THE RELATION BETWEEN FUNCTIONS SATISFYING A CERTAIN INTEGRAL EQUATION AND GENERAL WATSON TRANSFORMS

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1. In some work of Ramanujan[1] certain results are given which are equivalent to the following.

If

$$\int_0^\infty F(x)F(ux)dx = \frac{1}{1+u} \tag{1}$$

and

$$f(s) = \int_0^\infty F(x)x^{s-1}dx$$

then

$$f(s)f(1-s) = \frac{\pi}{\sin \pi s}. \tag{a}$$

Also, if

$$G(x) = \sqrt{\left(\frac{2}{\pi}\right)}\frac{F(ix) + F(-ix)}{2}$$

then the relations

$$\int_0^\infty A(x)\, G(xy)dx = B(y)$$

(b)

$$\int_0^\infty B(y)\, G(xy)dy = A(x)$$

are consequences of one another for arbitrary functions $A(x)$.

Examples can be given of both the truth and falsity of these results.

The function $F(x) = e^{-x}$ satisfies (1) and gives $f(s) = \Gamma(s)$ and $G(x) = \sqrt{(2/\pi)} \cos x$. Therefore (a) is true, and by Fourier integral theory the formulae (b) are consequences of one another for functions $A(x)$ with suitable properties.

On the other hand (1) is satisfied by

$$F(x) = \sqrt{\left(\frac{2}{\pi}\right)}\frac{x}{1+x^2}$$

which yields $f(s) = \sqrt{(\pi/2)} \sec \frac{1}{2}\pi s$ and $G(x) = 0$ for all values of x except $\pm i$. In this case (a) is true, but the formulae (b) are certainly not consequences of one another under any circumstances.

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[1]See, for example, *Ramanujan* by G. H. Hardy, Chapter 11, Formula *F*.

Taking the results as they stand, if (1) need hold for real values of u only, then $F(x)$ need be defined for real values of x only, and there is no obvious reason to suppose that $F(ix)$ or $G(x)$ can be defined in any sense.

2. The results proven in this paper show that if $F(x) \in L^2(0,\infty)$ and satisfies (1) then (a) is true and $F(x)$ is the value taken on the real axis by an analytic function $F(z)$, regular for $R(z) > 0$. Also, the formula

$$G_1(x) = \lim_{u\to 0} \sqrt{\left(\frac{2}{\pi}\right)} \int_0^x \frac{F(u+iv) + F(u-iv)\,dv}{2}$$

defines a function $G_1(x)$ for almost all positive x, and $G_1(x)/x$ is a Watson kernel, i.e. $G_1(x)/x \in L^2(0,\infty)$ and the relations

$$\frac{d}{dy}\int_0^\infty A(x)\,\frac{G_1(xy)}{x}\,dx = B(y)$$

and

$$\frac{d}{dx}\int_0^\infty B(y)\,\frac{G_1(xy)}{y}\,dy = A(x)$$

are consequences of one another for $A(x) \in L^2(0,\infty)$.

3. The tools most often used in the following work are L^2 Mellin transform theory and general Watson transform theory. The main results of the Mellin transform theory used are given[2] in T., Theorems 71 and 72 with $k = \frac{1}{2}$, while the Watson (or general) transform theory is described in T., Chapter VIII.

THEOREM 1. *Let $F(x)$ belong to the class $L^2(0,\infty)$ and satisfy* (1) *for all $u > 0$. Let $f(s)$ be the Mellin transform of $F(x)$ defined in the mean square sense for $R(s) = \frac{1}{2}$. Then*

$$f(s)f(1-s) = \Gamma(s)\Gamma(1-s) \tag{2}$$

for $R(s) = \frac{1}{2}$.

For, applying the Parseval formula for Mellin transforms (T., Theorem 72) to the left-hand side of (1), we have

$$\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} f(s)f(1-s)u^{-s}ds = \frac{1}{1+u}$$

since $f(s)u^{-s}$ is the Mellin transform of $F(ux)$. But

$$\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Gamma(s)\Gamma(1-s)u^{-s}ds = \frac{1}{1+u}$$

and so

$$\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \{f(s)f(1-s) - \Gamma(s)\Gamma(1-s)\}\, u^{-s}ds = 0.$$

Since $\Gamma(\frac{1}{2}+it)$ and $f(\frac{1}{2}+it)$ are $L^2(-\infty,\infty)$, $f(s)f(1-s) - \Gamma(s)\Gamma(1-s)$ is $L(\frac{1}{2}-i\infty, \frac{1}{2}+i\infty)$ and so the inverse form of T., Theorem 32, can be applied.

