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TWO NOTES ON FRAMES

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Abstract

The coproduct of two frames A, B (equivalently the product of two locales, see Isbell (1972a)) will be shown to be given by the lattice of Galois connections between A and B (the connection product in the sense of Isbell (1972b)). We show the inverse limit of a direct system of locales to be given by the set inverse limit of the underlying lattices under the bonding antimaps (see Dowker and Strauss (1975)). This implies the existence of infinite locale products.

We also realize the locale of frame congruences as a pullback in the category of locales.

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Let D be a distributive lattice and let a, b be elements of D. A relative pseudocomplement of a with respect to b is an element of x of D such that $y \wedge a \leqslant b$ if and only if $y \leqslant x$. Such an element is necessarily unique. A lattice is called Brouwerian if a relative pseudocomplement of a with respect to b, denoted $a * b$, exists for any pair of elements of D. For complete lattices this is equivalent to the join infinite distributivity condition $x \wedge \bigvee_\alpha y_\alpha = \bigvee_\alpha (x \wedge y_\alpha)$. A frame is a complete Brouwerian lattice, and a frame map is a lattice homomorphism of frames which preserves the maximum element and arbitrary unions. The open sets of any topological space form a frame and there is a contravariant functor from the category of topological spaces to the category of frames which takes any continuous map to the associated inverse image map on the associated frames. If the frame L is isomorphic to the lattice of open sets of the space X, we will sometimes abuse language and say that L 'is' the space X.

Frames have been studied in Dowker and Papert (1966), Isbell (1972a), Dowker and Strauss (1975) and Dowker and Strauss (1976). Following Isbell (1972a), we call the dual category to the category of frames the category of locales. Dowker and Strauss (1975) have exhibited a concrete realization of the category of locales as follows: the objects are the same as those of the category of frames, and to every frame map $f\colon L\to M$ is associated a locale map $g\colon M\to L$ defined by the formula $g(u)=\bigvee\{x\in L\mid f(x)\leqslant u\}$. Then f can be recovered from g by the formula

$$(\dagger)\qquad f(a)=\bigwedge\{u\in M\mid a\leqslant g(u)\}.$$

We call f and g adjoint maps; they satisfy $fg(u)\leqslant u$, $gf(a)\geqslant a$, and $f(a)\leqslant u$ if and only if $a\leqslant g(u)$. A function $g\colon M\to L$ is a locale map if and only if it satisfies:

(1) $g(\bigwedge_\alpha u_\alpha)=\bigwedge_\alpha g(u_\alpha)$,

(2) $g(u)=1$ if and only if $u=1$,

(3) $a*g(u)=g(f(a)*u)$,

where f is defined by (†).

Dowker and Strauss only state $\leqslant$ in (3); the opposite inequality follows from (1) as follows:

$$a\wedge g(f(a)*u)\leqslant g(f(a))\wedge g(f(a)*u)=g(f(a)\wedge(f(a)*u))\leqslant g(u)$$
$$g(f(a)*u)\leqslant a*g(u).$$

The equality (3) shows that the image of a locale map is a q-set in L (see Dowker and Strauss (1966)), that is a subset S of L closed under arbitrary intersections and such that if $l\in L$ and $s\in S$ then $l*s\in S$. Since the q-sets in L can be naturally identified with the sublocales of L this shows that a locale map admits a natural decomposition into the composite of a surjection and an injection.

This note is divided into two parts. In the first part, we give a lattice theoretic construction of the product of locales (coproduct of frames), different from that in Dowker and Strauss (1976). We show that the product is given by a 'tensor product' construction, the lattice of Galois connections, which has also been considered by Isbell (1972b) and Shmuely (1974). We also construct the inverse limit of a directed system of locales, thereby proving the existence of infinite products.

In the second part we consider the topological representation of distributive lattices (Stone (1937); Gratzer (1971)) in the case where the distributive lattice is a frame. We also exhibit the dual lattice $Q'(L)$ to the lattice of frame congruence relations of a frame L (Dowker and Papert (1966); Isbell (1972a)) as the pullback of a diagram

$$\begin{array}{ccc} Q'(L) & \longrightarrow & I(B(L)) \\ \downarrow & & \downarrow \\ L & \longrightarrow & I(L) \end{array}$$

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in the category of locales. Here $I(L)$ is the lattice of ideals of L or (by our abuse of language) 'is' the Stone representation space of L, and $I(B(L))$ 'is' the Stone representation space of the Boolean algebra generated by L (Gratzer (1971)), which is homeomorphic to the Stone representation space of L with the strong (Nerode (1959)) topology. We note in passing that the construction which assigns to a bounded distributive lattice its lattice of ideals is a monad (Maclane (1971)) in the category of bounded distributive lattices, and that the category of algebras over this monad is equivalent to the category of frames.

