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A FINITELY GENERATED MODULAR ORTHOLATTICE

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ABSTRACT. We discuss [2] of the same title and offer an alternative example. This example is a subalgebra of the ortholattice of closed subspaces of separable real Hilbert space.

In [2] an ultraproduct construction is used to provide an example of: *a 3-generator (with generators a, c, d) modular ortholattice containing an infinite independent sequence of nonzero pairwise perspective (and orthogonal) elements.*

The ultraproduct construction is correct but the subspaces E and F defined on page 243 of [2] do not have the properties claimed of them. In particular, $(E \cap A) + (F \cap A) \neq A$. An analysis, cf. [3], [4], of the, necessarily 2-distributive, modular lattice generated by the elements, $a, a^{\perp}, c, d$ may allow one to show that the perspectivities claimed in [2], or ones derived from these, exist in the ultraproduct. However a second, we feel, important property is not so easily restored: *The elements,* $0, 1, a, a^{\perp}, c, d$, *generate* $FM(J_1^4)$, cf. [1].

In this note we give an alternative example which has this property as well. We would like to emphasize that the underlying idea has not changed. The choice of bases for the generators, or less exactly, the idea of infinitely stretching a quadruple of defect -1 to get rid of all but one nontrivial join, has been lifted directly from [2].

Let H be a separable, real Hilbert space, and let A and $A^{\perp}$ be orthocomplementary infinite dimensional closed subspaces of H. Let $(e_i)_{i\in\mathbf{N}}$ be a orthonormal basis of $A^{\perp}$, and let $(f_i)_{i\in\mathbf{N}}$ be an orthonormal basis of A. Define C to be the closed subspace with orthogonal basis $\big(e_i + (1/2)f_{i+1}\big)_{i\in\mathbf{N}}$, and D the closed subspace with orthogonal basis $(e_i + f_i)_{i\in\mathbf{N}}$. It is easily seen that $\big(f_1, (1/2e_i - f_{i+1})_{i\in\mathbf{N}}\big)$ is an orthogonal basis of $C^{\perp}$ and that $\big((e_i - f_i)_{i\in\mathbf{N}}\big)$ is an orthogonal basis of $D^{\perp}$. Throughout the paper we will use the symbol $+$ to denote the addition of vectors and the sum of vector spaces. The symbol $\vee$ will be used for the closure of the sum of two vector spaces, i.e. their join in $L(H)$, the orthomodular lattice of closed subspaces of H. We write $\langle X\rangle$, with X a set of vectors, for the closed subspace generated by X.

LEMMA 1.

$$\begin{aligned} A + C = A + C^{\perp} = A + D = A + D^{\perp} = A^{\perp} + C^{\perp} = A^{\perp} + D \\ = A^{\perp} + D^{\perp} = C + D = C + D^{\perp} = C^{\perp} + D = C^{\perp} + D^{\perp} = H; \end{aligned}$$

and,

$$A^{\perp} \vee C = A^{\perp} + C = \langle f_1\rangle^{\perp}.$$

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PROOF. *We will provide proofs only for the last four sums to H, the proofs of the other equalities are more or less automatic.*

$C + D = H$: *Since it is easy to see that* $A^\perp + D = H$ *it is enough to show that* $A^\perp \subseteq C + D$. *Assume* $\sum_{i=1}^\infty \alpha_i e_i \in A^\perp$. *Define recursively numbers* β_i *by* $\beta_1 = \alpha_1$, $\beta_{i+1} = \alpha_{i+1} + (1/2)\beta_i$, $i \geq 1$. *Then,*

$$\begin{aligned}\sum_{i=1}^n \beta_i^2 &= \alpha_1^2 + \sum_{i=2}^n \big(\alpha_i + (1/2)\beta_{i-1}\big)^2\\ &= \sum_{i=1}^n \alpha_i^2 + \sum_{i=2}^n \alpha_i\beta_{i-1} + (1/4)\sum_{i=1}^{n-1}\beta_i^2\\ &\leq \sum_{i=1}^n \alpha_i^2 + (1/2)\sum_{i=2}^n \alpha_i^2 + (1/2)\sum_{i=1}^{n-1}\beta_i^2 + (1/4)\sum_{i=1}^{n-1}\beta_i^2\\ &\leq (3/2)\sum_{i=1}^n \alpha_i^2 + (3/4)\sum_{i=1}^n \beta_i^2.\end{aligned}$$