[2]The abbreviation T. is used for E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, Oxford, 1937.

This yields the formula $f(s)f(1-s) - \Gamma(s)\Gamma(1-s) = 0$ and proves Theorem 1.

Theorem 1 shows that the assertion (a) holds rigorously for functions of $L^2(0,\infty)$.

THEOREM 2. *Let $F(x)$ be a real function defined in $(0,\infty)$ satisfying the conditions of Theorem 1 and let $f(s)$ be defined as before. If*

$$k(s) = \frac{f(1-s)}{\Gamma(1-s)} \tag{3}$$

and

$$\frac{K_1(x)}{x} = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{k(s)}{1-s}\, x^{-s}ds \tag{4}$$

the integral being defined in the mean square sense, then $K_1(x)/x$ belongs to the class $L^2(0,\infty)$ and is the kernel of a Watson transform and

$$F(x) = x\int_0^\infty K_1(t)e^{-xt}dt = -x\frac{d}{dx}\int_0^\infty \frac{K_1(t)}{t}\, e^{-xt}dt \tag{5}$$

for almost all values of x. The function $F(x)$ is therefore almost everywhere the value taken on the real axis by a function $F(z) = F(x+iy)$ regular in the right half-plane.

The statement that $K_1(x)/x$ is the kernel of a Watson transform means that it has the same properties as the function $k_1(x)/x$ of T., Theorem 129.

Using the results of Theorem 1 we have

$$k(s)k(1-s) = \frac{f(s)f(1-s)}{\Gamma(s)\Gamma(1-s)} = 1. \tag{6}$$

Also, as $F(x)$ is a real function, $f(s)$ and therefore $k(s)$ take conjugate values for conjugate values of s and so

$$|k(\tfrac{1}{2}+it)| = |k(\tfrac{1}{2}+it)k(\tfrac{1}{2}-it)|^{\frac{1}{2}} = 1.$$

Therefore $K_1(x)/x$, as defined by (4), is a function of $L^2(0,\infty)$ and is a Watson kernel.

We also have for almost all x

$$\begin{aligned} F(x) &= \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} f(s)x^{-s}ds = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} k(1-s)\Gamma(s)x^{-s}ds \\ &= \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{k(1-s)}{s}\, s\Gamma(s)x^{-s}ds = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{k(1-s)}{s}\,\Gamma(s+1)x^{-s}ds \\ &= \int_0^\infty \frac{K_1(t)}{t}\, xte^{-xt}dt = x\int_0^\infty K_1(t)e^{-xt}dt. \end{aligned}$$

The first integral is the usual inverse Mellin transform, the second is obtained by using (3) and finally we apply the Parseval formula for Mellin transforms and use the fact that $k(s)/(1-s)$ is the transform of $K_1(t)/t$ and that $\Gamma(s+1)x^{-s}$ is the transform of xte^{-xt}.

The second formula for $F(x)$ in (5) can be obtained from the first by an appeal to general theory of reversion of the order of integration and differentiation, or by differentiation with respect to x of the relation

$$\int_x^\infty \frac{F(t)}{t}\,dt = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{k(1-s)\Gamma(s)}{s}\,x^{-s}ds$$
$$= \int_0^\infty \frac{K_1(t)}{t}\,e^{-xt}\,dt.$$

This relation is obtained by using the Parseval formula for Mellin transforms and the information that the transform of $F(t)$ is $k(1-s)\Gamma(s)$, the transform of the function equal to $1/t$ for $t \geqslant x$ and zero elsewhere is $-\,x^{s-1}/(s-1)$, and the transform of e^{-xt} is $\Gamma(s)x^{-s}$.