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Let A and B be complete distributive lattices; following Shmuely (1974), we denote by $A \otimes B$ the set of mappings $\varphi\colon A \to B$ which satisfy $\varphi(\bigvee a_\alpha) = \bigwedge \varphi(a_\alpha)$ where $\{a_\alpha\}$ is any subset of A, with the pointwise order.

Proposition 1. *If A and B are frames, then $A \otimes B$ is a frame.*

Proof. (1) $A \otimes B$ is a complete lattice since $(\bigwedge_\alpha \varphi_\alpha)(x) = \bigwedge_\alpha (\varphi_\alpha(x))$ defines a meet for the subset $\{\varphi_\alpha\}$ of $A \otimes B$.

(2) $A \otimes B$ is a Brouwerian lattice; for $f, \varphi \in A \otimes B$ define

$$q(x) = \bigwedge_{u \in A} f(u) * \varphi(u \wedge x).$$

Then

$$\begin{aligned} q\Big(\bigvee_\alpha x_\alpha\Big) &= \bigwedge_{u \in A} f(u) * \varphi\Big(u \wedge \bigvee_\alpha x_\alpha\Big) \\ &= \bigwedge_{u \in A} f(u) * \varphi\Big(\bigvee_\alpha u \wedge x_\alpha\Big) \\ &= \bigwedge_{u \in A} f(u) * \bigwedge_\alpha \varphi(u \wedge x_\alpha) \\ &= \bigwedge_{u \in A} \bigwedge_\alpha f(u) * \varphi(u \wedge x_\alpha) \\ &= \bigwedge_\alpha \bigwedge_{u \in A} f(u) * \varphi(u \wedge x_\alpha) \end{aligned}$$

and

$$q\Big(\bigvee_\alpha x_\alpha\Big) = \bigwedge_\alpha q(x_\alpha).$$

So $q \in A \otimes B$. But

$$\begin{aligned} (q \wedge f)(x) &= q(x) \wedge f(x) \\ &= f(x) \wedge \bigwedge_{u \in A} f(u) * \varphi(u \wedge x), \end{aligned}$$

$$(q \wedge f)(x) \leqslant f(x) \wedge (f(x) * \varphi(x)) \leqslant \varphi(x)$$

and so $q \wedge f \leqslant \varphi$.

Now suppose $r \wedge f \leqslant \varphi$ in $A \otimes B$ so that for all $x \in A$, $r(x) \wedge f(x) \leqslant \varphi(x)$. Then for all $u \in A$, $r(x) \wedge f(u) \leqslant r(x \wedge u) \wedge f(x \wedge u) \leqslant \varphi(x \wedge u)$ and so

$$r(x) \leqslant \bigwedge_{u \in A} f(u) * \varphi(x \wedge u) = q(x).$$

This shows that q is the relative pseudocomplement in $A \otimes B$ of f with respect to φ, that is $q = f * \varphi$ and $A \otimes B$ is Brouwerian.

We consider frame maps $\eta_1 \colon A \to A \otimes B$ and $\eta_2 \colon B \to A \otimes B$ defined as follows:

$$\eta_1(x)(y) = \begin{cases} 1_B & \text{if } y \leqslant x, \\ 0_B & \text{otherwise}, \end{cases}$$

$$\eta_2(z)(y) = \begin{cases} z & \text{if } y \neq 0_A, \\ 1_B & \text{if } y = 0_A, \end{cases}$$

for all $x, y \in A$, $z \in B$.

It is clear that $\eta_1(x)$ and $\eta_2(z)$ are in $A \otimes B$, that $\eta_1(x_1 \wedge x_2) = \eta_1(x_1) \wedge \eta_1(x_2)$, for x_1, x_2 in A, that $\eta_1(1_A) = 1_{A \otimes B}$, and that η_2 is a frame map. To show η_1 is a frame map, we must prove $\eta_1(\bigvee_\alpha x_\alpha) = \bigvee_\alpha \eta_1(x_\alpha)$ where the union on the right is taken in $A \otimes B$. If $\{x_\alpha\} \subset A$ we note that if $x \leqslant x_\alpha$ for some α then

$$\Big(\bigvee_\alpha \eta_1(x_\alpha)\Big)(x) = 1_B.$$

So

$$(\bigvee \eta_1(x_\alpha))\Big(\bigvee_\alpha x_\alpha\Big) = 1_B \quad \text{and} \quad \bigvee_\alpha \eta_1(x_\alpha) \geqslant \eta_1\Big(\bigvee_\alpha x_\alpha\Big).$$

Since the opposite inequality is clear, η_1 is a frame map. We note that the η_i commute with arbitrary intersections.

