\alpha(X_\alpha))(\tau) \subset B_{r^*}(0, X)$. Thus, for $t \in [c, d]$, $u \in B_r(0, \mathcal{PC}_\alpha(X_\alpha))$ and $0 < \varepsilon < (c - \mu)/2$,

$$(-A)^\alpha \Gamma_n^3 u(t) = T(\varepsilon)(-A)^\alpha \Gamma_n^3 u(t - \varepsilon) \subset T(\varepsilon) B_{r^*}(0, X),$$

from which we infer that $\cup_{s \in [c, d]} (-A)^\alpha \Gamma_n^3 B_r(0, \mathcal{PC}_\alpha(X_\alpha))(s) \subset T(\varepsilon) B_{r^*}(0, X)$. This proves the assertion since $T(\varepsilon)$ is compact.

Step 4. The set of functions $\Gamma_n^3 B_r(0, \mathcal{PC}_\alpha(X_\alpha)) = \{\Gamma_n^3 u : u \in B_r(0, \mathcal{PC}_\alpha(X_\alpha))\}$ is equicontinuous at $t \neq t_i$ and left equicontinuous $t = t_i$ in the norm of X_α.

The case $t \in (0, t_1]$ is trivial. Assume that $t \in (t_j, t_{j+1})$ for some $j \geq 1$. By using the fact that $(-A)^\alpha \Gamma_n^3 B_r(0, \mathcal{PC}_\alpha(X_\alpha))(t)$ is relatively compact in X, for $\varepsilon > 0$ we select $0 < \delta < t_{j+1} - t$ such that $\|(T(\theta) - I)x\| \leq \varepsilon$ for all $x \in (-A)^\alpha \Gamma_n^3 B_r(0, \mathcal{PC}_\alpha(X_\alpha))(t)$ and every $0 < \theta \leq \delta$. Then, for $u \in B_r(0, \mathcal{PC}_\alpha(X_\alpha))$ and $0 < h < \delta$,

$$\begin{aligned}
\|(-A)^\alpha \Gamma_n^3 u(t + h) - (-A)^\alpha \Gamma_n^3 u(t)\| &= \|(T(h) - I)(-A)^\alpha \Gamma_n^3 u(t)\| \\
&\leq \sup\{(T(h) - I)(-A)^\alpha \Gamma_n^3 v(s)\| : v \in B_r(0, \mathcal{PC}_\alpha(X_\alpha))\} \\
&\leq \varepsilon,
\end{aligned}$$

which shows that $(-A)^\alpha \Gamma_n^3 B_r(0, \mathcal{PC}_\alpha(X_\alpha))$ is right equicontinuous at t.

We now prove left equicontinuity for the case where $t \in (t_j, t_{j+1}]$. Let $t_j < \mu < c < t$. Since the set $\cup_{s \in [\mu, t]} (-A)^\alpha \Gamma_n^3 B_r(0, \mathcal{PC}_\alpha(X_\alpha))(s)$ is relatively compact in X, for $\varepsilon > 0$ we choose $0 < \delta < (c - \mu)/2$ such that $\|(T(\theta) - I)x\| \leq \varepsilon$ for all $x \in \cup_{s \in [\mu, t]} (-A)^\alpha \Gamma_n^3 B_r(0, \mathcal{PC}_\alpha(X_\alpha))(s)$ and each $0 < \theta < \delta$. Under these conditions, for $u \in B_r(0, \mathcal{PC}_\alpha(X_\alpha))$ and $0 < h < \delta$,

$$\begin{aligned}
\|(-A)^\alpha \Gamma_n^3 u(t - h) - (-A)^\alpha \Gamma_n^3 u(t)\| &= \|(I - T(h))(-A)^\alpha \Gamma_n^3 u(t - h)\| \\
&\leq \sup\{\|(I - T(h))(-A)^\alpha \Gamma_n^3 v(s)\| : s \in [\mu, t], v \in B_r(0, \mathcal{PC}_\alpha(X_\alpha))\} \\
&\leq \varepsilon,
\end{aligned}$$

which proves the left equicontinuity at t. This completes the proof that $\Gamma_n^3 B_r(0, \mathcal{PC}_\alpha(X_\alpha))$ is equicontinuous at $t \neq t_i$.

Step 5. For every $j \in \{1, \ldots, N\}$, $\lim_{t \downarrow t_j} (t - t_j)^\alpha (-A)^\alpha \Gamma_n^3 u(t) = 0$ uniformly for $u \in B_r(0, \mathcal{PC}_\alpha(X_\alpha))$.

For $u \in B_r(0, \mathcal{PC}_\alpha(X_\alpha))$ and $t \in (t_j, t_{j+1})$,

$$(t - t_j)^\alpha \|(-A)^\alpha \Gamma_n^3 u(t)\| \leq \frac{(t - t_j)^\alpha}{(t - t_j + \frac{1}{n})^\alpha} C_\alpha \sum_{i=1}^N \left(\frac{c_i(\frac{r}{\delta_{i-1}^\alpha})r}{\delta_{i-1}^\alpha} + d_i \right),$$

which proves the assertion since $(t - t_j)^\alpha / (t - t_j + 1/n)^\alpha \to 0$ as $t \downarrow t_j$.

