The Solution of Mathieu's Differential Equation.

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The determination of the harmonic functions of elliptic and hyperbolic cylinders depends on the solution of Mathieu's differential equation. This equation, it has been remarked by Professor Whittaker,* is the one which naturally comes up for study after the hypergeometric equation has been disposed of. Its solution presents difficulties which do not arise in connection with the hypergeometric equation or its degenerate cases, and it cannot, I think, be said that any discussion of the equation has yet been given with which the student of analysis can rest content. The treatment given below, though certainly incomplete at some points, seems to follow the lines along which a thoroughly successful theory may be hoped for.

Two independent solutions are obtained in terms of series which are proved to converge absolutely and uniformly for all values of the variable. The coefficients of these series are expressed in the form of multiple series which can in fact be summed in a finite number of terms involving known functions, but the actual carrying out of the summations is not effected for the general term.

The equation for the troublesome index with which students of the equation are familiar is found in an interesting and comparatively simple form. Here also the general term is left as a multiple series, but a method is explained by which this can be evaluated in terms of simple functions, at the cost indeed of serious algebraical labour, and the evaluation is actually completed for a few of the earlier terms.

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* Whittaker, Proc. of Fifth International Congress of Mathematicians, 1912, Vol. I., p. 366.

Several papers have recently appeared in our own *Proceedings*, Vols. XXXII. and XXXIII. In these papers other references will be found.

A second form of solution of Mathieu's equation is given in series involving Bessel functions. It is remarkable that the coefficients in these series are the same as in the first or standard form of the solution. A very valuable feature of the solutions in Bessel functions is that they exhibit at a glance the asymptotic character of the solutions for infinite values of the variable.

1. Elliptic cylindrical coordinates α and β are connected with ordinary rectangular coordinates x and y by the equations

$$\left.\begin{aligned} x &= c\cosh\alpha\,\cos\beta, \\ y &= c\sinh\alpha\,\sin\beta, \end{aligned}\right\} \qquad (1)$$

or, in one equation,

$$x + i\,y = c\cosh(\alpha + i\,\beta). \qquad (2)$$

The wave equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \kappa^2 V = 0 \qquad (3)$$

becomes

$$\frac{\partial^2 V}{\partial \alpha^2} + \frac{\partial^2 V}{\partial \beta^2} + \tfrac{1}{2}\kappa^2 c^2 (\cosh 2\alpha - \cos 2\beta)\, V = 0. \qquad (4)$$

If $u(\alpha)\, v(\beta)$ is a solution of this equation, we must have

$$\frac{d^2 u}{d\alpha^2} + (\tfrac{1}{2}\kappa^2 c^2 \cosh 2\alpha - s^2)\, u = 0, \qquad (5)$$

$$\frac{d^2 v}{d\beta^2} - (\tfrac{1}{2}\kappa^2 c^2 \cos 2\beta - s^2)\, v = 0, \qquad (6)$$

where s^2 is a constant.

Since (6) may be derived from (5) by changing α into $i\beta$, it is sufficient to deal with (5), which is in effect Mathieu's differential equation.

The problem attempted in the present paper is the complete solution of (5) on the supposition that the constants κ and s are given.

2. According to the theory of linear differential equations, a solution of (5) exists expressible in the form

$$u = \sum_{n=-\infty}^{\infty} c_n e^{(2n+\nu)\alpha}, \qquad (7)$$

where c_n and ν are constants to be determined.

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Substitution in (5) gives

$$\{(2n+\nu)^2 - s^2\}\, c_n + \tfrac{1}{4}\kappa^2 c^2 (c_{n-1} + c_{n+1}) = 0, \quad \text{.........} (8)$$

$$\text{or, if} \qquad c_n = (-1)^n C_n, \quad \text{.........} (8)'$$

$$C_{n+1} + C_{n-1} = \frac{(n + \frac{1}{2}\nu)^2 - \frac{1}{4}s^2}{\frac{1}{16}\kappa^2 c^2} C_n. \quad \text{.........} (9)$$

If we write

$$\left.\begin{aligned} \tfrac{1}{2}\nu &= \mu, \\ \tfrac{1}{2}s &= r, \\ \tfrac{1}{4}\kappa c &= \lambda, \end{aligned}\right\} \quad \text{.........} (10)$$

this becomes

$$C_{n+1} + C_{n-1} = \frac{(n+\mu)^2 - r^2}{\lambda^2} C_n. \quad \text{.........} (11)$$

It is the treatment of this recurrence equation which constitutes the distinctive feature of the following analysis.

In (11) the variable n is an integer, but it will be convenient to deal with the general difference equation

$$w_{z+1} + w_{z-1} = \frac{z^2 - r^2}{\lambda^2} w_z \quad \text{.........} (12)$$

in which z is a complex variable.

If w_z is any solution of (12), then obviously

$$C_n = w_{n+\mu} \quad \text{.........} (13)$$

$$\text{and} \qquad C_n = w_{-n-\mu} \quad \text{.........} (14)$$

are solutions of (11).

The method of procedure will be as follows.

We shall find a solution of (12), say $w_z = \phi(z)$, vanishing when the real part of z is infinite and positive. Then

$$\phi(n+\mu) = 0, \text{ when } n = +\infty;$$

$$\text{and} \quad \phi(-n-\mu) = 0, \text{ when } n = -\infty.$$

But we can choose μ so that the solutions $\phi(n+\mu)$ and $\phi(-n-\mu)$ of (11) are the same except for a constant factor; for the two solutions $\phi(n+\mu)$ and $A\,\phi(-n-\mu)$ are identical if they are equal to each other for two consecutive values of n, say $n = 0$ and $n = 1$; that is, if

$$\left.\begin{aligned} &\phi(1+\mu) = A\,\phi(-1-\mu), \\ \text{and} \quad &\phi(\mu) = A\,\phi(-\mu), \end{aligned}\right\} \quad \text{.........} (15)$$

so that the equation for μ is

$$\phi(\mu)\,\phi(-\mu-1) - \phi(-\mu)\,\phi(\mu+1) = 0. \quad \text{.........} (16)$$

3. We have now to find a solution $\phi(z)$ of the difference equation

$$\phi(z+1)+\phi(z-1)=\frac{z^2-r^2}{\lambda^2}\phi(z), \quad \text{.............. (17)}$$

such that $\phi(N+ai) \rightarrow 0$ when $N \rightarrow \infty$, where a and N are real.

Since $\phi(n+\mu)$ is the coefficient of the term of order n in a power series which we expect to converge rapidly, it is natural to assume tentatively that in (17) $\phi(z+1)$ is negligible in comparison with $\phi(z-1)$ when the real part of z is great and positive.

Thus, approximately,

$$\phi(z-1)=\frac{z^2-r^2}{\lambda^2}\phi(z), \quad \text{..................... (18)}$$

an equation of which one solution is

$$\phi(z)=\frac{\lambda^{2z}}{\Pi(z+r)\,\Pi(z-r)}, \quad \text{.................. (19)}$$

and this satisfies the condition at infinity.