The locale mappings $\pi_1 \colon A \otimes B \to A$ and $\pi_2 \colon A \otimes B \to B$ adjoint to η_1 and η_2 are defined by the formulae

$$\pi_1(\varphi) = \bigvee \{x \mid \varphi(x) = 1_B\},$$

$$\pi_2(\varphi) = \varphi(1_A)$$

as is verified without difficulty.

Theorem 1. *$A \otimes B$ is the frame coproduct of A and B with inclusion maps η_1 and η_2, equivalently $A \otimes B$ is the locale product of A and B with projection maps π_1 and π_2.*

Proof. Let $\psi \colon A \to Z$ and $\lambda \colon B \to Z$ be frame maps and $p \colon Z \to A$ and $g \colon Z \to B$ the adjoint maps to ψ respectively λ. We define $\psi \otimes \lambda \colon A \otimes B \to Z$ and $p \otimes g \colon Z \to A \otimes B$ by the formulae:

$$\psi \otimes \lambda(\varphi) = \bigvee_{x \in A} \psi(x) \wedge \lambda\varphi(x) \quad \text{and} \quad p \otimes g(z)(a) = g(\psi(a) * z).$$

Then $\psi\otimes\lambda$ will turn out to be a frame map and $p\otimes g$ its adjoint locale map. First we show that $\psi\otimes\lambda$ and $p\otimes g$ are adjoint. We have:

$$\begin{array}{lll}
\varphi \leqslant p\otimes g(z) & & \text{if and only if}\\
\varphi(x) \leqslant p\otimes g(z)(x) & \text{for all } x\in A & \text{if and only if}\\
\varphi(x) \leqslant g(\psi(x) * z) & \text{for all } x\in A & \text{if and only if}\\
\lambda\varphi(x) \leqslant \psi(x) * z & \text{for all } x\in A & \text{if and only if}\\
\psi(x)\wedge\lambda\varphi(x) \leqslant z & \text{for all } x\in A & \text{if and only if}\\
\psi\otimes\lambda(\varphi) = \bigvee_{x\in A}\psi(x)\wedge\lambda\varphi(x) \leqslant z. & &
\end{array}$$

Also

$$\psi\otimes\lambda(1_{A\otimes B}) = \bigvee_{x\in A}\psi(x)\wedge\lambda(1_B) \geqslant \psi(1_A)\wedge\lambda(1_B) = 1_Z$$

and

$$\begin{aligned}
\psi\otimes\lambda(\varphi_1\wedge\varphi_2) &= \bigvee_{x\in A}\psi(x)\wedge\lambda(\varphi_1\wedge\varphi_2)(x)\\
&= \bigvee_{x\in A}\psi(x)\wedge\lambda(\varphi_1(x)\wedge\varphi_2(x))\\
&= \bigvee_{x\in A}\psi(x)\wedge\lambda\varphi_1(x)\wedge\lambda\varphi_2(x)\\
&= \bigvee_{x,w\in A}\psi(x\wedge w)\wedge\lambda\varphi_1(x\wedge w)\wedge\lambda\varphi_2(x\wedge w),\\
\psi\otimes\lambda(\varphi_1\wedge\varphi_2) &\geqslant \bigvee_{x\in A}\bigvee_{w\in A}\psi(x)\wedge\psi(w)\wedge\lambda\varphi_1(x)\wedge\lambda\varphi_2(w),\\
\psi\otimes\lambda(\varphi_1\wedge\varphi_2) &\geqslant \Big[\bigvee_{x\in A}\psi(x)\wedge\lambda\varphi_1(x)\Big]\wedge\Big[\bigvee_{w\in A}\psi(w)\wedge\lambda\varphi_2(w)\Big],\\
\psi\otimes\lambda(\varphi_1\wedge\varphi_2) &\geqslant [\psi\otimes\lambda(\varphi_1)]\wedge[\psi\otimes\lambda(\varphi_2)]
\end{aligned}$$

but the opposite inequality is clear and so

$$\psi\otimes\lambda(\varphi_1\wedge\varphi_2) = [\psi\otimes\lambda(\varphi_1)]\wedge[\psi\otimes\lambda(\varphi_2)]$$

and $\psi\otimes\lambda$ is a frame map whence $p\otimes g$ is a locale map.