From Steps 1–5 and Lemma 1.2 it follows that Γ_n is completely continuous from $B_r(0, \mathcal{PC}_\alpha(X_\alpha))$ into $B_r(0, \mathcal{PC}_\alpha(X_\alpha))$ and from Schauder's fixed point criteria there exists a fixed point v_n of Γ_n. Let $u_n : [0, a] \to X$ be defined by $u_n(0) = x_0$ and $u_n(t) = v_n(t)$ for $t \in (0, a]$. It is easy to see that $u_n(\cdot)$ is a solution of the integral equation (2.6) and $u(\cdot)$ is continuous at zero. In order to prove that $u_n \in \mathcal{PC}(X)$, we note that

$$\int_{t_i}^{t_{i+1}} \|f(\tau, u_n(\tau))\| \, d\tau \leq \mathfrak{L} := \|m_f\|_{L^{q'}([0,a])} \left(\int_0^a W_f^q \left(\frac{r}{\tau^\alpha} \right) d\tau \right)^{1/q}, \tag{2.10}$$

for all $i \in \{0, \dots, N\}$. Using (2.10) and (2.7), for $t \in (t_j, t_{j+1}]$ it follows that

$$\|u_n(t)\| \leq C_0\|x_0\| + C_0 \sum_{i=0}^{j} \int_{t_i}^{t_{i+1}} \|f(\tau, u_n(\tau))\| \, d\tau + C_0^2 \sum_{i=1}^{j} \left(\frac{c_i(\frac{r}{\delta_{i-1}^\alpha})r}{\delta_{i-1}^\alpha} + d_i \right)$$

$$\leq C_0\|x_0\| + C_0 N \mathfrak{L} + \frac{C_0}{C_\alpha} r, \tag{2.11}$$

which implies that $\|u_n\|_{\mathcal{PC}(X)} \leq C_0(\|x_0\| + N\mathfrak{L} + C_0 r/C_\alpha)$ and $u_n \in \mathcal{PC}(X)$. This completes the proof. □

REMARK 2.10. In the remainder of this section, we always assume that the assumptions in Theorem 2.9 are fulfilled. Next, for $n \in \mathbb{N}$ we use the notation u_n to represent a given solution of the integral equation (2.6). For convenience, we rewrite the functions u_n in the form $u_n = \sum_{i=1}^{3} u_n^i$ where $u_n^1(t) = T(t)x_0$, $u_n^2(t) = \int_0^t T(t-s)f(s, u_n(s)) \, ds$ and $u_n^3(t) = T(1/n) \sum_{t_i < t} T(t - t_i)I_i(u_n(t_i))$.

We now consider some properties of the set $\{u_n : n \in \mathbb{N}\}$. The proof of Lemma 2.11 can be deduced from the proof of Theorem 2.9. We include it for completeness.

LEMMA 2.11. *If the assumptions in Theorem 2.9 are satisfied, then*
(a)   *the set $\{u_n^i : i = 1, 2, 3, n \in \mathbb{N}\}$ is bounded in $\mathcal{PC}(X)$ and $\{u_n^i : i = 1, 2, 3, n \in \mathbb{N}\} \subset B_r(0, \mathcal{PC}_\alpha(X_\alpha))$,*
(b)   *the set $\{u_n^2 : n \in \mathbb{N}\}$ is relatively compact in $C([0, a]; X_\alpha)$,*
(c)   *for $j \in \{1, \dots, N\}$ and $b \in (t_j, t_{j+1}]$, the set $\{u_{n_{|[b,t_{j+1}]}}^3 : n \in \mathbb{N}\}$ is relatively compact in $C([b, t_{j+1}]; X_\alpha)$.*

PROOF. By noting that the number r in the statement of Theorem 2.9 is independent of $n \in \mathbb{N}$, from (2.9) we infer that

$$\|u_n\|_{\alpha,j} \leq \|u_n^1\|_{\alpha,j} + \|u_n^2\|_{\alpha,j} + \|u_n^3\|_{\alpha,j}$$

$$\leq C_\alpha\|x_0\| + C_\alpha Q\|m_f\|_{L^{q'}([0,a])} \left(\int_0^a W_f\left(\frac{2^\alpha r}{s^\alpha}\right) ds \right)^{1/q}$$

$$+ C_\alpha C_0 \sum_{i=1}^{j} \left(\frac{c_i(\frac{r}{\delta_{i-1}^\alpha})r}{\delta_{i-1}^\alpha} + d_i \right),$$

which from condition (2.7) implies that $\{u_n^i : i = 1, 2, 3, \ n \in \mathbb{N}\} \subset B_r(0, \mathcal{PC}_\alpha(X_\alpha))$. Moreover, since the estimates (2.10) and (2.11) are independent of $n \in \mathbb{N}$, from (2.11) we have

$$\|u_n\|_{\mathcal{PC}(X)} \leq \|u_n^1\|_{\mathcal{PC}(X)} + \|u_n^2\|_{\mathcal{PC}(X)} + \|u_n^3\|_{\mathcal{PC}(X)} \leq C_0\|x_0\| + C_0 N \mathfrak{L} + \frac{C_0^2}{C_\alpha} r,$$

where $\mathfrak{L} := \|m_f\|_{L^{q'}([0,a])} (\int_0^a W_f^q(r/\tau^\alpha) \, d\tau)^{1/q}$. This completes the proof of (a).