Looking now for a solution of (17) which shall have (19) for its asymptotic form when $R(z)$, (the real part of z), tends to $+\infty$, we write in (17)

$$\phi(z)=\frac{\lambda^{2z}}{\Pi(z+r)\,\Pi(z-r)}v_z, \quad \text{................ (20)}$$

and obtain for v_z the equation

$$\frac{\lambda^4}{(z+r)(z+r+1)(z-r)(z-r+1)}v_{z+1}+v_{z-1}=v_z,$$

or, changing z into $z+1$,

$$v_z-v_{z+1}=-\frac{\lambda^4}{(z+r+1)(z+r+2)(z-r+1)(z-r+2)}v_{z+2}. \quad \text{...... (21)}$$

We shall find a solution of (21) in the form of a series of ascending powers of λ^4, say

$$v_z=1-\lambda^4A_z^{(1)}+\lambda^8A_z^{(2)}-\lambda^{12}A_z^{(3)}+\ldots, \quad \text{........... (22)}$$

where the coefficients $A_z^{(1)}$, $A_z^{(2)}$, ... all tend to 0 as $R(z)$ tends to $+\infty$.*

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* It will be found interesting to apply a similar process to the difference equation for the Bessel function $J_z(\lambda)$, viz.,

$$J_{z+1}(\lambda)+J_{z-1}(\lambda)=\frac{2z}{\lambda}J_z(\lambda).$$

4. For brevity write

$$a_z = \frac{1}{(z+r+1)(z+r+2)(z-r+1)(z-r+2)}. \quad \ldots\ldots (23)$$

Then (21) is

$$v_z - v_{z+1} = -\lambda^4 a_z v_{z+2}. \quad \ldots\ldots (24)$$

From (22)

$$v_{z+1} = 1 - \lambda^4 A_{z+1}^{(1)} + \lambda^8 A_{z+1}^{(2)} \ldots, \quad \ldots\ldots (25)$$

$$-\lambda^4 a_z v_{z+2} = \quad -\lambda^4 a_z \quad + \lambda^8 a_z A_{z+2}^{(1)} \ldots, \quad \ldots\ldots (26)$$

so that

$$v_z - v_{z+1} = \quad -\lambda^4 (A_z^{(1)} - A_{z+1}^{(1)}) + \lambda^8 (A_z^{(2)} - A_{z+1}^{(2)}) \ldots, \quad \ldots . (27)$$

From (24), (26), (27), by equating coefficients of the various powers of λ,

$$A_z^{(1)} - A_{z+1}^{(1)} = a_z, \quad \ldots\ldots (28)$$

$$A_z^{(2)} - A_{z+1}^{(2)} = a_z A_{z+2}^{(1)}, \quad \ldots\ldots (29)$$

$$A_z^{(3)} - A_{z+1}^{(3)} = a_z A_{z+2}^{(2)}, \quad \ldots\ldots (30)$$

and so on.

A solution of (28) is

$$A_z^{(1)} = a_z + a_{z+1} + a_{z+2} + \ldots, \quad \ldots\ldots (31)$$

and this is easily proved to tend to 0 as $R(z)$ tends to $+\infty$.

Similarly

$$A_z^{(2)} = a_z A_{z+2}^{(1)} + a_{z+1} A_{z+3}^{(1)} + a_{z+2} A_{z+4}^{(1)} + \ldots \quad \ldots\ldots (32)$$

$$\left.\begin{aligned} &= a_z (a_{z+2} + a_{z+3} + a_{z+4} + \ldots) \\ &\quad + a_{z+1} (a_{z+3} + a_{z+4} + a_{z+5} + \ldots) \\ &\quad + a_{z+2} (a_{z+4} + a_{z+5} + a_{z+6} + \ldots) \\ &\quad + \ldots\ldots\ldots, \end{aligned}\right\} \ldots\ldots (33)$$

$$A_z^{(3)} = a_z A_{z+2}^{(2)} + a_{z+1} A_{z+3}^{(2)} + a_{z+2} A_{z+4}^{(2)} + \ldots, \quad \ldots\ldots (34)$$

and so on.

More formally,

$$\left.\begin{aligned} A_z^{(1)} &= \sum_{p_1=0}^{\infty} a_{z+p_1}, \\ A_z^{(2)} &= \sum_{p_1=0}^{\infty} \Big(a_{z+p_1} \sum_{p_2=2}^{\infty} a_{z+p_1+p_2}\Big), \\ A_z^{(3)} &= \sum_{p_1=0}^{\infty} \Big\{a_{z+p_1} \sum_{p_2=2}^{\infty} \Big(a_{z+p_1+p_2} \sum_{p_3=2}^{\infty} a_{z+p_1+p_2+p_3}\Big)\Big\}, \\ A_z^{(q)} &= \sum_{p_1=0}^{\infty} a_{z+p_1} \sum_{p_2=2}^{\infty} a_{z+p_1+p_2} \cdots \sum_{p_q=2}^{\infty} a_{z+p_1+p_2+\ldots+p_q} \end{aligned}\right\} \ldots\ldots (35)$$

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5. We shall now show that the repeated series of (35) are absolutely convergent and that with the definitions (35) of $A_z^{(1)}$, $A_z^{(2)}$, etc., the series for v_z in (22) is absolutely convergent.

Put $$| a_{z+q} | = m_{z+q} . \qquad \qquad (36)$$

Also put $B_z^{(q)}$ for the repeated series obtained from $A_z^{(q)}$ by replacing each a by its modulus.

Thus $B_z^{(q)}$ is the sum of all the q-ary products of the quantities

$$m_z , \; m_{z+1} , \; m_{z+2} , \ldots$$

with the omission of all products in which any two of the suffixes differ by unity. Thus $B_z^{(q)}$ is less than the sum of all the q-ary products, without restriction, of

$$m_z , \; m_{z+1} , \; m_{z+2} , \ldots$$

and therefore, as is obvious by the Multinomial Theorem, less than

$$\frac{1}{q\,!} \left(m_z + m_{z+1} + m_{z+2} + \ldots \right)^q ,$$

which, from the definition (23) of a_z, is clearly finite.

Put $$m_z + m_{z+1} + m_{z+2} + \ldots = M.$$

Thus $$B_z^{(q)} < \frac{1}{q\,!} M^q , \qquad \qquad (37)$$

and $$| A_z^{(q)} | < \frac{1}{q\,!} M^q . \qquad \qquad (38)$$

Hence, if we put $$| \lambda^4 | = L \text{ in } (22), \qquad \qquad (39)$$

$$| v_z | < 1 + LM + \frac{L^2 M^2}{2\,!} + \ldots$$

$$< e_{LM} \qquad \qquad (40)$$

so that the series (22) converges absolutely.

We see now that the repeated series which defines $A_z^{(q)}$ in (35) may be treated as a multiple series, and the terms taken in any order.

Again, when $R(z) \rightarrow +\infty$, clearly $M \rightarrow 0$, so that $v_z \rightarrow 1$, and $\phi(z)$ as defined by (20) and (22) has actually

$$\frac{\lambda^{2z}}{\Pi(z+r)\,\Pi(z-r)}$$

for its asymptotic form as $R(z) \rightarrow +\infty$.

Finally, in the series (7), when $c_n = (-1)^n \phi(n+\mu)$, the general term for a large positive n has the asymptotic form

$$\frac{(-1)^n \lambda^{2n+2\mu}}{\Pi(n+\mu+r)\,\Pi(n+\mu-r)} e^{(2n+2\mu)\,a} ,$$

and the series therefore converges at the upper end, ($n \to +\infty$), absolutely and uniformly for every α.