The integrals over the range $(\frac{1}{2}-i\infty,\ \frac{1}{2}+i\infty)$ occurring in this transformation are mean square integrals in Mellin transform theory, but, as $k(1-s)$ is bounded and $\Gamma(\frac{1}{2}+it) = O(e^{\frac{-\pi|t|}{2}})$, they are absolutely convergent as well, and so may be taken in the ordinary sense.

Corollary 1. *The condition that $F(x)$ be real in Theorem 2 may be dispensed with if $k(s)$ as defined by* (3) *is bounded on $R(s) = \frac{1}{2}$.*

For in the argument used in proving Theorem 2 the condition that $F(x)$ be real is only used to prove the boundedness of $k(s)$ on $R(s) = \frac{1}{2}$.

Corollary 2. *Any function $F(x)$ defined by* (5), *where $K_1(t)/t$ is a Watson kernel, will satisfy* (1).

If $K_1(t)/t$ is a Watson kernel, the final part of the argument may be reversed and we obtain

$$F(x) = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} k(1-s)\Gamma(s)x^{-s}ds$$

where $k(s)k(1-s) = 1$. Thus $F(x)$ is the Mellin transform of $k(1-s)\Gamma(s)$, and applying the Parseval relation,

$$\int_0^\infty F(x)F(ux)dx = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} k(s)\Gamma(1-s)k(1-s)\Gamma(s)u^{-s}ds$$
$$= \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Gamma(s)\Gamma(1-s)u^{-s}ds = \frac{1}{1+u}.$$

Theorem 3. *Let $F(x)$ satisfy the conditions of Theorem 2 and let $F(z)$ be its continuation, regular for $R(z) > 0$. Then if*

$$G_1(u,x) = \sqrt{\left(\frac{2}{\pi}\right)}\int_0^x \frac{F(u+iv)+F(u-iv)}{2}\,dv,$$

the limit as $u \to +0$ of $G_1(u,x)/x$ exists for almost all x in $(0,\infty)$ and defines a function $G_1(x)/x$ which belongs to the class $L^2(0,\infty)$ and is the kernel of a Watson transform. The Mellin transform of $G_1(x)/x$ is $\sqrt{(2/\pi)}\,f(s)\cos\frac{1}{2}\pi s/(1-s)$ where $f(s)$ is the Mellin transform of $F(x)$.

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We have

$$G_1(u,x) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^x \frac{F(u+iv)+F(u-iv)}{2}\,dv$$
$$= \frac{1}{i}\sqrt{\left(\frac{2}{\pi}\right)} \int_{u-ix}^{u+ix} \frac{F(w)}{2}\,dw$$

and using (5), with $u > 0$,

$$iG_1(u,x) = \tfrac{1}{2}\sqrt{\left(\frac{2}{\pi}\right)} \int_{u-ix}^{u+ix} w\,dw \int_0^\infty K_1(t)e^{-wt}dt$$

(7) $$= \tfrac{1}{2}\sqrt{\left(\frac{2}{\pi}\right)} \int_{u-ix}^{u+ix} \left(-w\frac{d}{dw}\int_0^\infty \frac{K_1(t)}{t}e^{-wt}dt\right)dw$$

$$= \tfrac{1}{2}\sqrt{\left(\frac{2}{\pi}\right)}\left\{\left[-w\int_0^\infty \frac{K_1(t)}{t}e^{-wt}dt\right]_{u-ix}^{u+ix} + \int_{u-ix}^{u+ix} dw\int_0^\infty \frac{K_1(t)}{t}e^{-wt}dt\right\}.$$

In order to calculate the limit of $G_1(u,x)$ as $u \to 0$ we refer to T., Theorems 94 and 95. After a change of variable amounting to the rotation of the complex plane through a right angle, these theorems show that if $\Phi(x)$ has a Fourier transform $\varphi(x)$ which is null for $x < 0$, then

$$\Phi(u+ix) = \frac{1}{\sqrt{(2\pi)}}\int_0^\infty \varphi(t)e^{-(u+ix)t}dt$$

converges in mean square as well as almost everywhere to $\Phi(x)$ as $u \to 0$.