Furthermore,

$$\begin{aligned}
\psi\otimes\lambda(\eta_1(a)) &= \bigvee_{x\in A}\psi(x)\wedge\lambda(\eta_1(a)(x))\\
&= \bigvee_{x\leqslant a}\psi(x)\wedge\lambda(1_B)\\
&= \psi(a)
\end{aligned}$$

and

$$\begin{aligned}
\psi\otimes\lambda(\eta_2(b)) &= \bigvee_{x\in A}\psi(x)\wedge\lambda(\eta_2(b)(x))\\
&= \bigvee_{0\neq x\in A}\psi(x)\wedge\lambda(b)\\
&= \lambda(b).
\end{aligned}$$

Now let $c\colon A\otimes B\to Z$ be any frame map such that $c\eta_1=\psi$ and $c\eta_2=\lambda$. Any element of $A\otimes B$ is a union of elements of the form $\eta_1(x)\wedge\eta_2(y)$. So in order to

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check that $c=\psi\otimes\lambda$ we need only check that $c\eta_1(x)=\psi\otimes\lambda(\eta_1(x))$ and $c\eta_2(y)=\psi\otimes\lambda(\eta_2(y))$. But both of these follow immediately from the formulas above.

To prove the existence of infinite products it will now suffice to show the existence of inverse limits of directed systems of locales with all bonding maps surjective. We note that any directed inverse system of locales has the same inverse limit as one with all bonding maps surjective since we may replace each term L_α of the inverse system by the intersection of the images of the bonding maps into L_α.

Now let $[L_\alpha]_{\alpha\in D}$ be a directed inverse system of locales with surjective bonding maps $g_{\alpha\beta}\colon L_\beta\to L_\alpha$ for $\alpha<\beta$ satisfying $g_{\alpha\beta}\circ g_{\beta\gamma}=g_{\alpha\gamma}\colon L_\gamma\to L_\alpha$. Let $L=\lim L_\alpha$ where the inverse limit is in the sense of sets. Since all the $g_{\alpha\beta}$ commute with infinite intersections, the set L has an order which admits infinite intersections and L is a complete lattice. Then the set maps $g_\alpha\colon L\to L_\alpha$ commute with infinite intersections. Let $b,u\in L$ and define:

$$q_\beta=\bigwedge_{\alpha\geqslant\beta} g_{\beta\alpha}(g_\alpha(b)*g_\alpha(u)). \tag{‡}$$

Note that if $\alpha>\tau>\beta$ then

$$\begin{aligned} g_{\beta\alpha}(g_\alpha(b)*g_\alpha(u)) &= g_{\beta\tau}\,g_{\tau\alpha}(g_\alpha(b)*g_\alpha(u)),\\ g_{\beta\alpha}(g_\alpha(b)*g_\alpha(u)) &\leqslant g_{\beta\tau}(g_{\tau\alpha}\,g_\alpha(b)*g_{\tau\alpha}\,g_\alpha(u)),\\ g_{\beta\alpha}(g_\alpha(b)*g_\alpha(u)) &\leqslant g_{\beta\tau}(g_\tau(b)*g_\tau(u)) \end{aligned}$$

so that the intersection in (‡) may be taken over any cofinal subset of D all of whose elements are greater than β. Now if $\mu>\beta$, then

$$\begin{aligned} g_{\beta\mu}(q_\mu) &= g_{\beta\mu}\Big(\bigwedge_{\alpha\geqslant\mu} g_{\mu\alpha}(g_\alpha(b)*g_\alpha(u))\Big)\\ &= \bigwedge_{\alpha\geqslant\mu} g_{\beta\mu}\,g_{\mu\alpha}(g_\alpha(b)*g_\alpha(u))\\ &= \bigwedge_{\alpha\geqslant\mu} g_{\beta\alpha}(g_\alpha(b)*g_\alpha(u))\\ &= \bigwedge_{\alpha\geqslant\beta} g_{\beta\alpha}(g_\alpha(b)*g_\alpha(u)) = q_\beta \end{aligned}$$

by the above remark. Thus the q_β represent an element $q\in L$. Now

$$\begin{aligned} &g_\beta(q\wedge b)=q_\beta\wedge g_\beta(b),\\ &g_\beta(q\wedge b)=g_\beta(b)\wedge\bigwedge_{\alpha\geqslant\beta} g_{\beta\alpha}(g_\alpha(b)*g_\alpha(u)),\\ &g_\beta(q\wedge b)\leqslant g_\beta(b)\wedge(g_\beta(b)*g_\beta(u)),\\ &g_\beta(b\wedge q)\leqslant g_\beta(u) \end{aligned}$$

and so

$$b\wedge q\leqslant u.$$

We will show that $q=b*u$ in L. Let $x\wedge b\leqslant u$ in L. Then

$$\begin{aligned} g_\alpha(x)\wedge g_\alpha(b)&\leqslant g_\alpha(u)\\ g_\alpha(x)&\leqslant g_\alpha(b)*g_\alpha(u) \end{aligned}$$