As for the nature of the convergence at the lower end, ($n \to -\infty$), we have, as explained at the end of § 2,

$$\phi(n+\mu) = \frac{\phi(\mu)}{\phi(-\mu)}\,\phi(-n-\mu)$$
$$= \frac{\phi(\mu)}{\phi(-\mu)}\,\frac{\lambda^{-2n-2\mu}}{\Pi(-n-\mu+r)\,\Pi(-n-\mu-r)},$$

asymptotically, and the general term of (7) for a large negative n has the asymptotic form

$$(-1)^n\,\frac{\phi(\mu)}{\phi(-\mu)}\,\frac{\lambda^{-2n-2\mu}}{\Pi(-n-\mu+r)\,\Pi(-n-\mu-r)}\,e^{(2n+2\mu)\alpha}, \quad \ldots(42)$$

so that the convergence is absolute and uniform at this end also.

The convergence at the lower end depends, of course, on μ being a root of (16). We have now to consider in some detail the character of the equation for μ.

6. We take, according to the definitions (20), (22), (23), (35),

$$\phi(z) = \frac{\lambda^{2z}}{\Pi(z+r)\,\Pi(z-r)}\,\{1 - \lambda^4 A_z^{(1)} + \lambda^8 A_z^{(2)} - \ldots\}, \quad \ldots\ldots\ldots\ldots\ldots(43)$$

and consider the function $f(z)$ defined as

$$f(z) = \phi(z)\,\phi(-z-1) - \phi(-z)\,\phi(z+1). \quad \ldots\ldots\ldots\ldots (44)$$

We have, by (17), for every z

$$\phi(z+1) + \phi(z-1) = \frac{z^2 - r^2}{\lambda^2}\,\phi(z),$$

so that also

$$\phi(-z+1) + \phi(-z-1) = \frac{z^2 - r^2}{\lambda^2}\,\phi(-z).$$

Eliminate $\dfrac{z^2 - r^2}{\lambda^2}$ between these. Thus

$$\phi(-z)\,\phi(z+1) + \phi(-z)\,\phi(z-1)$$
$$= \phi(z)\,\phi(-z+1) + \phi(z)\,\phi(-z-1),$$

or
$$\phi(z)\,\phi(-z-1) - \phi(-z)\,\phi(z+1)$$
$$= \phi(z-1)\,\phi(-z) - \phi(-z+1)\,\phi(z).$$

By the definition (44) the right-hand member here is $f(z-1)$ and also $f(-z)$.

Hence $$f(z) = f(z-1) = f(-z), \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (45)$$

and $f(z)$ is an even periodic function of z, with period unity.

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Thus

$$f(z) = \underset{n=-\infty}{\text{Lt.}} f(n+z)$$

$$= \underset{n=-\infty}{\text{Lt.}} \{\phi(n+z)\,\phi(-n-z-1) - \phi(-n-z)\,\phi(n+z+1)\} \quad ..(46)$$

Now, (20),

$$\phi(z) = \frac{\lambda^{2z}}{\Pi(z+r)\,\Pi(z-r)}\, v_z,$$

so that $f(z)$

$$= \underset{n=-\infty}{\text{Lt.}} \left\{ \frac{\lambda^{-2}}{\Pi(n+z+r)\,\Pi(n+z-r)\,\Pi(-n-z-1+r)\,\Pi(-n-z-1-r)}\, v_{n+z}\, v_{-n-z-1} - \frac{\lambda^{2}}{\Pi(-n-z+r)\,\Pi(-n-z-r)\,\Pi(n+z+1+r)\,\Pi(n+z+1-r)}\, v_{-n-z}\, v_{n+z+1} \right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (47)$$

But, (§ 5),

$$\underset{n=-\infty}{\text{Lt.}} v_{-n-z} = \underset{n=-\infty}{\text{Lt.}} v_{-n-z-1} = 1. \quad \ldots\ldots\ldots\ldots\ldots\ldots (48)$$

Also (22),

$$v_{n+z} = 1 - \lambda^4 A_{n+z}^{(1)} + \lambda^8 A_{n+z}^{(2)} - \ldots, \quad \ldots\ldots\ldots\ldots\ldots (49)$$

and, as we see by writing $n+z$ for z in (35), and then changing p_1 into $p_1 - n$, $A_{n+z}^{(q)}$ differs from $A_z^{(q)}$ only in having n to ∞ instead of 0 to ∞ for the limits of the p_1 summation.

Write

$$\underset{n=-\infty}{\text{Lt.}} A_{n+z}^{(q)} = A_{\infty, z}^{(q)}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (50)$$

It is easily proved, after the manner of § 5, that this limit is finite, and that $\underset{n=-\infty}{\text{Lt.}} v_{n+z}$ may be taken term by term on the right of (49).

Hence

$$\left.\begin{aligned} \underset{n=-\infty}{\text{Lt.}} v_{n+z} &= \underset{n=-\infty}{\text{Lt.}} v_{n+z+1} \\ &= 1 - \lambda^4 A_{\infty, z}^{(1)} + \lambda^8 A_{\infty, z}^{(2)} - \ldots \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (51)$$

Again,

$$\Pi(x)\,\Pi(-x-1) = -\frac{\pi}{\sin x\pi}. \quad \ldots\ldots\ldots\ldots\ldots (52)$$

By using (48), (51), and (52) in (47) we find

$$f(z) = \frac{\sin(z+r)\pi \sin(z-r)\pi}{\pi^2\lambda^2}\left(1 - \lambda^4 A_{\infty, z}^{(1)} + \lambda^8 A_{\infty, z}^{(2)} - \ldots\right) \quad \ldots\ldots (53)$$

The equation (16) for μ is $f(\mu) = 0$, so that

$$\sin(\mu+r)\pi \sin(\mu-r)\pi \left(1 - \lambda^4 A_{\infty, \mu}^{(1)} + \lambda^8 A_{\infty, \mu}^{(2)} - \ldots\right) = 0. \quad \ldots (54)$$

7. We have still to examine the nature of the functions $A_{\infty,\mu}^{(1)}$, $A_{\infty,\mu}^{(2)}$, etc. It will be shown that

$$A_{\infty,\mu}^{(q)} = \frac{F_q(r)}{\sin(\mu+r)\pi \sin(\mu-r)\pi}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (55)$$

where $F_q(r)$ is a function of r only of the form

$$F_q(r) = C_q(r)\cos 2r\pi + S_q(r)\sin 2r\pi, \quad \ldots\ldots\ldots\ldots \quad (56)$$

where $C_q(r)$ and $S_q(r)$ are *rational* functions of r. The equation (54) for μ therefore takes the form

$$\sin(\mu+r)\pi\sin(\mu-r)\pi - \lambda^4 F_1(r) + \lambda^8 F_2(r) - \ldots = 0, \quad \ldots\ldots \quad (57)$$

or

$$\left.\begin{aligned} \cos 2\mu\pi = \cos 2r\pi - 2\lambda^4\{C_1(r)\cos 2r\pi + S_1(r)\sin 2r\pi\} \\ + 2\lambda^8\{C_2(r)\cos 2r\pi + S_2(r)\sin 2r\pi\} - \ldots \end{aligned}\right\} \quad \ldots \quad (58)$$

The values of $F_1(r)$ and $F_2(r)$ will be given explicitly, and a method explained by which $F_q(r)$ could be found for any assigned numerical value of q. The process for $q=3$ and $q=4$ could be carried out without much trouble, but for higher values of q the algebraical calculations become more and more laborious.