Here $\Phi(x)$ is taken to be

$$\int_0^\infty \frac{K_1(t)}{t}e^{-ixt}dt$$

where the integration is in the mean square sense, and its transform $\varphi(x)$ is $\sqrt{(2\pi)}K_1(x)/x$ for $x \geqslant 0$ and zero for $x < 0$. Hence, as $u \to 0$,

$$\int_0^\infty \frac{K_1(t)}{t}e^{-(u+ix)t}dt = \Phi(u+ix)$$

converges in the mean square sense and also in the ordinary sense almost everywhere to

$$\int_0^\infty \frac{K_1(t)}{t}e^{-ixt}dt.$$

But mean square convergence over the infinite range in x implies mean convergence with index 1 over any finite range, and so

$$\lim_{u\to 0}\int_{-x}^{x}\left|\int_0^\infty \frac{K_1(t)}{t}e^{-(u+iy)t}dt - \int_0^\infty \frac{K_1(t)}{t}e^{-iyt}dt\right|dy = 0$$

or $$\lim_{u\to 0}\int_{u-ix}^{u+ix} dw\int_0^\infty \frac{K_1(t)}{t}e^{-wt}dt = \int_{-ix}^{+ix} dw\int_0^\infty \frac{K_1(t)}{t}e^{-wt}dt.$$

Therefore the limit of $iG_1(u,x)$ as $u \to 0$ is

$$\tfrac{1}{2}\sqrt{\left(\frac{2}{\pi}\right)}\left\{-ix\int_0^\infty \frac{K_1(t)}{t}e^{ixt}dt - ix\int_0^\infty \frac{K_1(t)}{t}e^{-ixt}dt + \int_{-ix}^{+ix} dw\int_0^\infty \frac{K_1(t)}{t}e^{-wt}dt\right\}$$

$$= -i\sqrt{\left(\frac{2}{\pi}\right)}\left\{x\int_0^\infty \frac{K_1(t)}{t}\cos xt\,dt - \int_0^x dy\int_0^\infty \frac{K_1(t)}{t}\cos yt\,dt\right\}.$$

Hence

$$(8)\qquad \frac{G_1(x)}{x} = \lim_{u\to 0}\frac{G_1(u,x)}{x}$$

$$= \sqrt{\left(\frac{2}{\pi}\right)}\frac{1}{x}\int_0^x dy\int_0^\infty \frac{K_1(t)}{t}\cos yt\,dt - \sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty \frac{K_1(t)}{t}\cos xt\,dt$$

or

$$(9)\qquad \frac{G_1(x)}{x} = \sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty \frac{K_1(t)}{t}\frac{\sin xt}{xt}dt - \sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty \frac{K_1(t)}{t}\cos xt\,dt.$$

Another and more direct method of deriving (8) and (9) from (7) is as follows. In the first expression in (7) for $iG_1(u,x)$, if we reverse the order of integration (as is justified by uniform convergence), carry out the integration with respect to w and then rearrange we obtain

$$G_1(u,x) = \sqrt{\left(\frac{2}{\pi}\right)}\left(u\int_0^\infty \frac{K_1(t)}{t}e^{-ut}\sin xt\,dt - x\int_0^\infty \frac{K_1(t)}{t}e^{-ut}\cos xt\,dt + \int_0^\infty \frac{K_1(t)}{t}\frac{\sin xt}{t}e^{-ut}dt\right).$$

Now applying the Parseval relation for cosine transforms to the integral in the second term of this expression, we have

$$\sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty \frac{K_1(t)}{t}e^{-ut}\cos xt\,dt = \frac{1}{\pi}\int_0^\infty K_c(y)\left\{\frac{u}{u^2+(x+y)^2}+\frac{u}{u^2+(x-y)^2}\right\}dy$$

where $K_c(y)$ is the cosine transform of $K_1(t)/t$ and

$$\tfrac{1}{2}\sqrt{\left(\frac{2}{\pi}\right)}\left\{\frac{u}{u^2+(x+y)^2}+\frac{u}{u^2+(x-y)^2}\right\}$$

is the cosine transform of $e^{-ut}\cos xt$. But by the theory of the Cauchy singular integral (T., Theorem 13 and 1-17),

$$\lim_{u\to 0}\frac{1}{\pi}\int_0^\infty K_c(y)\frac{u}{u^2+(x+y)^2}dy = 0$$

and

$$\lim_{u\to 0}\frac{1}{\pi}\int_0^\infty K_c(y)\frac{u}{u^2+(x-y)^2}dy = K_c(x)$$

for almost all positive x if $K_c(y)/(1+y^2) \in L(0,\infty)$, which is true since $K_c(y) \in L^2(0,\infty)$. Therefore

$$\lim_{u\to 0} \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty \frac{K_1(t)}{t} e^{-ut} \cos xt dt = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty \frac{K_1(t)}{t} \cos xt dt$$

where the integral on the right-hand side is taken in the mean square sense.