We now prove (b). Let Γ^2 be the map introduced in the proof of Theorem 2.9. From the estimate (2.9) we infer that $\Gamma^2 B_r(0, \mathcal{PC}_\alpha(X_\alpha)) \subset B_r(0, \mathcal{PC}_\alpha(X_\alpha))$, which from

Steps 1 and 2 permits us to conclude that $\Gamma^2 B_r(0, \mathcal{PC}_\alpha(X_\alpha))$ is relatively compact in $C([0, a]; X_\alpha)$. Since the definition of Γ^2 is independent of n, $u_n^2 = \Gamma^2 u_n$ for all $n \in \mathbb{N}$ and $\{u_n^2 : n \in \mathbb{N}\} \subset B_r(0, \mathcal{PC}_\alpha(X_\alpha))$, we have that $\{u_n^2 : n \in \mathbb{N}\} \subset \Gamma^2 B_r(0, \mathcal{PC}_\alpha(X_\alpha))$ which implies that $\{u_n^2 : n \in \mathbb{N}\}$ is relatively compact in $C([0, a]; X_\alpha)$. This proves (b).

We divide the proof of (c) into two steps.

Step 1. For $b \in (t_j, t_{j+1}]$, the set $\{u_n^3(s) : n \in \mathbb{N}, \ s \in [b, t_{j+1}]\}$ is relatively compact in X_α.

Let $t \in [b, t_{j+1}]$ and $0 < \vartheta \leq (b - t_j)/2$. Since $\{u_n : n \in \mathbb{N}\} \subset B_r(0, \mathcal{PC}_\alpha(X_\alpha))$, from the definition of the norm in $\mathcal{PC}_\alpha(X_\alpha)$ we have that $\|u_n(s - \vartheta)\|_\alpha \leq r/(s - \vartheta - t_j)^\alpha \leq \mu := 2^\alpha r/(b + t_j)^\alpha$ for all $s \in [b, t_{j+1}]$. Using this fact, for $n \in \mathbb{N}$ and $t \in [b, t_{j+1}]$,

$$(-A)^\alpha u_n^3(t) = T(\vartheta)(-A)^\alpha T\left(\frac{1}{n}\right) \sum_{i=1}^{j} T(t - \vartheta - t_i) I_i(u(t_i))$$

$$= T(\vartheta)(-A)^\alpha u_n^3(t - \vartheta) \in T(\vartheta) B_\mu(0, X),$$

which implies that $\{(-A)^\alpha u_n^3(s) : n \in \mathbb{N}, \ s \in [b, t_{j+1}]\} \subset T(\vartheta) B_\mu(0, X)$. This proves the assertion since $T(\vartheta)$ is compact.

Step 2. For every $b \in (t_j, t_{j+1}]$, the set $\{u_n^3 : n \in \mathbb{N}\}$ is an equicontinuous subset of $C([b, t_{j+1}]; X_\alpha)$.

Assume that $t \in [b, t_{j+1})$. By using the fact that $\{(-A)^\alpha u_n^3(t) : n \in \mathbb{N}\}$ is relatively compact in X, for given $\varepsilon > 0$ we select $0 < \delta < t_{j+1} - t$ such that $\|T(s)x - T(s')x\| \leq \varepsilon$ for all $x \in \{(-A)^\alpha u_n^3(t) : n \in \mathbb{N}\}$ and every $s, s' \in [0, a]$ with $0 < |s - s'| < \delta$. Then, for $0 < h < \delta$ and $n \in \mathbb{N}$,

$$\|(-A)^\alpha u_n^3(t + h) - (-A)^\alpha u_n^3(t)\| = \|(T(h) - I)(-A)^\alpha u_n^3(t)\| \leq \varepsilon,$$

which shows that $\{u_n^3 : n \in \mathbb{N}\}$ is right equicontinuous at $t \in [b, t_{j+1})$.

We now prove the left equicontinuity for the case where $t \in (b, t_{j+1}]$. Since the set $\{(-A)^\alpha u_n^3(s) : n \in \mathbb{N}, \ s \in [(b + t_j)/2, t_{j+1}]\}$ is relatively compact in X (see Step 1), for $\varepsilon > 0$ there exists $0 < \delta < (b - t_j)/2$ such that $\|(T(s) - I)x\| \leq \varepsilon$ for all $x \in \{(-A)^\alpha u_n^3(s) : n \in \mathbb{N}, \ t \in [(b + t_j)/2, t_{j+1}]\}$ and each $0 < s < \delta$. Then, for $0 < h < \delta$ and $p \in \mathbb{N}$ it follows that

$$\|(-A)^\alpha u_p^3(t - h) - (-A)^\alpha u_p^3(t)\| = \|(I - T(h))(-A)^\alpha u_n^3(t - h)\|$$

$$\leq \sup\left\{\|(I - T(h))(-A)^\alpha u_n^3(s)\| : n \in \mathbb{N}, \ s \in \left[\frac{b + t_j}{2}, t_{j+1}\right]\right\} \leq \varepsilon,$$

which proves that $\{u_n^3 : n \in \mathbb{N}\}$ is left equicontinuous at t.

Finally, from Steps 1 and 2 it follows that $\{u_{n|_{[b,t_{j+1}]}}^3 : n \in \mathbb{N}\}$ is relatively compact in $C([b, t_{j+1}], X_\alpha)$. $\qquad\square$

In the next lemma, we use the notation introduced in (1.4).