8. By (35), (50),

$$A_{\infty,\mu}^{(1)} = \sum_{n=-\infty}^{\infty} a_{n+\mu}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (59)$$

where

$$a_{n+\mu} = \frac{1}{(n+\mu+r+1)(n+\mu+r+2)(n+\mu-r+1)(n+\mu-r+2)},$$

which, in partial fractions,

$$= \frac{1}{2r(2r-1)}\frac{1}{n+\mu+r+1} - \frac{1}{2r(2r+1)}\frac{1}{n+\mu+r+2}$$
$$+ \frac{1}{2r(2r+1)}\frac{1}{n+\mu-r+1} - \frac{1}{2r(2r-1)}\frac{1}{n+\mu-r+2}.$$

But $$\sum_{n=-\infty}^{\infty}\left(\frac{1}{n+\theta} - \frac{1}{n+\phi}\right) = \pi(\cot\theta\pi - \cot\phi\pi). \quad \ldots\ldots\ldots\ldots \quad (60)$$

Hence

$$A_{\infty,\mu}^{(1)} = \frac{\pi}{r(2r-1)(2r+1)}\{\cot(\mu+r)\pi - \cot(\mu-r)\pi\} \ldots\ldots\ldots\ldots (61)$$

$$= -\frac{\pi}{r(2r-1)(2r+1)}\frac{\sin 2r\pi}{\sin(\mu+r)\pi\sin(\mu-r)\pi}, \quad \ldots\ldots\ldots\ldots (61')$$

which agrees with (55) and (56).

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9. We have by (35), (50),

$$A_{\infty,\mu}^{q} = \sum_{p_1=-\infty}^{\infty} \sum_{p_2=2}^{\infty} \sum_{p_3=2}^{\infty} \cdots \sum_{p_q=2}^{\infty} a_{\mu+p_1} a_{\mu+p_1+p_2} \ldots a_{\mu+p_1+p_2+\ldots\ldots+p_q}. \quad \ldots. \tag{62}$$

In this absolutely convergent multiple series the p_1 summation can be taken first.

The function of μ summed in (62) has $4q$ simple poles, for $2q$ of which $\mu+r$ is an integer, and $\mu-r$ for the other $2q$. If the sum of the residues of the function at the poles for which $\mu-r$ is an integer is σ_q, then the sum of the residues at the poles for which $\mu+r$ is an integer is $-\sigma_q$, since the sum of all the residues is zero.

Also each residue, and therefore σ_q, is independent of p_1. Hence

$$A_{\infty,\mu}^{(q)} = \pi\{\cot(\mu-r)\pi - \cot(\mu+r)\pi\} \sum_{p_2=2}^{\infty} \sum_{p_3=2}^{\infty} \cdots \sum_{p_q=2}^{\infty} \sigma_q. \quad \ldots..(63)$$

Write now A_q for $A_{\infty,\mu}^{(q)}$ and Σ_q for the multiple series on the right of (63), so that Σ_q is a function of r only.

Thus $\quad A_q = \pi\{\cot(\mu-r)\pi - \cot(\mu+r)\pi\} \Sigma_q. \quad \ldots\ldots\ldots\ldots(64)$

Σ_q is therefore the residue of the function A_q of μ at any one of the poles for which $\mu-r$ is an integer. This suggests the following method of turning the difficulty of a direct evaluation of the multiple series Σ_q.

10. The q-ple series in (62) may be regarded (*cf.* § 5) as the sum of all the q-ary products of the quantities

$$\ldots a_{\mu-1},\ a_\mu,\ a_{\mu+1},\ \ldots,$$

with the omission of all products containing two consecutive a's. Hence the coefficient of $a_{\mu+n}$ in A_q is the remainder when from A_{q-1} is subtracted all terms containing one or more of $a_{\mu+n-1}$, $a_{\mu+n}$, $a_{\mu+n+1}$.

Consider A_q as a linear function of the independent variables

$$\ldots a_{\mu-1},\ a_\mu,\ a_{\mu+1},\ \ldots.$$

Then the coefficient of $a_{\mu+n}$ in A_q is $\dfrac{\partial A_q}{\partial a_{\mu+n}}$.

For brevity write t_n for $a_{\mu+n}$.

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Thus

$$\left.\begin{aligned}\frac{\partial A_q}{\partial t_n} = A_{q-1} - \left(t_{n-1}\frac{\partial A_{q-1}}{\partial t_{n-1}} + t_n \frac{\partial A_{q-1}}{\partial t_n} + t_{n+1}\frac{\partial A_{q-1}}{\partial t_{n+1}}\right)\\ + t_{n-1}\,t_{n+1}\frac{\partial^2 A_{q-1}}{\partial t_{n-1}\,\partial t_{n+1}},\end{aligned}\right\}\ \ldots\ldots \quad (65)$$

for, in subtracting from A_{q-1} the terms containing t_{n-1}, those containing t_n, and those containing t_{n+1}, we are subtracting twice those which contain the product $t_{n-1}\,t_{n+1}$.

We can use (65) as a reduction formula by repeated application of which $\frac{\partial A_q}{\partial t_n}$ is finally expressed in terms of A_{q-1}, A_{q-2}, ..., A_1, and a certain finite number of the t's adjacent to t_n, viz.,

$$t_{n-q},\ t_{n-q+1},\ \ldots,\ t_{n+q}. \quad \text{We take } A_0 = 1.$$

We find

$$\begin{aligned}\frac{\partial A_q}{\partial t_n} &= A_{q-1} - (t_{n-1} + t_n + t_{n+1})\,A_{q-2}\\ &\quad + \{t_n^2 + 2\,(t_{n-1} + t_{n+1})\,t_n + t_{n-2}\,t_{n-1} + t_{n-1}^2 + t_{n-1}\,t_{n+1}\\ &\quad + t_{n+1}^2 + t_{n+1}\,t_{n+2}\}\,A_{q-3} + \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (66)\\ &= A_{q-1} + X_1^{(n)}\,A_{q-2} + X_2^{(n)}\,A_{q-3} + \ldots + X_{q-2}^{(n)}\,A_1 + X_{q-1}^{(n)}\ldots, \quad \ldots\ldots \quad (67)\end{aligned}$$

where $X_r^{(n)}$ is a rational integral homogeneous function of

$$t_{n-r},\ t_{n-r+1},\ \ldots,\ t_{n+r} \text{ of degree } r.$$

A recurrence formula connecting $X_1^{(n)}$, $X_2^{(n)}$, ..., $X_{q-1}^{(n)}$ may be found by applying (67) to the case where all the t's vanish except those which appear there explicitly, that is to say, those from t_{n-q+1} to t_{n+q-1}, the values of A_{q-1}, A_{q-2}, etc., being written down for this case by inspection from the property stated in the first sentence of this § 10.

11. *Value of* $A_{\infty,\,\mu}^{(2)}$.