If the cosine is replaced by a sine a similar result holds.

The above is just a proof that the integrals of the L^2 cosine and sine transforms are Abel summable to the same value as is obtained by mean square methods.

These results give the limits as $u \to 0$ of the first two terms in the expression for $G_1(u,x)$, while the limit of the third is just the Abel sum of a convergent integral, since $K_1(t)/t$ and $\sin xt/t$ are both $L^2(0,\infty)$. As the limit of the first term is zero our final result is (9) as before.

Now the first term in (8) is of the form

$$\beta(x) = \frac{1}{x}\int_0^x \alpha(y)dy$$

where $\alpha(y)$, being the cosine transform of a function of $L^2(0,\infty)$, is also $L^2(0,\infty)$. Hence $\beta(x)$ is of the class $L^2(0,\infty)$ (T., p. 396). The second term of (8) is the cosine transform of $K_1(t)/t$ (a mean square integral) and is also $L^2(0,\infty)$ and so $G_1(x)/x$ is $L^2(0,\infty)$ as well.

In order to complete the proof of Theorem 3 it must now be shown that $G_1(x)/x$ as defined by (8) or (9) is the kernel of a Watson transform. Three methods of proof will be used, the first by finding the Mellin transform of $G_1(x)/x$, the second by using the properties of the resultants of Watson kernels, and the third by using a known property of the kernels themselves.

Using the Parseval relation for Mellin transforms,

$$\int_0^\infty \frac{K_1(t)}{t}\frac{\sin xt}{xt} dt = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{k(1-s)}{s} \sin \tfrac{1}{2}\pi(s-1)\Gamma(s-1)x^{-s}ds$$

$$= \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{k(1-s)}{s} \cos \tfrac{1}{2}\pi s \frac{\Gamma(s)}{1-s} x^{-s}ds$$

as the Mellin transforms of $K_1(t)/t$ and $\sin xt/xt$ are $k(s)/(1-s)$ and $\sin \frac{1}{2}\pi(s-1)$ $\Gamma(s-1)x^{-s}$ respectively.

Similarly,

$$\int_0^\xi dx \int_0^\infty \frac{K_1(t)}{t} \cos xt dt = \int_0^\infty \frac{K_1(t)}{t}\frac{\sin \xi t}{t} dt$$

$$= \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{k(1-s)}{s} \cos \tfrac{1}{2}\pi s \frac{\Gamma(s)}{1-s} \xi^{1-s}ds.$$

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But

$$\int_0^\xi dx \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{k(1-s)}{s} \cos \tfrac{1}{2}\pi s\ \Gamma(s)x^{-s}ds$$

$$= \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{k(1-s)}{s} \cos \tfrac{1}{2}\pi s\ \frac{\Gamma(s)}{1-s}\ \xi^{1-s}\ ds$$

as the Mellin transform of the function equal to unity in $(0, \xi)$ and zero otherwise is ξ^s/s and the transform of

$$\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{k(1-s)}{s} \cos \tfrac{1}{2}\pi s\Gamma(s)x^{-s}ds$$

is $k(1-s)\cos \frac{1}{2}\pi s\ \Gamma(s)/s$. Therefore, differentiating the last two formulae with respect to ξ,

$$\int_0^\infty \frac{K_1(t)}{t}\cos xtdt = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{k(1-s)}{s} \cos \tfrac{1}{2}\pi s\ \Gamma(s)x^{-s}ds$$

for almost all x. Hence

$$\frac{G_1(x)}{x} = \sqrt{\left(\frac{2}{\pi}\right)}\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{k(1-s)}{s} \cos \tfrac{1}{2}\pi s\ \Gamma(s)\left(\frac{1}{1-s}-1\right)x^{-s}ds$$