Thus, $\sum_{i=1}^n \beta_i^2 \leq 6\sum_{i=1}^n \alpha_i^2$ *and hence,* $\sum_{i=1}^\infty \beta_i^2 < \infty$. *It follows that* $\sum_{i=1}^\infty \beta_i\big(e_i + (1/2)f_{i+1}\big)$ *exists and belongs to C and that* $\sum_{i=2}^\infty -(1/2)\beta_{i-1}(e_i + f_i)$ *exists and belongs to D. But,*

$$\begin{aligned}\sum_{i=1}^\infty \beta_i(e_i + (1/2)f_{i+1}) + \sum_{i=2}^\infty(-1/2)\beta_{i-1}(e_i + f_i))\\ &= \beta_1 e_1 + \sum_{i=2}^\infty\big(\beta_i - (1/2)\beta_{i-1}\big)e_i\\ &= \sum_{i=1}^\infty \alpha_i e_i \in C + D.\end{aligned}$$

Thus, $C + D = H$.

$C + D^\perp = H$: Again it is enough to show that $A^\perp \subseteq C + D^\perp$. Assume $\sum_{i=1}^\infty \alpha_i e_i \in A^\perp$. Define numbers β_i by $\beta_1 = \alpha_1$, $\beta_{i+1} = \alpha_{i+1} - (1/2)\beta_i$, $i \geq 1$. Almost verbatim the same way as in the last proof it follows that $\sum_{i=1}^\infty \beta_i\big(e_i + (1/2)f_{i+1}\big)$ exists and belongs to C and that $\sum_{i=2}^\infty (1/2)\beta_{i-1}(e_i - f_i)$ exists and belongs to $D^\perp$. The sum of these two vectors is $\sum_{i=1}^\infty \alpha_i e_i \in C + D^\perp$.

$C^\perp + D = H$: It is enough to show that $A^\perp \subseteq C^\perp + D$. Assume $\sum_{i=1}^\infty \alpha_i e_i \in A^\perp$. Let K be an upper bound of the sequence $(|\alpha_i|)_{i\in\mathbf{N}}$. Then, for natural numbers $i < n$, $\sum_{j=i}^n |(-1)^{i+j-1}(1/2^{j-i})\alpha_j| \leq K\sum_{j=0}^\infty (1/2^j) = 2K$. We may thus define numbers β_i by $\beta_i = \sum_{j=i}^\infty (-1)^{i+j-1}(1/2^{j-i})\alpha_j$ and the $|\beta_i|$ have the bound $2K$. Note that $2\beta_i + \beta_{i+1} =$

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$-2\alpha_i$. It follows that

$$\begin{aligned}\sum_{i=1}^{n}\beta_i^2 &= \sum_{i=1}^{n}\left(\alpha_i + (1/2)\beta_{i+1}\right)^2\\ &= \sum_{i=1}^{n}\alpha_i^2 + \sum_{i=1}^{n}\alpha_i\beta_{i+1} + (1/4)\sum_{i=2}^{n+1}\beta_i^2\\ &\leq \sum_{i=1}^{n}\alpha_i^2 + (1/2)\sum_{i=1}^{n}\alpha_i^2 + (1/2)\sum_{i=2}^{n+1}\beta_i^2 + (1/4)\sum_{i=2}^{n+1}\beta_i^2\\ &\leq (3/2)\sum_{i=1}^{n}\alpha_i^2 + (3/4)\sum_{i=1}^{n}\beta_i^2 + 3K^2\end{aligned}$$

and hence that $\sum_{i=1}^{\infty}\beta_i^2 < \infty$. Note that the $3K^2$ is a bound of $(3/4)\beta_{n+1}^2$. As before we obtain that $\beta_1 f_1 + \sum_{i=2}^{\infty}\beta_i\left(f_i - (1/2)e_{i-1}\right)$ exists and belongs to $C^\perp$, that $\sum_{i=1}^{\infty} -\beta_i(e_i + f_i)$ exists and belongs to D, and that their sum is $\sum_{i=1}^{\infty}\alpha_i e_i$.