By (64)

$$A_2 = A_{\infty,\,\mu}^{(2)} = \pi\{\cot(\mu - r)\,\pi - \cot(\mu + r)\,\pi\}\,\Sigma_2. \quad \ldots\ldots \quad (68)$$

Thus Σ_2 is the residue of A_2 at the pole $\mu = r$. Now the terms of A_2 which are infinite for $\mu = r$ are those involving $a_{\mu-1}$ and $a_{\mu-2}$. As at (65) the coefficient of $a_{\mu-n}$ in A_2 is

$$A_1 - (a_{\mu-n-1} + a_{\mu-n} + a_{\mu-n+1}).$$

Also

$$a_{\mu-1} = \frac{1}{(\mu - r)(\mu - r + 1)(\mu + r)(\mu + r + 1)},$$

$$a_{\mu-2} = \frac{1}{(\mu - r)(\mu - r - 1)(\mu + r)(\mu + r - 1)}.$$

Hence

$$\left.\begin{aligned} \Sigma_2 = {} & \frac{1}{2r(2r+1)}\left[A_1 - (a_{\mu-2} + a_{\mu-1} + a_\mu)\right]_{\mu=r} \\ & - \frac{1}{2r(2r-1)}\left[A_1 - (a_{\mu-3} + a_{\mu-2} + a_{\mu-1})\right]_{\mu=r} \end{aligned}\right\} \text{..... (69)}$$

The functions in square brackets are easily evaluated for $\mu = r$ by expansion in ascending powers of $\mu - r$, the value of A_1 being as given in (61).

We thus find

$$\begin{aligned} \Sigma_2 = {} & -\frac{\pi \cot 2r\pi}{r^2(2r-1)^2(2r+1)^2} \\ & - \frac{1}{2r^2(2r-1)(2r+1)^2}\left(1 + \frac{1}{2r} + \frac{1}{2r+1}\right) \\ & - \frac{1}{2r^2(2r-1)^2(2r+1)}\left(1 - \frac{1}{2r} - \frac{1}{2r-1}\right) \\ & + \frac{1}{4r(2r-1)^2(2r-2)} - \frac{1}{4r(2r+1)^2(2r+2)}. \text{ .... (70)} \end{aligned}$$

12. From (65) explicit expressions in terms of series that can be summed can be derived from $A_1, A_2, A_3, \ldots$ in succession.

Since A_q is a homogeneous function of q dimensions of the variables t, we have

$$\begin{aligned} qA_q &= \Sigma\left(t_n \frac{\partial A_q}{\partial t_n}\right) \\ &= A_{q-1}\Sigma t_n - \Sigma\left\{t_n\left(t_{n-1}\frac{\partial A_{q-1}}{\partial t_{n-1}} + t_n \frac{\partial A_{q-1}}{\partial t_n} + t_{n+1}\frac{\partial A_{q-1}}{\partial t_{n+1}}\right)\right\} \\ &\qquad + \Sigma(t_{n-1} t_n t_{n+1}) \\ &= A_{q-1}\Sigma t_n - \Sigma\left\{t_n(t_{n-1} + t_n + t_{n+1})\frac{\partial A_{q-1}}{\partial t_n}\right\} \\ &\qquad\qquad + \Sigma\left(t_{n-1} t_n t_{n+1} \frac{\partial^2 A_{q-1}}{\partial t_{n-1}\partial t_{n+1}}\right). \text{ ...(71)} \end{aligned}$$

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Thus

$$2A^2 = A_1 \Sigma t_n - \Sigma t_n (t_{n-1} + t_n + t_{n+1})$$
$$= A_1 \Sigma t_n - \Sigma t_n^2 - 2 \Sigma t_n t_{n+1}$$
$$= (\Sigma t_n)^2 - \Sigma t_n^2 - 2 \Sigma t_n t_{n+1} \quad \ldots\ldots (72)$$

$$3A_3 = A_2 \Sigma t_n - \Sigma \{ t_n (t_{n-1} + t_n + t_{n+1}) (\Sigma t_n - t_{n-1} - t_n - t_{n+1}) \} + \Sigma t_{n-1} t_n t_{n+1}$$

or
$$\left. \begin{aligned} A_3 = \tfrac{1}{6} (\Sigma t_n)^3 - \tfrac{1}{2} \Sigma t_n \Sigma t_n^2 - \Sigma t_n \Sigma t_n t_{n+1} + \tfrac{1}{3} \Sigma t_n^3 \\ + \Sigma t_n^2 t_{n+1} + \Sigma t_n t_{n+1}^2 + \Sigma t_n t_{n+1} t_{n+2}, \end{aligned} \right\} \quad \ldots\ldots (73)$$

and so on without difficulty.

These formulae are obviously analogous to the expressions for the coefficients of a rational integral equation in terms of sums of powers of the roots, to which in fact they reduce if we take every alternate t_n to be zero.

All such summations as those of (72), (73) are easily effected by the partial fraction method of § 8 in terms of $\cot(\mu - r)\pi$, $\cot(\mu + r)\pi$ and their successive μ-derivatives, the coefficients being rational functions of r.

In the above the summations are all from $n = -\infty$ to $n = \infty$. The same formulae (72), (73), etc., still hold when the n-summation is taken from a finite value of n to $n = \infty$; we have only to suppose all the variables t_n to be zero when n is less than this finite number. The formulae therefore hold when instead of A_p we take the function $A_z^{(p)}$ of § 4; t_n being there a_{z+n}, and the n-summations from 0 to ∞.

We thus see that the functions $A_z^{(p)}$ are all expressible in finite terms by means of ψ functions (Gauss' function $\psi(x)$ being $\frac{d}{dx} \log \Pi(x)$), and their successive derivatives.

13. A modification of the method just explained leads to a proof of (56). From (67)

$$\left. \begin{aligned} qA_q = A_{q-1} \Sigma t_n + A_{q-2} \Sigma (t_n X_1^{(n)}) + \ldots \\ + A_1 \Sigma (t_n X_{q-2}^{(n)}) + \Sigma (t_n X_{q-1}^{(n)}) \end{aligned} \right\} \ldots\ldots (74)$$

By expressing $t_n X_r^{(n)}$ in partial fractions we find $\Sigma (t_n X_r^{(n)})$ as a linear function of $\cot(\mu - r)\pi$, $\cot(\mu + r)\pi$ and of the first r μ-derivatives of these. Moreover, since the sum of the residues of $t_n X_r^{(n)}$ is zero, the coefficients of $\cot(\mu - r)\pi$ and $\cot(\mu + r)\pi$ are equal but of opposite sign.

Now the p^{th} μ-derivative of $\cot(\mu+a)\pi$ is a function of μ of the form $\{1+\cot^2(\mu+a)\pi\}$ multiplied by a rational integral function of $\cot(\mu+a)\pi$ of degree $p-1$.