LEMMA 2.12. *If the assumptions in Theorem 2.9 are fulfilled, then $\{u_n^3 : n \in \mathbb{N}\}$ is relatively compact in $\mathcal{PC}(X)$.*

PROOF. To prove this result, we use Lemma 1.1. From Lemma 2.11 we know that $\{u_n^i, u_n : i = 1, 2, 3, \ n \in \mathbb{N}\}$ is bounded in $\mathcal{PC}(X)$ and $\{u_{n|[b,t_{j+1}]}^3 : n \in \mathbb{N}\}$ is relatively compact in $C([b, t_{j+1}]; X)$ for every j and all $b \in (t_j, t_{j+1}]$. Thus, to complete the proof it is sufficient to show that $\{\widetilde{(u_n^3)}_j : n \in \mathbb{N}\}$ is right equicontinuous at t_j and $\{\widetilde{(u_n^3)}_j(t_j) : n \in \mathbb{N}\}$ is relatively compact in X for all $j \in \{1, \ldots, N\}$.

Let $j \in \{1, \ldots, N\}$. From Lemma 2.11(b) and (c), the set $P = \{u_n(t_i) : i = 1, \ldots, N, n \in \mathbb{N}\}$ is relatively compact in X, which implies that $Q = \{T(1/n)x : n \in \mathbb{N}, x \in \cup_{i=1}^N I_i(P)\}$ is also relatively compact in X. In particular, there exists $R > 0$ such that $Q \subset B_R(0, X)$.

Step 1. The set $\{\widetilde{(u_n^3)}_j(t_j) : n \in \mathbb{N})\}$ is relatively compact in X.

For $p \in \mathbb{N}$, we have that

$$\widetilde{(u_p^3)}_j(t_j) = \sum_{i=1}^{j-1} T(t_j - t_i)T\left(\frac{1}{p}\right)I_i(u_p(t_i)) + T\left(\frac{1}{p}\right)I_j(u_p(t_j)) \in \sum_{i=1}^{j-1} T(t_j - t_i)B_R(0, X) + Q,$$

and hence $\{\widetilde{(u_n^3)}_j(t_j) : n \in \mathbb{N})\} \subset \sum_{i=1}^{j-1} T(t_j - t_i)B_R(0, X) + Q$. This proves the assertion since the operators $T(t_j - t_i)$ are compact.

Step 2. The set $\{\widetilde{(u_n^3)}_j : n \in \mathbb{N})\}$ is right equicontinuous at $t = t_j$ in the norm of X.

Since $\{\widetilde{(u_n^3)}_j(t_j) : n \in \mathbb{N})\}$ is relatively compact in X, for $\varepsilon > 0$ there exists $0 < \delta < t_{j+1} - t_j$ such that $\|(T(s) - I)x\| \leq \varepsilon$ for all $x \in \{\widetilde{(u_n^3)}_j(t_j) : n \in \mathbb{N}\}$ and each $0 < s < \delta$. Then, for $0 < h < \delta$ and $p \in \mathbb{N}$,

$$\begin{aligned}
\|\widetilde{(u_p^3)}_j(t_j + h) - \widetilde{(u_p^3)}_j(t_j)\| &= \left\| T(h) \sum_{i=1}^{j} T(t_j - t_i)T\left(\frac{1}{p}\right)I_i(u_p(t_i)) - \widetilde{(u_p^3)}_j(t_j) \right\| \\
&= \|(T(h) - I)\widetilde{(u_p^3)}_j(t_j)\| \\
&\leq \sup_{n \in \mathbb{N}} \|(T(h) - I)\widetilde{(u_n^3)}_j(t_j)\| \\
&\leq \varepsilon,
\end{aligned}$$

which shows that $\{\widetilde{(u_n^3)}_j : n \in \mathbb{N})\}$ is right equicontinuous at $t = t_j$.

From Steps 1 and 2 we infer that $\{\widetilde{(u_n^3)}_j : n \in \mathbb{N})\}$ is relatively compact in $C([t_j, t_{j+1}], X)$. This completes the proof. □

We are now in a position to prove the main result of this work.

THEOREM 2.13. *Suppose that the assumptions in Theorem 2.9 are fulfilled. Then there exists a mild solution $u \in \mathcal{PC}_\alpha(X_\alpha)$ of the impulsive problem (1.1)–(1.3).*

Proof. From Lemmas 2.11 and 2.12 there exist a subsequence $(u_{n_i})_{i \in \mathbb{N}}$ of $(u_n)_{n \in \mathbb{N}}$ and $u \in \mathcal{PC}(X)$ such that $\|u_{n_i} - u\|_{\mathcal{PC}(X)} \to 0$ as $i \to \infty$ and $u_{n_i|_{[b, t_j]}} \to u$ in $C([b, t_{j+1}], X_\alpha)$ for all $j \in \{0, \ldots, N\}$ and every $b \in (t_j, t_{j+1})$. Moreover, from Lemma 2.11 we know that $(t - t_j)^\alpha \|(-A)^\alpha u_n(t)\| \leq r$ for all $n \in \mathbb{N}$, $j \in \{0, \ldots, N\}$ and each $t \in (t_j, t_{j+1}]$, which implies that $(t - t_j)^\alpha \|(-A)^\alpha u(t)\| \leq r$ for all $t \in (t_j, t_{j+1}]$ and every $j \in \{0, \ldots, N\}$. This proves that $u \in B_r(0, \mathcal{PC}_\alpha(X_\alpha))$. Finally, a standard application of the Lebesgue dominated convergence theorem allows us to conclude that $u(\cdot)$ is a mild solution. □

Next, we consider two easy consequences of Theorem 2.13.