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and if $\alpha > \beta$, $g_\beta(x) = g_{\beta\alpha} g_\alpha(x) \leqslant g_{\beta\alpha}(g_\alpha(b) * g_\alpha(u))$. So

$$g_\beta(x) \leqslant \bigwedge_{\alpha \geqslant \beta} g_{\beta\alpha}(g_\alpha(b) * g_\alpha(u)) = g_\beta(q)$$

and so $x \leqslant q$ and $q = b * u$ in L. Thus L is a locale.

Note that the frame maps $f_{\beta\alpha}$ adjoint to the $g_{\alpha\beta}$ satisfy $g_{\alpha\beta} f_{\beta\alpha} = \mathrm{Id}_{L_\alpha}$. Also we may define $f_\alpha\colon L_\alpha \to L$ by the formula $[f_\alpha(x)]_\beta = f_{\beta\alpha}(x)$ since if $\beta > \mu > \alpha$ we have

$$\begin{aligned} g_{\mu\beta}([f_\alpha(x)]_\beta) &= g_{\mu\beta} f_{\beta\alpha}(x) \\ &= g_{\mu\beta} f_{\beta\mu} f_{\mu\alpha}(x) \\ &= f_{\mu\alpha}(x) \\ &= [f_\alpha(x)]_\mu. \end{aligned}$$

Then f_α is the adjoint to g_α (defined by (†)), since $g_\alpha f_\alpha(x) = x$ and if $\beta > \alpha$,

$$[f_\alpha g_\alpha(y)]_\beta = f_{\beta\alpha} g_\alpha(y) = f_{\beta\alpha} g_{\alpha\beta} g_\beta(y) \leqslant g_\beta(y)$$

and so

$$f_\alpha g_\alpha(y) \leqslant y.$$

Now

$$\begin{aligned} [f_\alpha(x \wedge z)]_\beta &= f_{\beta\alpha}(x \wedge z) \\ &= f_{\beta\alpha}(x) \wedge f_{\beta\alpha}(z), \\ [f_\alpha(x \wedge z)]_\beta &= [f_\alpha(x)]_\beta \wedge [f_\alpha(z)]_\beta \end{aligned}$$

and

$$f_\alpha(x \wedge z) = f_\alpha(x) \wedge f_\alpha(z).$$

Thus the f_α are frame maps and so the g_α are locale maps.

Now if $p_\alpha\colon P \to L_\alpha$ are locale maps such that $g_{\alpha\beta} p_\beta = p_\alpha$ then the p_α lift uniquely to $p\colon P \to L$ which preserves arbitrary intersections. By uniqueness of adjoints the adjoint $r\colon L \to P$ satisfies $r f_\alpha = r_\alpha$ where the r_α are adjoint to the p_α. If $x \in L$ then $x = \bigvee_\alpha f_\alpha g_\alpha(x)$. So.

$$\begin{aligned} r(x \wedge y) &= r\Big(\bigvee_\alpha f_\alpha g_\alpha(x \wedge y)\Big) \\ &= \bigvee_\alpha r f_\alpha g_\alpha(x \wedge y) \\ &= \bigvee_\alpha r_\alpha g_\alpha(x \wedge y) \\ &= \bigvee_\alpha (r_\alpha g_\alpha(x) \wedge r_\alpha g_\alpha(y)) \\ &= \bigvee_{\alpha,\beta} (r_\alpha g_\alpha(x) \wedge r_\beta g_\beta(y)) \end{aligned}$$

since if $\psi < \mu$, $r_\mu g_\mu(x) \geqslant r_\mu f_{\mu\psi} g_{\psi\mu} g_\mu(x) = r_\psi\, g_\psi(x)$. So

$$\begin{aligned} r(x \wedge y) &= \Big(\bigvee_\alpha r_\alpha g_\alpha(x)\Big) \wedge \Big(\bigvee_\beta r_\beta g_\beta(y)\Big) \\ &= r\Big(\bigvee_\alpha f_\alpha g_\alpha(x)\Big) \wedge r\Big(\bigvee_\beta f_\beta g_\beta(y)\Big), \\ r(x \wedge y) &= r(x) \wedge r(y) \end{aligned}$$

and r is a frame map. So p is a locale map.