$C^\perp + D^\perp = H$: The proof is almost verbatim the same as the last proof with the definition $\beta_i = \sum_{j=i}^{\infty}(1/2^{j-i})\alpha_j$, and the formula $\beta_i = \alpha_i + (1/2)\beta_{i+1}$. This completes the proof of lemma 1.

Let $M = \left\{\{0\}, A, A^\perp, C, C^\perp, D, D^\perp, H\right\}$ and let I be the p-ideal in $L(H)$ consisting of all finite-dimensional subspaces of H. Let K consist of all closed subspaces of H which are congruent modulo I to some $S \in M$. In other words: A closed subspace X in H belongs to K iff there exists $S \in M$ such that $(X \vee S) \wedge (X \wedge S)^\perp$ is finite-dimensional. Since any two elements of M are incongruent modulo I it follows that for every $X \in K$ there exists exactly one $S \in M$ such that $X \equiv S \bmod I$. We may thus define

$$d(X) = \dim\left((X \vee S) \wedge (X \wedge S)^\perp\right).$$

Note that for every $X \in K$, $d(X) = d(X^\perp)$. Since M is closed under orthocomplementation, K is also closed under orthocomplementation. Since the join and meet in $L(H)$ of any two elements in K is congruent to some element in M it follows that K is a sub-ortholattice of $L(H)$. The decisive fact now is:

LEMMA 2. *For all $X, Z \in K$: $X \vee Z = X + Z$.*

PROOF. Note that the claim is trivially true if one of X, Z is finite dimensional. Let $\prec$ be the covering relation in $L(H)$. We proceed by induction on $n = d(X) + d(Z)$. If $n = 0$ the claim is trivial or reduces to one of the claims of Lemma 1. If $n \geq 1$ we may assume by symmetry that $d(X) \geq 1$ and $X \equiv S \bmod I$ with $S \in M$. Assume first that $S \cap X < X$. Choose $Y \in L(H)$ such that $S \cap X \leq Y \prec X$. Then $S \cap X = S \cap Y$. Since the congruence class of S is modular it follows that $S \vee Y < S \vee X$ and hence that $S \vee Y \prec S \vee X$, and $d(Y) = d(X) - 1$. By inductive hypothesis $Y \vee Z = Y + Z$. Since $X \cap Y^\perp$ is one-dimensional this gives $X + Z = (X \cap Y^\perp) + Y + Z = (X \cap Y^\perp) + (Y + Z) = (X \cap Y^\perp) + (Y \vee Z) = (X \cap Y^\perp) \vee Y \vee Z = X \vee Z$, settling the case $S \cap X < X$.

We may thus assume that $X < S$. Choose $Y \in L(H)$ such that $X \prec Y \leq S$. By inductive hypothesis we obtain $Y \vee Z = Y + Z$ and $Y^\perp \vee Z^\perp = Y^\perp + Z^\perp$. If $X \cap Z < Y \cap Z$ then

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$X < X + (Y \cap Z) \leq Y$ and hence $X + (Y \cap Z) = Y$. Thus $X + Z = X + (Y \cap Z) + Z = Y + Z = Y \vee Z \geq X \vee Z$, and $X + Z = X \vee Z$. We may thus also assume that $X \cap Z = Y \cap Z$. Since the lattice $V(H)$ of (not necessarily closed) subspaces of H is modular it follows that $Y + Z = Y \vee Z$ covers $X + Z$ in $V(H)$. If $X \vee Z \neq X + Z$ we would obtain $X + Z < X \vee Z \leq Y \vee Z = Y + Z$ and thus $X \vee Z = Y \vee Z$. This would give