Corollary 2.14. *Assume that H_2 and H_3 are satisfied, the semigroup $(T(t))_{t \geq 0}$ is compact and there are positive constants $\gamma \in (0, 1/\alpha q)$, $\mu_i, \tilde{c}_i$ $i = 1, \ldots, N$, such that $W_f(s) = s^\gamma$ and $c_i(s) = \tilde{c}_i s^{\mu_i}$ for all $s \geq 0$ and every $i = 1, \ldots, N$. Suppose that $\lim_{h \downarrow 0} \|m_f\|_{L^{q'}([d, d+h])} h^{1/q - \alpha(\gamma+1)} = 0$ for all $d \in [0, a)$ and $\lim_{h \downarrow 0} \|m_f\|_{L^{q'}([c-h, c])} h^{1/q - \alpha(\gamma+1)} = 0$ uniformly for c in compact subsets of $(0, a]$ and there is $r > 0$ such that*

$$C_\alpha \left(\|x_0\| + Q\|m_f\|_{L^{q'}([0,a])} \frac{2^{\alpha\gamma} a^{1/q - \alpha\gamma}}{(1 - q\alpha\gamma)^{1/q}} r^\gamma + C_0 \sum_{i=1}^{N} \left(\frac{\tilde{c}_i r^{\mu_i + 1}}{\delta_{i-1}^{\alpha(\mu_i+1)}} + d_i \right) \right) \leq r.$$

Then there exists a mild solution $u \in \mathcal{PC}_\alpha(X_\alpha)$ of (1.1)–(1.3). In particular, if $1/q - \alpha(\gamma + 1) > 0$ and r is as above, then there exits a mild solution $u \in \mathcal{PC}_\alpha(X_\alpha)$.

Corollary 2.15. *Assume that conditions H_2 and H_3 are satisfied, there are constants $\tilde{c}_i$, $i = 1, \ldots, N$, such that $c_i(s) \leq \tilde{c}_i$ for all $i = 1, \ldots, N$ and each $s > 0$, the semigroup $(T(t))_{t \geq 0}$ is compact and the function $\tau \to W_f^q(1/\tau^\alpha)$ is integrable on $[0, a]$. Suppose, in addition, that $\lim_{h \downarrow 0} \|m_f\|_{L^{q'}([c, c+h])} (1/h^\alpha)(\int_0^h W_f^q(1/s^\alpha) \, ds)^{1/q} = 0$ for all $c \in [0, a)$, $\lim_{h \to 0} \|m_f\|_{L^{q'}([d-h, d])} (1/h^\alpha)(\int_0^h W_f^q((1/s^\alpha)) \, ds)^{1/q} = 0$ for each $d \in (0, a]$ and*

$$C_\alpha Q\|m_f\|_{L^{q'}([0,a])} \limsup_{r \to \infty} \frac{1}{r} \left(\int_0^a W_f^q \left(\frac{2^\alpha r}{s^\alpha} \right) ds \right)^{1/q} + C_0 \sum_{i=1}^{N} \frac{\tilde{c}_i}{\delta_{i-1}^\alpha} < 1.$$

Then there exists a mild solution $u \in \mathcal{PC}_\alpha(X_\alpha)$ of (1.1)–(1.3). In particular, if $W_f(\cdot)$ is bounded and $C_\alpha C_0 \sum_{i=1}^{N} \tilde{c}_i / \delta_{i-1}^\alpha < 1$, then there exits a mild solution $u \in \mathcal{PC}_\alpha(X_\alpha)$ of (1.1)–(1.3).

We complete this section by studying the existence of a classical solution. We assume that the conditions in Theorem 2.6 are satisfied and that $u \in \mathcal{PC}_\alpha(X_\alpha)$ is a mild solution of (1.1)–(1.3). For completeness, we note that a function $v \in C([c, d]; X)$ is called a classical solution of

$$\begin{aligned} w'(t) &= Aw(t) + \xi(t), \quad t \in (c, d], (\xi \in L^1([c, d]; X)) \\ w(c) &= z \in X, \end{aligned} \tag{2.12}$$

if $v(t) \in D(A)$ for all $t \in (c, d]$, $Av(\cdot)$ is continuous on $(c, d]$, $v(c) = z$ and $v(\cdot)$ satisfies the equation (2.12).

In the following theorem we establish conditions under which the mild solution $u(\cdot)$ of (1.1)–(1.3) is a classical solution.

PROPOSITION 2.16. *Assume that there are $\beta \in (0, 1)$ and a function $\mathcal{K}_f \in C([0, \infty); \mathbb{R}^+)$ such that*

$$\|f(t, x) - f(s, y)\| \le \mathcal{K}_f(l)(|t - s|^\beta + \|(-A)^\alpha x - (-A)^\alpha y\|),$$

for all $t, s \in [0, a]$, $x, y \in B_l(0, X_\alpha)$ and each $l > 0$. Then $u(\cdot)$ is a classical solution.