Write

$$\left.\begin{aligned} x &= \cot(\mu-r)\pi, \\ y &= \cot(\mu+r)\pi. \end{aligned}\right\} \qquad (75)$$

Thus (74) becomes

$$\left.\begin{aligned} qA_q = A_{q-1}\,&C_1(x-y) \\ &+A_{q-2}\{(x^2+1)P_0(x)+(y^2+1)Q_0(y)+C_2(x-y)\} \\ &+A_{q-3}\{(x^2+1)P_1(x)+(y^2+1)Q_1(y)+C_3(x-y)\} \\ &+\cdots\cdots\cdots\cdots \\ &+A_1\{(x^2+1)P_{q-3}(x)+(y^2+1)Q_{q-3}(y)+C_{q-1}(x-y)\} \\ &+\quad\{(x^2+1)P_{q-2}(x)+(y^2+1)Q_{q-2}(y)+C_q(x-y)\}, \end{aligned}\right\} \qquad (76)$$

where $P_n(x)$, $Q_n(y)$ are rational integral functions of x, y respectively of degree n, the coefficients in which are functions of r, and C_n is a function of r only.

Also by (64) A_n has the form $B_n(x-y)$, where $B_n=\pi\,\Sigma_n$ is a function of r only.

Thus

$$\left.\begin{aligned} 0 = &-qB_q(x-y) \\ &+B_{q-1}(x-y)\,C_1(x-y) \\ &+B_{q-2}(x-y)\{(x^2+1)P_0(x)+(y^2+1)Q_0(y)+C_2(x-y)\} \\ &+\cdots\cdots\cdots\cdots \\ &+\qquad\{(x^2+1)P_{q-2}(x)+(y^2+1)Q_{q-2}(y)+C_q(x-y)\} \end{aligned}\right\}. \qquad (77)$$

Here the rational integral function of x and y on the right is not necessarily *identically* zero, for x and y are not independent. In fact,

$$\cot 2r\pi = \cot(\overline{\mu+r}\,\pi - \overline{\mu-r}\,\pi)$$
$$= \frac{xy+1}{x-y},$$

so that

$$xy-(x-y)\cot 2r\pi+1=0. \qquad (78)$$

Thus all we can infer is that the right of (77) is divisible by

$$xy-(x-y)\cot 2r\pi+1.$$

It will still be so divisible after we subtract

$$\{xy-(x-y)\cot 2r\pi+1\}\,\{P_{q-2}(x)+Q_{q-2}(y)\}$$

from its last line, which then becomes

$$(x-y)\,[xP_{q-2}(x)-yQ_{q-2}(y)+\{P_{q-2}(x)+Q_{q-2}(y)\}\cot 2r\pi+C_q].$$

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The modified right of (77) is then divisible by $x-y$, and the quotient is still divisible by $xy-(x-y)\cot 2r\pi+1$; in fact, it is identically zero, since it is the sum of a function of x alone and a function of y alone.

In the identically null quotient put $x=y=i$, and we get

$$-qB_q+i\{P_{q-2}(i)-Q_{q-2}(i)\}+\{P_{q-2}(i)+Q_{q-2}(i)\}\cot 2r\pi+C_q=0$$

or, say,

$$B_q=C_q(r)\cot 2r\pi+S_q(r). \quad\text{......} \quad (79)$$

Thus

$$A_q=B_q\{\cot(\mu-r)\pi-\cot(\mu+r)\pi\}$$

$$=\frac{C_q(r)\cos 2r\pi+S_q(r)\sin 2r\pi}{\sin(\mu-r)\pi\,\sin(\mu+r)\pi}, \quad\text{......} \quad (80)$$

which proves (55) and (56).

14. *Summary of fundamental results.*

In the preceding paragraphs we have defined a solution of Mathieu's equation

$$\frac{d^2u}{d\alpha^2}+(\tfrac{1}{2}\kappa^2c^2\cosh 2\alpha-s^2)u=0, \quad\text{......} \quad (81)$$

say,

$$u=J(\nu, s, \kappa c, \alpha), \quad\text{......} \quad (82)$$

where

$$J(\nu, s, \kappa c, \alpha)=\sum_{n=-\infty}^{\infty}(-1)^n\phi(n+\tfrac{1}{2}\nu)e^{(2n+\nu)\alpha}, \quad\text{......} \quad (83)$$

where, if

$$\left.\begin{aligned}\tfrac{1}{2}s&=r,\\ \tfrac{1}{4}\kappa c&=\lambda,\end{aligned}\right\} \quad\text{......} \quad (84)$$

the function $\phi(z)$ is defined as

$$\phi(z)=\frac{\lambda^{2z}}{\Pi(z+r)\Pi(z-r)}\{1-\lambda^4A_z^{(1)}+\lambda^8A_z^{(2)}-\lambda^{12}A_z^{(3)}+\dots\}, \quad\text{...} \quad (85)$$

in which

$$A_z^{(q)}=\sum_{p_1=0}^{\infty}\sum_{p_2=2}^{\infty}\sum_{p_3=2}^{\infty}\cdots\sum_{p_q=2}^{\infty}a_{z+p_1}\,a_{z+p_1+p_2}\cdots a_{z+p_1+p_2+\dots+p_q}, \quad\text{...} \quad (86)$$

and

$$a_z=\frac{1}{(z+r+1)(z+r+2)(z-r+1)(z-r+2)}. \quad\text{......} \quad (87)$$

If

$$\tfrac{1}{2}\nu=\mu, \quad\text{......} \quad (88)$$

the equation for μ is

$$\phi(\mu)\phi(-\mu-1)-\phi(-\mu)\phi(\mu+1)=0, \quad\text{......} \quad (89)$$

which reduces to the form

$$\sin(\mu+r)\pi\sin(\mu-r)\pi\,\{1-\lambda^4A_{\infty,\mu}^{(1)}+\lambda^8A_{\infty,\mu}^{(2)}-\dots\}=0, \quad\text{...} \quad (90)$$

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where $A_{\infty,\mu}^{(q)}$ differs from $A_{\mu}^{(q)}$ as defined by (86) only in having the p_1-summation from $p_1 = -\infty$ to $p_1 = \infty$.

It is proved that

$$A_{\infty,\mu}^{(q)} = \frac{C_q(r)\cos 2r\pi + S_q(r)\sin 2r\pi}{\sin(\mu - r)\pi \sin(\mu + r)\pi}, \quad \ldots\ldots\ldots\ldots\ldots \quad (91)$$

where $C_q(r)$ and $S_q(r)$ are rational functions of r. These are found explicitly for $r = 1$ and $r = 2$, and a process is explained by which they may be calculated for any assigned value of q.

The equation for ν is

$$\left.\begin{aligned}\cos\nu\pi = \cos s\pi &- 2\left(\tfrac{1}{4}\kappa c\right)^4\left\{C_1\left(\tfrac{1}{2}s\right)\cos s\pi + S_1\left(\tfrac{1}{2}s\right)\sin s\pi\right\}\\ &+ 2\left(\tfrac{1}{4}\kappa c\right)^8\left\{C_2\left(\tfrac{1}{2}s\right)\cos s\pi + S_2\left(\tfrac{1}{2}s\right)\sin s\pi\right\}\\ &- \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}\right\}. \quad (92)$$

If ν_0 be any one value of ν, the solution of this equation is

$$\nu = 2N \pm \nu_0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (93)$$

where N is any integer.

But obviously (93) does not give more than two distinct solutions, say from $\nu = \nu_0$ and $\nu = -\nu_0$, for in (83) increase of ν by 2 and diminution of n by 1 only changes the sign of the function.

Further, when ν_0 is an integer, ν_0 and $-\nu_0$ differ by twice an integer, and the solutions coincide. A second solution for this case may be found by a well-known process. We prefer to define it, however, in connection with another form into which the solution of Mathieu's equation may be put, a form which has important advantages, and to which we now proceed.