$$
\begin{aligned}
\big((X^\perp \cap Y) \cap (X \vee Z)\big)^\perp &= X \vee Y^\perp \vee (X^\perp \cap Z^\perp) \\
&= (X \vee Y^\perp) + (X^\perp \cap Z^\perp) \quad (\text{since } X \vee Y^\perp \prec H) \\
&= X + Y^\perp + (X^\perp \cap Z^\perp) \quad (\text{since } X \text{ and } Y^\perp \text{ are orthogonal}) \\
&= X + [(Y^\perp + Z^\perp) \cap X^\perp] \quad (\text{by the modularity of } V(H)) \\
&= X + [(Y^\perp \vee Z^\perp) \cap X^\perp] \quad (\text{by inductive hypothesis}) \\
&= X + [(X^\perp \vee Z^\perp) \cap X^\perp] \quad (\text{since } X \cap Z = Y \cap Z) \\
&= X + X^\perp = H,
\end{aligned}
$$

and hence,

$$\{0\} = X^\perp \cap Y \cap (X \vee Z) = X^\perp \cap Y \cap (Y \vee Z) = X^\perp \cap Y \neq \{0\}.$$

Thus $X \vee Z = X + Z$, completing the proof.

It follows from Lemma 2 that K is a sublattice of $V(H)$ and hence modular. The same is thus true for the sub-ortholattice L of $L(H)$ generated by $\{A, C, D\}$, since it is a sublattice of K.

LEMMA 3. *L is a covering sublattice of K. In particular, L is atomistic.*

PROOF. Let $Y_1 = A^\perp \vee C, Z_i = (A \wedge Y_i) \vee (D \wedge Y_i)$, and $Y_{i+1} = (A^\perp \wedge Z_i) \vee (C \wedge Z_i)$. It is easy to show that $Z_n = \langle (e_i, f_i)_{i \leq n} \rangle^\perp$, and $Y_n = \langle (e_i, f_i)_{i \leq n-1}, f_n \rangle^\perp$. If we define $W_1 = Y_1$, $W_{2n} = Y_n^\perp \vee Z_n$, and $W_{2n+1} = Z_n^\perp \vee Y_{n+1}$, then $(W_i)_{i \in \mathbf{N}}$ is a sequence of coatoms in $L(H)$ with $\wedge_{i \in \mathbf{N}} W_i = 0$. Hence for nonzero $X \in L$ there exists $i \in \mathbf{N}$ so that $X \not\leq W_i$. It follows that X covers $X \wedge W_i$ in $L(H)$. Therefore the atoms of L are atoms of $L(H)$ and hence atoms of K. From this it follows that the coverings in L are coverings in K and the proof is complete.

LEMMA 4. *L is subdirectly irreducible.*

PROOF. In a modular ortholattice perspectivity between atoms is a transitive relation. It thus follows from Lemma 6.7 of [5] that the p-ideal generated by an atom x of a modular ortholattice consists of the set of all finite joins of atoms which are perspective to x. The p-ideal I of L consisting of the finite dimensional subspaces is generated by the atom $A \wedge C^\perp$ since identifying this atom with 0 produces the MOL $M03$, as does modding out all the finite-dimensional subspaces of L. Hence all the atoms of L are pairwise perspective and, since L is atomistic, I is the smallest nontrivial p-ideal of L. In fact it is easy to see that it is the only proper nontrivial p-ideal of K.

This gives our main result:

THEOREM 5. *There exists a sub-ortholattice of $L(H)$ which is subdirectly irreducible modular atomistic and generated by elements a, c, d satisfying*

$$\begin{aligned} a \vee d = a^{\perp} \vee d = a \vee d^{\perp} = a^{\perp} \vee d^{\perp} = a \vee c^{\perp} = a^{\perp} \vee c^{\perp} \\ = a \vee c = c \vee d = c^{\perp} \vee d = c \vee d^{\perp} = c^{\perp} \vee d^{\perp} = 1 \end{aligned}$$

and,

$$a^{\perp} \vee c \neq 1,$$

i.e. the lattice generated by the elements a, $a^{\perp}$, c, d in L is isomorphic to $FM(J_4)$.

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