PROOF. To prove the assertion, we show that each function $\widetilde{u}_j$ (see (1.4)) is a classical solution of

$$z'(t) = Az(t) + f(t, z(t)), \quad t \in (t_j, t_{j+1}], \quad z(t_j) = u(t_j^+). \tag{2.13}$$

To begin, we study the case $j = 0$. Let $a_1 \in (0, t_1)$. We affirm that $u(a_1) \in X_\mu$ for all $\mu \in (0, 1)$. To prove our claim, we note that $u_{|[a_1/2, t_1]} \in C([a_1/2, t_1], X_\alpha)$, which implies that the function $\|f(\cdot, u(\cdot))\|$ is bounded on $[a_1/2, t_1]$. Using this fact, for $\mu \in (0, 1)$ we see that

$$\|(-A)^\mu u(a_1)\| \le \left\|(-A)^\mu T\left(a_1 - \frac{a_1}{2}\right)u\left(\frac{a_1}{2}\right)\right\| + \int_{a_1/2}^{a_1} \|(-A)^\mu T(a_1 - \tau)f(\tau, u(\tau))\| \, d\tau$$

$$\le \frac{2^\mu C_\mu}{a_1^\mu}\left\|u\left(\frac{a_1}{2}\right)\right\| + C_\mu \|f(\cdot, u(\cdot))\|_{C([a_1/2, t_1]; X)} \frac{(\frac{a_1}{2})^{1-\mu}}{1 - \mu},$$

which shows $\|(-A)^\mu u(a_1)\|$ is finite. This proves that $u(a_1) \in X_\mu$.

Let $r_1 > 0$ and $k \in \mathbb{N}$ such that $\|u\|_{C([a_1/2, t_1]; X_\alpha)} \le r_1$ and $\vartheta = C_\alpha \mathcal{K}_f(r_1) \times b^{1-\alpha}/(1 - \alpha) < 1$ where $b = (t_1 - a_1)/k$. Under these conditions, for $\mu_1 \in (\alpha, 1)$, $t \in [a_1, a_1 + b)$ and $h > 0$ such that $t + h \in [a_1, a_1 + b]$,

$$\|(-A)^\alpha u(t + h) - (-A)^\alpha u(t)\|$$

$$\le \|(-A)^\alpha T(t - a_1)(T(h)u(a_1) - u(a_1))\|$$

$$+ \int_{a_1}^{a_1+h} \|(-A)^\alpha T(t + h - \tau)f(\tau, u(\tau))\| \, d\tau$$

$$+ \int_{a_1}^{t} \|(-A)^\alpha T(t - \tau)\| \|f(\tau + h, u(\tau + h)) - f(\tau, u(\tau))\| \, d\tau$$

$$\le \|T(t - a_1)\| \int_0^h \|(-A)^{1+\alpha-\mu_1} T(\tau)(-A)^{\mu_1} u(a_1)\| \, d\tau$$

$$+ C_\alpha \|f(\cdot, u(\cdot))\|_{C([a_1, t_1]; X)} \int_{a_1}^{a_1+h} \frac{d\tau}{(a_1 + h - \tau)^\alpha}$$

$$+ \int_{a_1}^{t} \frac{C_\alpha \mathcal{K}_f(r_1)}{(t - \tau)^\alpha}(h^\beta + \|(-A)^\alpha u(\tau + h) - (-A)^\alpha u(\tau)\|) \, d\tau$$

$$\leq C_0 C_{1+\alpha-\mu_1} \|(-A)^{\mu_1} u(a_1)\| \int_0^h \frac{d\tau}{\tau^{1+\alpha-\mu_1}}$$

$$+ C_\alpha \|f(\cdot, u(\cdot))\|_{C([a_1, t_1]; X)} \frac{h^{1-\alpha}}{1-\alpha}$$

$$+ C_\alpha \mathcal{K}_f(r_1) h^\beta \frac{a^{1-\alpha}}{1-\alpha} + C_\alpha \mathcal{K}_f(r_1) \int_{a_1}^t \frac{\|(-A)^\alpha u(\tau+h) - (-A)^\alpha u(\tau)\|}{(t-\tau)^\alpha} \, d\tau$$

$$\leq D_1 h^{\mu_2} + C_\alpha \mathcal{K}_f(r_1) \frac{b^{1-\alpha}}{1-\alpha} \sup_{\theta \in [a_1, b]} \|(-A)^\alpha u(\tau+h) - (-A)^\alpha u(\tau)\|,$$

where $\mu_2 = \min\{\mu_1 - \alpha, 1 - \alpha, \beta\}$ and D_1 is a positive constant independent of $t \in [a_1, a_1 + b]$. From this inequality we infer that $\|(-A)^\alpha u(t+h) - (-A)^\alpha u(t)\| \leq D_1 h^{\mu_2}/(1-\vartheta)$ for all $t \in (a_1, a_1 + b]$, $u \in C^{\mu_2}([a_1, a_1 + b]; X_\alpha)$ and $f(\cdot, u(\cdot)) \in C^{\mu_2}([a_1, a_1 + b]; X)$. Now, from [20, Theorem 4.3.2] it follows that the mild solution of the problem