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In this section we will discuss the representation spaces of frames, considered as distributive lattices. References for representation spaces of distributive lattices are Stone (1937) and Gratzer (1971). If L is a distributive lattice we denote by $B(L)$ the Boolean algebra generated by L (Gratzer (1971), Chapter 10) and $R(L)$ will denote the representation space of L. We note that in addition to its usual compact T_0 topology σ, $R(L)$ has a stronger compact Hausdorff topology τ, the strong topology (see Nerode (1959)) such that $(R(L), \tau)$ is naturally the representation space for the Boolean algebra $B(L)$ and the identity mapping $(R(L), \tau) \to (R(L), \sigma)$ is induced by the lattice mapping $L \to B(L)$. A σ-continuous map $R(L_1) \to R(L_2)$ is induced by a lattice mapping $L_2 \to L_1$ if and only if it is τ-continuous. There is a natural isomorphism ρ from the lattice $I(L)$ of ideals of L to the lattice of open sets of $R(L)$.

PROPOSITION 2. *The distributive lattice L is a frame if and only if, in R(L) the τ-closure of a σ-open set is σ-open. In this case* $\rho(\langle \vee i \rangle) = \tau\text{-closure } \rho(i)$ *for any ideal i of L.*

PROOF. If in $R(L)$ the τ-closure of a σ-open set is σ-open, then L must be complete, for to any ideal i of L corresponds the σ-open set $\rho(i)$ of $R(L)$ and if l is an upper bound for i then $\rho(\langle l \rangle)$ is τ-closed and thus contains the τ-closure of $\rho(i)$, which τ-closure must therefore correspond to a least upper bound for i. We must show the identity $y \wedge \bigvee_\alpha x_\alpha = \bigvee_\alpha (y \wedge x_\alpha)$ in L. Clearly the left side is greater than or equal to the right. So we must show $\bigvee_\alpha (y \wedge x_\alpha) \geqslant y \wedge \bigvee_\alpha x_\alpha$ or equivalently,

$$\tau\text{-closure}\Big[\bigcup_\alpha (\rho(y) \cap \rho(x_\alpha))\Big] \supset \rho(y) \cap \tau\text{-closure}\Big[\bigcup_\alpha \rho(x_\alpha)\Big].$$

But this follows immediately from the fact that $\rho(y)$ is τ-open in $R(L)$.

Now suppose that L is a frame, let i be an ideal of L, and let l be the least upper bound of i. Then $\rho(\langle l \rangle)$ is τ-closed and thus contains the τ-closure of $\rho(i)$. Let p be a point of $R(L)$ and suppose that N is a τ-open set with $p \in N$ and $N \cap \rho(i) = \varnothing$. Then N contains a subset $W = \rho(\langle a \rangle) \cap [R(L) - \rho(\langle b \rangle)]$ with $p \in W$ since such sets form a base for the topology τ. Then $\rho(i) \cap \rho(\langle a \rangle) \cap [R(L) - \rho(\langle b \rangle)]$ is empty so $\rho(i) \cap \rho(\langle a \rangle) \subset \rho(\langle b \rangle)$ and if $x \in i$ then $x \wedge a \leqslant b$. Then since L is a frame $(\vee i) \wedge a \leqslant b$ and so $\rho(\langle l \rangle) \cap \rho(\langle a \rangle) \subset \rho(\langle b \rangle)$ whence $p \notin \rho(\langle l \rangle)$. It follows that $\rho(\langle l \rangle)$ is the τ-closure of $\rho(i)$ which is therefore weakly open.

PROPOSITION 3. *Let L_1 and L_2 be frames and let $\varphi: L_1 \to L_2$ be a homomorphism of bounded distributive lattices, and let $R(\varphi): R(L_2) \to R(L_1)$ be the associated map of representation spaces. Then φ is a frame map if and only if for any open set U of $R(L_1)$ we have τ_2-closure$((R\varphi)^{-1}(U)) = (R\varphi)^{-1}(\tau_1$-closure $U)$.*

PROOF. Clear.

LEMMA 1. *Let α and β be topologies on a set X. Then the following are equivalent:*
(1) *The α-closure of a β-open subset of X is β-open.*
(2) *The β-closure of an α-open subset of X is α-open.*

PROOF. We prove only (1) implies (2). Let T be α-open and let W be its β-closure. Then $[\alpha\text{-closure}(X-W)] \cap T$ is empty since T is α-open. But $X-[\alpha\text{-closure}(X-W)]$ is a β-closed set containing T and contained in the β-closure W of T so

$$W = X - [\alpha\text{-closure}(X-W)]$$

which is α-open.

COROLLARY 1. *L is a frame if and only if in $R(L)$ the σ-closure of any τ-open set* is τ-open.