15. *Solutions of Mathieu's equation in terms of series of Bessel functions.*

In (5) write temporarily $\frac{1}{2}\kappa c = k$, and the equation becomes

$$\frac{d^2 u}{d\alpha^2} + \left(k^2 e^{2\alpha} + k^2 e^{-2\alpha} - s^2\right) u = 0. \quad \ldots\ldots\ldots\ldots\ldots \quad (94)$$

If

$$\left.\begin{aligned}J_n &\equiv J_n(ke^{-\alpha}),\\ J_t &\equiv J_t(ke^{\alpha}),\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (95)$$

where J denotes the ordinary Bessel function,

then

$$\frac{d^2 J_n}{d\alpha^2} + \left(k^2 e^{-2\alpha} - n^2\right) J_n = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (96)$$

and

$$\frac{d^2 J_t}{d\alpha^2} + \left(k^2 e^{-2\alpha} - t^2\right) J_t = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (97)$$

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Hence

$$\frac{d^2}{d\alpha^2}(J_n J_t) + (k^2 e^{2\alpha} + k^2 e^{-2\alpha} - n^2 - t^2) J_n J_t$$

$$= 2\frac{dJ_n}{d\alpha}\frac{dJ_t}{d\alpha}$$

$$= -2k^2 J_n' J_t' \text{ .... } (98)$$

Thus

$$\frac{d^2}{d\alpha^2}(J_n J_t) + (k^2 e^{2\alpha} + k^2 e^{-2\alpha} - s^2) J_n J_t$$

$$= -2k^2 J_n' J_t' + (n^2 + t^2 - s^2) J_n J_t \text{ .... } (99)$$

Now,

$$\left.\begin{aligned} 2J_n' &= J_{n-1} - J_{n+1}, \\ 2J_t' &= J_{t-1} - J_{t+1}, \end{aligned}\right\} \text{ .... } (100)$$

so that

$$4J_n' J_t' = (J_{n-1} - J_{n+1})(J_{t-1} - J_{t+1}) \text{ .... } (101)$$

Also

$$\left.\begin{aligned} \frac{2n}{ke^{-\alpha}} J_n &= J_{n-1} + J_{n+1}, \\ \frac{2t}{ke^{\alpha}} J_t &= J_{t-1} + J_{t+1}, \end{aligned}\right\} \text{ .... } (102)$$

(100) and (102) being fundamental formulae in Bessel functions, so that

$$\frac{4nt}{k^2} J_n J_t = (J_{n-1} + J_{n+1})(J_{t-1} + J_{t+1}) \text{ .... } (103)$$

From (101) and (103)

$$2J_n' J_t' = -\frac{2nt}{k^2} J_n J_t + J_{n-1} J_{t-1} + J_{n+1} J_{t+1} \text{ .... } (104)$$

Hence the right-hand member of (99)

$$= \{(n+t)^2 - s^2\} J_n J_t - k^2 (J_{n-1} J_{t-1} + J_{n+1} J_{t+1}) \text{ ...... } (105)$$

Put $t = n + \nu$ and try in (94)

$$u = \sum_{n=-\infty}^{\infty} C_n J_n(\tfrac{1}{2}\kappa c e^{-\alpha}) J_{n+\nu}(\tfrac{1}{2}\kappa c e^{\alpha}) \text{ .... } (106)$$

We require by (105)

$$\sum_{n=-\infty}^{\infty} C_n \left[\{(2n+\nu)^2 - s^2\} J_n J_t - \tfrac{1}{4}\kappa^2 c^2 (J_{n-1} J_{t-1} + J_{n+1} J_{t+1})\right] = 0.$$

$$\text{............ } (107)$$

In this series the coefficient of $J_n J_t$ will vanish if

$$C_n \{(2n+\nu)^2 - s^2\} = \tfrac{1}{4}\kappa^2 c^2 (C_{n+1} + C_{n-1}), \text{ .... } (108)$$

which is the same as (9).

Thus we may take $C_n = \phi(n + \frac{1}{2}\nu)$, with ν as before.

The series

$$\sum_{n=-\infty}^{\infty} \phi(n + \tfrac{1}{2}\nu) J_n(\tfrac{1}{2}\kappa c e^{-\alpha}) J_{n+\nu}(\tfrac{1}{2}\kappa c e^{\alpha}) \text{ .... } (109)$$

can be seen in a moment to be absolutely and uniformly convergent. It therefore defines a solution of Mathieu's equation.

16. Since the function (109) is the product of $e^{\nu\alpha}$ by a uniform function of $e^{2\alpha}$, this solution must be a constant multiple of the solution $J(\nu, s, \kappa c, \alpha)$ defined in (83), say,

$$\sum_{n=-\infty}^{\infty} \phi(n + \tfrac{1}{2}\nu) J_n(\tfrac{1}{2}\kappa c e^{-\alpha}) J_{n+\nu}(\tfrac{1}{2}\kappa c e^{\alpha}) = C\, J(\nu, s, \kappa c, \alpha). \quad (110)$$

The constant C may be determined by assigning any special value to α, say $\alpha = 0$. This leads to a complicated expression for C, and it turns out that we get the value of C in a much simpler form by considering the limiting forms of the two sides of (110) as $R(\alpha)$ tends to $+\infty$.* It is convenient to keep the imaginary part of α constant, and such that $\frac{1}{2}\kappa c e^{\alpha}$ is a positive pure imaginary, say, iy, with phase $\frac{1}{2}\pi$.

Then in (109)

$$J_{n+\nu}(\tfrac{1}{2}\kappa c e^{\alpha}) = J_{n+\nu}(iy)$$

$$\doteq i^{n+\nu} e^{y} / \sqrt{(2\pi y)}, \text{ .... } (111)$$

for any fixed n, where the symbol $\doteq$ means "is asymptotically equal to."

Also since $J_{-n} = (-1)^n J_n$, the value of the other J factor in the general term of (109) is small if n differs from 0, and its order of smallness increases by 1 as n increases by 1 numerically. The most important term of (109) is therefore that for $n = 0$. If we assume that it is more important than the *sum* of all the other terms, it follows that the asymptotic form of (109) is

$$\phi(\tfrac{1}{2}\nu) J_\nu(iy) = \phi(\tfrac{1}{2}\nu)\, i^{\nu} e^{y} / \sqrt{(2\pi y)}. \text{ .... } (112)$$

---

* The process of determining the limiting forms of functions defined by series with general terms so complicated as those of (110) is, of course, delicate and difficult, and although I believe the conclusions stated in the remaining part of the paper to be true, I am far from claiming that I give adequate proof of them.

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On the other side of (110) consider now the asymptotic form of $J(\nu, s, \kappa c, \alpha)$.

In (83) the term of order n, when n is large and positive,

$$=(-1)^n \phi(n+\tfrac{1}{2}\nu)\, e^{(2n+\nu)\alpha}$$

$$\doteq(-1)^n \frac{(\frac{1}{4}\kappa c\, e^{\alpha})^{2n+\nu}}{\Pi\left(n+\frac{\nu+s}{2}\right)\Pi\left(n+\frac{\nu-s}{2}\right)}, \quad \text{by (41)}$$

$$\doteq i^{\nu}\frac{(\frac{1}{2}y)^{2n+\nu}}{\Pi\, n\, \Pi\,(n+\nu)}, \quad (\text{since } \Pi\,(n+\theta) \doteq n^{\theta}\,\Pi\,(n)),$$

and this is just the n^{th} term in the expansion of $J_\nu(iy)$.