$$= \sqrt{\left(\frac{2}{\pi}\right)}\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{f(s)\cos \frac{1}{2}\pi s}{1-s}\ x^{-s}ds$$

almost everywhere, and so the Mellin transform of $G_1(x)/x$ is $\sqrt{(2/\pi)}f(s)\cos \frac{1}{2}\pi s/(1-s)$. But if $g(s) = \sqrt{(2/\pi)}f(s)\cos\frac{1}{2}\pi s$ then

$$g(s)g(1-s) = (2/\pi)f(s)f(1-s)\cos\tfrac{1}{2}\pi s\cos\tfrac{1}{2}\pi(1-s) = 1$$

by (2) for $R(s) = \frac{1}{2}$. Therefore, if $g(s)/(1-s)$ is the transform of $G_1(x)/x$, $g(s)g(1-s) = 1$ for $R(s) = \frac{1}{2}$.

Also, using the fact that $f(s)$ takes conjugate values for conjugate values of s, $|g(\frac{1}{2}+it)| = 1$ for t in $(-\infty, \infty)$.

Therefore, by T., Theorem 129, $G_1(x)/x$ is the kernel of a Watson transform and the Theorem is proved.

For the second method of proof, we first transform (8) by using T., Theorem 69. This Theorem yields the following result: if $K_1(t)/t$ has the cosine transform

$$\sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty \frac{K_1(t)}{t}\cos ytdt$$

then $\displaystyle\int_t^\infty \frac{1}{u}\frac{K_1(u)}{u}du$ has the cosine transform

$$\sqrt{\left(\frac{2}{\pi}\right)}\frac{1}{x}\int_0^x dy\int_0^\infty \frac{K_1(t)}{t}\cos ytdt.$$

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Therefore

$$\sqrt{\left(\frac{2}{\pi}\right)}\frac{1}{x}\int_0^x dy\int_0^\infty \frac{K_1(t)}{t}\cos ytdt = \sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty \cos xtdt\int_t^\infty \frac{K_1(u)}{u^2}\,du$$

and so, using (8),

$$\frac{G_1(x)}{x} = \sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty \cos xt\left(\int_t^\infty \frac{K_1(u)}{u^2}\,du - \frac{K_1(t)}{t}\right)dt. \tag{10}$$

Now if $L_1(x)/x$, $M_1(x)/x$ and $N_1(x)/x$ are Watson kernels with Mellin transforms $l(s)/(1-s)$, $m(s)/(1-s)$, and $n(s)/(1-s)$, and if $B_1(x)/x$ is the M transform of $L_1(x)/x$ and $C_1(x)/x$ the N transform of $B_1(x)/x$, then the Mellin transform of $B_1(x)$ is $l(1-s)m(s)/s$ and that of

$$C_1(x)/x \text{ is } l(s)m(1-s)n(s)/(1-s)$$

(T., 8.6). But if $p(s) = l(s)m(1-s)n(s)$, then $p(s)p(1-s) = 1$ and so $C_1(x)/x$ is the kernel of a Watson transform.

Now let

$$\frac{L_1(x)}{x} = \begin{cases} 0 & (0 < x < 1), \\ 1/x & (1 < x) \end{cases} \qquad l(s) = 1,$$

$$\frac{M_1(x)}{x} = \frac{K_1(x)}{x}, \qquad m(s) = k(s),$$

$$\frac{N_1(x)}{x} = \sqrt{\left(\frac{2}{\pi}\right)}\frac{\sin x}{x}, \qquad n(s) = \sqrt{\left(\frac{2}{\pi}\right)}\Gamma(s)\cos\tfrac{1}{2}\pi s,$$

where $K_1(x)/x$ is the kernel in the formula for $G_1(x)/x$. Then

$$\frac{B_1(x)}{x} = \frac{d}{dx}\int_1^\infty \frac{K_1(xt)}{t^2}\,dt$$
$$= \frac{d}{dx}\,x\int_x^\infty \frac{K_1(t)}{t^2}\,dt = \int_x^\infty \frac{K_1(t)}{t^2}\,dt - \frac{K_1(x)}{x}$$