PROPOSITION 4. *Let $\varphi: L_1 \to L_2$ be a bounded distributive lattice homomorphism between the frames L_1 and L_2. Then φ is a frame map if and only if whenever W is τ_2-open in $R(L_2)$, then the σ_1-closure of $R(\varphi)(W)$ is τ_1-open.*

PROOF. If φ is a frame map, then for any U which is σ_1-open in $R(L_1)$, we have $(R\varphi)^{-1}(\tau_1\text{-closure}(U)) = \tau_2\text{-closure}((R\varphi)^{-1}(U))$. Now let

$$U = R(L_1) - [\sigma_1\text{-closure}(R\varphi(W))]$$

which is σ_1-open. Then:

$$(R\varphi)^{-1}(U) \cap W = \varnothing,$$

$$\tau_2\text{-closure}((R\varphi)^{-1}(U)) \cap W = \varnothing,$$

$$(R\varphi)^{-1}(\tau_1\text{-closure}(U)) \cap W = \varnothing = \tau_1\text{-closure}(U) \cap R\varphi(W),$$

$$\tau_1\text{-closure}(U) \cap \sigma_1\text{-closure}(R\varphi(W)) = \varnothing,$$

and so

$$U = \tau_1\text{-closure}(U)$$

and

$$\sigma_1\text{-closure}(R\varphi(W)) \text{ is } \tau_1\text{-open}.$$

Now suppose that for all τ_2-open subsets W of $R(L_2)$ the σ_1-closure of $R\varphi(W)$ is τ_1-open. Let U be a σ_1-open subset of $R(L_1)$ and let

$$W = R(L_2) - [\tau_2\text{-closure}((R\varphi)^{-1}(U))],$$

which is τ_2-open. Then:

$$(R\varphi)^{-1}(U) \cap W = \varnothing,$$
$$U \cap R\varphi(W) = \varnothing,$$
$$U \cap \sigma_1\text{-closure}(R\varphi(W)) = \varnothing,$$
$$\tau_1\text{-closure}(U) \cap \sigma_1\text{-closure}(R\varphi(W)) = \varnothing,$$
$$(R\varphi)^{-1}(\tau_1\text{-closure}(U)) \cap W = \varnothing,$$
$$(R\varphi)^{-1}(\tau_1\text{-closure}(U)) \subset \tau_2\text{-closure}((R\varphi)^{-1}(U))$$

and so

$$(R\varphi)^{-1}(\tau_1\text{-closure}(U)) = \tau_2\text{-closure}((R\varphi)^{-1}(U))$$

since $R\varphi$ is τ-continuous.

We can now characterize the lattice of frame quotients $Q(L)$ of a frame L. Let $\varphi: L \to L_1$ be a surjective frame map. Then every σ_1-compact open subset of $R(L_1)$ is the inverse image under $R\varphi$ of a σ-compact open subset of $R(L)$. It follows that the relative τ and σ topologies of the image of $R\varphi$ are isomorphic by φ to the τ_1 and σ_1 topologies of $R(L_1)$. Now since $(R(L_1), \tau_1)$ is Hausdorff, $R\varphi$ is injective on points, and $R(L_1)$ can be identified with a τ-closed subset of $R(L)$ with the induced σ and τ topologies. Proposition 4 shows that the σ-closure in $R(L)$ of a relatively τ-open subset of $R(L_1)$ is τ-open. Suppose conversely that a τ-closed subset S of $R(L)$ has the property that any relatively τ-open subset of S has τ-open σ-closure in $R(L)$. Then since S is τ-closed it is the representation space of some distributive lattice quotient T of L, T is a frame by Corollary 1, and quotient map $L \to T$ is a frame map since it satisfies the conditions of Proposition 4. Therefore:

Proposition 5. *$Q(L)$ is naturally isomorphic to the lattice of τ-closed subsets S of $R(L)$ such that the σ-closure in $R(L)$ of a relatively τ-open subset of S is τ-open.*

The following corollary is essentially proved in Dowker and Papert (1966).

Corollary 2. *The dual $Q'(L)$ of $Q(L)$ is a q-set in $I(B(L))$ and $I(B(L))$ is naturally isomorphic to the dual lattice to the lattice of distributive lattice congruence relations on L.*

Proof. The isomorphism of $I(B(L))$ with the dual of the lattice of distributive lattice congruence relations on L is proved in Gratzer (1971). The isomorphism assigns to a congruence relation C the complement of the image of $R\psi$ where ψ is the quotient by the congruence relation.

Let $\{S_\alpha\}$ be a collection of subsets of $R(L)$ satisfying the condition of Proposition 5. The τ-closure S of $\bigcup_\alpha S_\alpha$ will be shown to satisfy the same condition. Let U be relatively τ-open in S. Then σ-closure$(U \cap S_\alpha) = U_\alpha$ is τ-open in $R(L)$ for all α.