From this it is not difficult to deduce that, with α as supposed just before (111),

$$J(\nu, s, \kappa c, \alpha) \doteq J_\nu(iy).$$

Hence $$C=\phi(\tfrac{1}{2}\nu), \qquad \text{.........................} \quad (113)$$

and (110) becomes, for all values of α,

$$J(\nu, s, \kappa c, \alpha)=\frac{1}{\phi(\frac{1}{2}\nu)}\sum_{n=-\infty}^{\infty} \phi(n+\tfrac{1}{2}\nu)\, J_n(\tfrac{1}{2}\kappa c e^{-\alpha})\, J_{n+\nu}(\tfrac{1}{2}\kappa c e^{\alpha}). \qquad ...(114)$$

It may be noted as remarkable that while the function on the left here and therefore that on the right also only changes in sign when ν is increased by a multiple of 2, the general term on the right is completely altered when this change is made, and we have really an infinite number of different series of the type on the right of (114), all equal in value, and all equal to the solution $J(\nu, s, \kappa c, \alpha)$ of Mathieu's equation.

17. Besides the solutions $J(\pm\nu, s, \kappa c, \alpha)$ it is obvious from equation (5) itself that there are two other solutions

$$J(\pm\nu, s, \kappa c, -\alpha).$$

The relations of these to the former solutions are easily determined.

Thus

$$J(\nu, s, \kappa c, -\alpha)=\sum_{n=-\infty}^{\infty} \phi(n+\tfrac{1}{2}\nu)\, e^{-(2n+\nu)\alpha},$$

or, changing n into $-n$,

$$=\sum_{n=-\infty}^{\infty} \phi(-n+\tfrac{1}{2}\nu)\, e^{(2n-\nu)\alpha}$$

$$=\frac{\phi(\frac{1}{2}\nu)}{\phi(-\frac{1}{2}\nu)}\sum_{n=-\infty}^{\infty} \phi(n-\tfrac{1}{2}\nu)\, e^{(2n-\nu)\alpha}$$

(§ 2 near 15),

so that

$$J(\nu, s, \kappa c, -\alpha) = \frac{\phi(\frac{1}{2}\nu)}{\phi(-\frac{1}{2}\nu)} J(-\nu, s, \kappa c, \alpha). \qquad (115)$$

18. We can now (*cf.* end of § 14) define a solution of Mathieu's equation, analogous to the Bessel function of the second kind, which will always be distinct from $J(\nu, s, \kappa c, \alpha)$.

In (114) write $-\nu$ for ν, and change the sign of n.

Thus

$$\begin{aligned} J(-\nu, s, \kappa c, \alpha) &= \sum_{n=-\infty}^{\infty} \frac{\phi(-n-\frac{1}{2}\nu)}{\phi(-\frac{1}{2}\nu)} e^{in\pi} J_n(\tfrac{1}{2}\kappa c e^{-\alpha}) J_{-n-\nu}(\tfrac{1}{2}\kappa c e^{\alpha}) \\ &= \sum_{n=-\infty}^{\infty} \frac{\phi(n+\frac{1}{2}\nu)}{\phi(\frac{1}{2}\nu)} e^{in\pi} J_n(\tfrac{1}{2}\kappa c e^{-\alpha}) J_{-n-\nu}(\tfrac{1}{2}\kappa c e^{\alpha}). \quad \dots \ (116) \end{aligned}$$

Multiply (114) by $e^{-i\nu\pi}$, subtract from (116), and finally multiply by $\frac{\pi}{2 \sin \nu\pi}$. Thus

$$\left.\begin{aligned} &\frac{\pi}{2 \sin \nu\pi} \{J(-\nu, s, \kappa c, \alpha) - e^{-i\nu\pi} J(\nu, s, \kappa c, \alpha)\} \\ &= \sum_{n=-\infty}^{\infty} \frac{\phi(n+\frac{1}{2}\nu)}{\phi(\frac{1}{2}\nu)} J_n(\tfrac{1}{2}\kappa c e^{-\alpha}) \cdot \frac{\pi}{2 \sin (n+\nu)\pi} \\ &\qquad \{J_{-n-\nu}(\tfrac{1}{2}\kappa c e^{\alpha}) - e^{-i(n+\nu)} J_{n+\nu}(\tfrac{1}{2}\kappa c e^{\alpha})\} \end{aligned}\right\} \dots \ (117)$$

The Bessel function of the second kind $G_m(z)$ is defined as

$$G_m(z) = \frac{\pi}{2 \sin m\pi} \{J_{-m}(z) - e^{-im\pi} J_m(z)\}. \qquad (118)$$

By analogy, take as definition

$$G(\nu, s, \kappa c, \alpha) = \frac{\pi}{2 \sin \nu\pi} \{J(-\nu, s, \kappa c, \alpha) - e^{-i\nu\pi} J(\nu, s, \kappa c, \alpha)\}. \qquad (119)$$

Then (117) is

$$G(\nu, s, \kappa c, \alpha) = \sum_{n=-\infty}^{\infty} \frac{\phi(n+\frac{1}{2}\nu)}{\phi(\frac{1}{2}\nu)} J_n(\tfrac{1}{2}\kappa c e^{-\alpha}) G_{n+\nu}(\tfrac{1}{2}\kappa c e^{\alpha}). \quad \dots \ (120)$$

Cf. with (114).

19. It need scarcely be said that the really striking feature of the series on the right of (114) and (120) is that, although they are highly convergent analytical series, they serve the purpose, usually

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achieved by the use of asymptotic series, of determining to any required order of approximation the forms of the functions $J(\nu, s, \kappa c, \alpha)$ and $G(\nu, s, \kappa c, \alpha)$ as $R(\alpha)$ tends to $+\infty$. In each case the term for $n=0$ is of lower order in the infinitesimal $e^{-\alpha}$ than any other term, and may be taken as giving the asymptotic value of the function. Moreover, by using the analytical ascending power expansion of $J_n(\frac{1}{2}\kappa c e^{-\alpha})$ and the asymptotic expansions of $J_{n+\nu}(\frac{1}{2}\kappa c e^{\alpha})$ and $G_{n+\nu}(\frac{1}{2}\kappa c e^{\alpha})$, and collecting the terms of one and the same order, we obtain asymptotic expansions not involving Bessel functions and involving the constant ν only in a very simple way, since the functions ϕ must drop out in virtue of the difference equation connecting them.

It is also easy to find the asymptotic forms of the solutions J and G for $R(\alpha)$ tending to $-\infty$ by making use of equation (115).

The asymptotic solutions referred to in this article have been given by Maclaurin (*Trans. Camb. Phil. Soc.*, Vol. 17, 1899) and Marshall (*Amer. Journ. Math.*, Vol. 31, 1909).

These writers do not succeed in finding the exact numerical relationships between the various asymptotic solutions. These are easily found by the theory just briefly sketched. But a fuller treatment of these and other asymptotic expansions must be left over for the present.