$$z'(t) = Az(t) + f(t, u(t)), \quad t \in [a_1, a_1 + b], \quad z(a_1) = u(a_1), \tag{2.14}$$

is a classical solution and from the uniqueness of mild solutions of (2.14) we obtain that $u = w$ on $[a_1, a_1 + b]$, $Au_{|[a_1, a_1+b]}$ is continuous on $(a_1, b]$, $u(t_1 + b) \in D(A)$ and $u_{|[a_1, a_1+b]}$ is a classical solution of

$$z'(t) = Az(t) + f(t, z(t)), \quad t \in [a_1, a_1 + b], z(a_1) = u(a_1).$$

Arguing as above, we can show that $u_{[a_1+b, a_1+2b]}$ is a classical solution of

$$z'(t) = Az(t) + f(t, z(t)), \quad t \in (a_1 + b, a_1 + 2b], z(a_1 + b) = u(a_1 + b),$$

$Au_{[a_1+b, a_1+2b]} \in C([a_1 + b, a_1 + 2b]; X)$ and $u(a_1 + 2b) \in D(A)$. Continuing this procedure, we can prove that $u(\cdot)$ is a classical solution of

$$z'(t) = Az(t) + f(t, z(t)), \quad t \in (a_1, t_1], z(a_1) = u(a_1).$$

Since a_1 is arbitrary, from the above steps it follows that $u_{|[0, t_1]}$ is a classical solution of

$$z'(t) = Az(t) + f(t, z(t)), \quad t \in (0, t_1], z(0) = x_0.$$

Repeating the process described on each interval $[t_j, t_{j+1}]$, we prove that $\widetilde{u}_j(\cdot)$ is a classical solution of (1.1)–(1.3). This completes the proof. □

3. Applications

In this section, $X = L^2([0, \pi])$ and $A : D(A) \subset X \to X$ is the operator given by $Ax = x''$ with domain $D(A) := \{x \in X : x'' \in X, \ x(0) = x(\pi) = 0\}$. It is well known that A is the generator of an analytic, compact semigroup $(T(t))_{t \geq 0}$ on X, and A has discrete spectrum with eigenvalues $-n^2$, $n \in \mathbb{N}$, and associated normalised

eigenvectors $z_n(\xi) = (2/\pi)^{1/2}\sin(n\xi)$. Moreover, the set of functions $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis of X, $T(t)x = \sum_{n=1}^{\infty} e^{-n^2 t}\langle x, z_n\rangle z_n$ and $\|T(t)\| \le e^{-t}$ for all $x \in X$ and every $t \ge 0$. In addition, $(-A)^{-1/2}x = \sum_{n=1}^{\infty}(1/n)\langle x, z_n\rangle z_n$ for $x \in X$, $(-A)^{1/2}x = \sum_{n=1}^{\infty} n\langle x, z_n\rangle z_n$ for $x \in D((-A)^{1/2}) = \{x \in X : \sum_{n=1}^{\infty} n\langle x, z_n\rangle z_n \in X\}$, $\|(-A)^{-1/2}\|_{\mathcal{L}(X)} = 1$ and $\|(-A)^{1/2}T(t)\| \le \frac{1}{\sqrt{2}}e^{-t/2}t^{-1/2}$ for all $t > 0$.

To begin, we study the existence of solutions for the impulsive problem

$$\frac{\partial}{\partial t}w(t,\xi) = \frac{\partial^2}{\partial\xi^2}w(t,\xi) + \mu(t)w(t,\xi), \quad (t,\xi) \in [0,a] \times [0,\pi], \qquad (3.1)$$

$$w(t,0) = w(t,\pi) = 0, \quad t \in [0,a], \qquad (3.2)$$

$$w(0,\xi) = z(\xi), \quad \xi \in [0,\pi], \qquad (3.3)$$

$$\triangle w(t_i,\xi) = \theta_i \frac{\partial}{\partial\xi}w(t_i,\xi), \quad \xi \in [0,\pi], i = 1,\dots,N, \qquad (3.4)$$

where $0 < t_1 < \cdots < t_N \le a$, θ_i are fixed real numbers, $z \in X$ and $\mu : [0,a] \to \mathbb{R}$ is a suitable function.

To represent (3.1)–(3.4) in the abstract form (1.1)–(1.3), we assume that $\mu \in L^{q'}([0,a];\mathbb{R})$ for some $q \in [1,2)$, where q' denotes the conjugate of q ($1/q + 1/q' = 1$). We introduce the functions $f : [0,a] \times X \to X$ and $I_i : X_{1/2} \to X$ given by $f(t,x)(\xi) = \mu(t)x(\xi)$ and $I_i(x) = \theta_i(\partial x/\partial\xi)$. It is easy to see that $I_i \in \mathcal{L}(X_{1/2}, X)$ for all i and there exists $C > 0$ such that $\|I_i\|_{\mathcal{L}(X_{1/2},X)} \le C|\theta_i|$ for all $i = 1,\dots,N$. We also note $\|f(t,x)\| \le m_f(t)W_f(\|(-A)^\alpha x\|)$ with $m_f(t) = |\mu(t)|$ and $W_f(s) = |s|$.