Since $\bigcup_\alpha S_\alpha$ is τ-dense in S, and U is τ-open in S, we have

$$\tau\text{-closure}\left(\bigcup_\alpha (U \cap S_\alpha)\right) \supset U;$$

so

$$\sigma\text{-closure}(U) = \sigma\text{-closure}\left(\bigcup_\alpha (U \cap S_\alpha)\right) = \sigma\text{-closure}\left(\bigcup_\alpha U_\alpha\right)$$

which is the σ-closure of a τ-open set hence τ-open. This shows that the subset $Q'(L)$ of $I(B(L))$ is closed under arbitrary intersections. To show it is a q-set we must show that if T is a τ-closed subset of $R(L)$ and S satisfies the conditions of Proposition 5, then τ-closure$((R(L)-T)\cap S)$ satisfies the conditions of Proposition 5. In other words, if S is a τ-closed set such that any relatively τ-open subset of S has τ-open σ-closure in $R(L)$ then the τ-closure M of a relatively τ-open subset N of S has the same property.

For this let U be relatively τ-open in M. Since N is τ-dense in M we have $U \subset \tau$-closure$(N \cap U)$ and so σ-closure$(U) = \sigma$-closure$(N \cap U)$. But $N \cap U$ is relatively τ-open in S and so σ-closure$(U) = \sigma$-closure$(N \cap U)$ is τ-open in $R(L)$.

We now consider the diagram of locales

(#)
$$\begin{array}{ccc} & & I(B(L)) \\ & & \downarrow \\ L & \longrightarrow & I(L) \end{array}$$

Our main result will be that the locale pullback of the diagram is naturally isomorphic to $Q'(L)$, the dual of the lattice of quotients of L (see Dowker and Strauss 1975)). We will identify $I(B(L))$ with the lattice of τ-closed subsets of $R(L)$, ordered by reverse inclusion, and $I(L)$ with the sublattice of σ-closed subsets of $R(L)$.

The set pullback of the diagram (#) is the set S of τ-closed subsets s of $R(L)$ such that the σ-closure of s is τ-open. Let $\{s_\alpha\}_{\alpha\in J}$ be a subset of S. Then the τ-closure s of $\bigcup_\alpha s_\alpha$ is in S, since

$$\sigma\text{-closure}(s) = \sigma\text{-closure}\left(\bigcup_\alpha s_\alpha\right) = \sigma\text{-closure}\left(\bigcup_\alpha (\sigma\text{-closure}(s_\alpha))\right)$$

is the σ-closure of a τ-open set hence τ-open. For the diagram (#) considered as a diagram of lattices this means that:

LEMMA 2. *The set pullback of* (#) *is a subset S of $I(B(L))$ which is closed under arbitrary intersections.*

Next we show:

LEMMA 3. *Let T be a subset of a frame F which is closed under arbitrary intersections. Then the set $R = \{t \in T \mid a * t \in T$ for all $a \in F\}$ is the largest q-set of F contained in T.*

PROOF. R clearly contains any q-set of F which is contained in T. Now if

$\{u_\alpha\} \subset R$ then if $a \in F$ we have $a * \bigwedge_\alpha u_\alpha = \bigwedge_\alpha a * u_\alpha$ which is an intersection of elements of T hence in T and so $\bigwedge_\alpha u_\alpha$ is in R. If $u \in R$, $a \in F$ and $x \in F$ then $x * (a * u) = (x \wedge a) * u \in T$ and so $a * u \in R$ and R is a q-set.

THEOREM 2. *The locale pullback of the diagram (#) is naturally isomorphic to $Q'(L)$.*

PROOF. Lemma 3 implies that the largest q-set of $I(B(L))$ contained in the set pullback S of (#) is the set $Q \subset I(B(L))$ where

$$Q = \{s \in S \mid a * s \in S \quad \text{for all } a \in I(B(L))\},$$

in other words the set of $s \in I(B(L))$ (which we identify with the lattice of τ-closed subsets of $R(L)$), such that the σ-closure of any relatively τ-open subset of s is again in S. But this is exactly the characterization of $Q(L)$ given by Proposition 5. Now L is just the image of $Q'(L)$ in $I(L)$.

We conclude that if A is an arbitrary locale, then any map of A into (#) factors uniquely through

$$\begin{array}{ccc} Q'(L) & \longrightarrow & I(B(L)) \\ \downarrow & & \downarrow \\ L & \longrightarrow & I(L) \end{array}$$

so that $Q'(L)$ is the pullback of (#).

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