In the next result, we say that $u \in \mathcal{PC}(X)$ is a mild solution (respectively, a classical solution) of (3.1)–(3.4) if $u(\cdot)$ is a mild solution (respectively, a classical solution) of the associated abstract problem (1.1)–(1.3). In this result, and in the remainder of this section, $Q := N + 2^{1/2+1/q}/|1 - q/2|^{1/q}$.

PROPOSITION 3.1. *Assume that*

$$\frac{1}{\sqrt{2}}\left(Q\frac{\sqrt{2}a^{1/q-1/2}}{|1 - q/2|^{1/q}}\|\mu\|_{L^{q'}([0,a])} + C\sum_{i=1}^{N}\frac{|\theta_i|}{\delta_{i-1}^{1/2}}\right) < 1.$$

Then there exists a unique mild solution $u \in \mathcal{PC}_{1/2}(X_{1/2})$ of (3.1)–(3.4). If, in addition, $\mu \in C^\beta([0,a];\mathbb{R})$ for some $\beta \in (0,1)$, then $u(\cdot)$ is a classical solution.

PROOF. The existence of a mild solution $u \in \mathcal{PC}_{1/2}(X_{1/2})$ follows from Corollary 2.7. If $\mu \in C^\beta([0,a];\mathbb{R})$, then the function $f(\cdot)$ satisfies the assumptions of Proposition 2.16 which implies that $u(\cdot)$ is a classical solution. $\square$

Similar to the previous example, in the next problems we say that $u \in \mathcal{PC}(X)$ is a mild solution if $u(\cdot)$ is a mild solution of the associated problem (1.1)–(1.3). Consider

the problem

$$\frac{\partial}{\partial t}w(t,\xi) = \frac{\partial^2}{\partial \xi^2}w(t,\xi) + \mu(t)|w(t,\xi)|^{\gamma}, \quad (t,\xi) \in [0,a] \times [0,\pi], \tag{3.5}$$

$$w(t,0) = w(t,\pi) = 0, \quad t \in [0,a], \tag{3.6}$$

$$w(0,\xi) = z(\xi), \quad \xi \in [0,\pi], \tag{3.7}$$

$$\triangle w(t_i,\xi) = \theta_i \frac{\partial}{\partial \xi}w(t_i,\xi), \quad \xi \in [0,\pi], \tag{3.8}$$

where $t_i, \theta_i, \mu(\cdot)$ are as above and $\gamma \in (1,2)$. In Proposition 3.2 we establish the existence of a mild solution for (3.5)–(3.8) via Corollary 2.15.

PROPOSITION 3.2. *Assume that* $C\sum_{i=1}^{N}\theta_i\delta_{i-1}^{-1/2} < 1$, $\lim_{h\downarrow 0}\|\mu\|_{L^{q'}([d,d+h])}h^{1/q-\alpha(\gamma+1)} = 0$ *for all* $d \in [0,a)$ *and* $\lim_{h\downarrow 0}\|\mu\|_{L^{q'}([c-h,c])}h^{1/q-\alpha(\gamma+1)} = 0$ *uniformly for c in compact subsets of* $(0,a]$. *Then there exists a mild solution* $u \in \mathcal{PC}_{1/2}(X_{1/2})$ *of (3.5)–(3.8). In particular, if* $q \in (1,2/(\gamma+1))$ *and* $C\sum_{i=1}^{N}\theta_i\delta_{i-1}^{-1/2} < 1$, *then there exists a mild solution for (3.5)–(3.8).*

We finish this section by studying the existence of solutions for the problem

$$\frac{\partial}{\partial t}w(t,\xi) = \frac{\partial^2}{\partial \xi^2}w(t,\xi) + \mu(t)w(t,\xi), \quad (t,\xi) \in [0,a] \times [0,\pi], \tag{3.9}$$

$$w(t,0) = w(t,\pi) = 0, \quad t \in [0,a], \tag{3.10}$$

$$w(0,\xi) = z(\xi), \quad \xi \in [0,\pi], \tag{3.11}$$

$$\triangle w(t_i,\xi) = \theta_i w(t_i,\xi)|w(t_i,\xi)|^{\gamma_i}, \quad \xi \in [0,\pi], \tag{3.12}$$

where μ, θ_i, μ are as above and $\gamma_i \in (0,1)$ for all $i = 1, \ldots, N$.

To treat this system, we define the functions $I_i : X_{1/2} \to X$ by $I_i(x) = \theta_i x|x|^{\gamma_i}$. The functions I_i are well defined and there exists $C > 0$ such that $\|I_i x\| \le C\, l^{\gamma_i}\|(-A)^{1/2}x\|$ for all $x \in B_l(0, X_{1/2})$ and every $l > 0$. The next result follows from Theorem 2.6.

PROPOSITION 3.3. *Assume that the above conditions are satisfied and there are* $r > 0$ *and* $\Theta \in (0,1)$ *such that*

$$\frac{1}{\sqrt{2}}\Big(Q\frac{\sqrt{2}a^{1/q-1/2}}{|1-q/2|^{1/q}}\|\mu\|_{L^{q'}([0,a])} + C\sum_{i=1}^{N}\frac{\theta_i r^{\gamma_i}}{\delta_{i-1}^{(\gamma_i+1)/2}}\Big) < \Theta.$$

If $\|z\| < \sqrt{2}r(1-\Theta)$, *then there exists a unique mild solution of system (3.9)–(3.12).*

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