and

$$\frac{C_1(x)}{x} = \sqrt{\left(\frac{2}{\pi}\right)}\frac{d}{dx}\int_0^\infty \frac{\sin xt}{t}\left(\int_t^\infty \frac{K_1(u)}{u^2}\,du - \frac{K_1(t)}{t}\right)dt$$
$$= \sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty \cos xt\left(\int_t^\infty \frac{K_1(u)}{u^2}\,du - \frac{K_1(t)}{t}\right)dt$$

where the final integral is to be taken in the mean square sense. The final transformation is valid since the last two expressions are only different forms for the cosine transform of a function of $L^2(0,\infty)$.

But this is just $G_1(x)/x$. Therefore $G_1(x)/x$ is the kernel of a Watson transform and its Mellin transform is the Mellin transform of $C_1(x)/x$,

i.e. $$l(s)m(1-s)n(s)/(1-s) = \sqrt{\left(\frac{2}{\pi}\right)}k(1-s)\Gamma(s)\cos\tfrac{1}{2}\pi s/(1-s)$$
$$= \sqrt{\left(\frac{2}{\pi}\right)}f(s)\cos\tfrac{1}{2}\pi s/(1-s).$$

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For the third method of proof, we write formula (10) in the form

$$\frac{G_1(x)}{x} = \sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty \cos xt dt \int_t^\infty \frac{K_1(u) - K_1(t)}{u^2}\, du$$

where the inner integral is a function of t of the class $L^2(0,\infty)$ and therefore the outer integral is a mean square integral. Hence the cosine transform of $G_1(x)/x$ is

$$\int_t^\infty \frac{K_1(u) - K_1(t)}{u^2}\, du$$

and therefore that of $G_1(ax)/ax$ is

$$\frac{1}{a}\int_{t/a}^\infty \frac{K_1(u) - K_1(t/a)}{u^2}\, du.$$

Using the Parseval relation for cosine transforms,

$$I(a) = \int_0^\infty \frac{G_1(t)}{t}\frac{G_1(at)}{t}\, dt = \int_0^\infty dt \int_t^\infty \frac{K_1(u)-K_1(t)}{u^2}\, du \int_{t/a}^\infty \frac{K_1(v)-K_1(t/a)}{v^2}\, dv$$

$$= a\int_0^\infty \frac{dt}{t^2}\int_1^\infty \frac{K_1(ut) - K_1(t)}{u^2}\, du \int_1^\infty \frac{K_1(tv/a) - K_1(t/a)}{v^2}\, dv.$$

The order of integration in the final triple integral may be changed as this integral is absolutely convergent. For

$$\int_0^\infty \frac{dt}{t^2}\int_1^\infty \frac{|K_1(ut)|+|K_1(t)|}{u^2}\, du \int_1^\infty \frac{|K_1(tv/a)|+|K_1(t/a)|}{v^2}\, dv$$

$$= \frac{1}{a}\int_0^\infty dt \int_t^\infty \frac{|K_1(u)|+|K_1(t)|}{u^2}\, du \int_{t/a}^\infty \frac{|K_1(v)|+|K_1(t/a)|}{v^2}\, dv$$

$$= \frac{1}{a}\int_0^\infty dt \left\{\int_t^\infty \frac{|K_1(u)|}{u^2}\, du + \frac{|K_1(t)|}{t}\right\}\left\{\int_{t/a}^\infty \frac{|K_1(v)|}{v^2}\, dv + a\frac{|K_1(t/a)|}{t}\right\}$$

and since $K_1(t)/t$ is $L^2(0,\infty)$ so is $\int_t^\infty \frac{|K_1(u)|}{u^2}\, du$ (T., Theorem 69 as before) and the integrand of the t integral is the product of two functions of $L^2(0,\infty)$ and is therefore $L(0,\infty)$.

Changing the order of integration,

$$I(a) = a\int_1^\infty \frac{du}{u^2}\int_1^\infty \frac{dv}{v^2}\int_0^\infty \frac{\{K_1(ut) - K_1(t)\}\{K_1(tv/a) - K_1(t/a)\}}{t^2}\, dt$$

$$= a\int_1^\infty \frac{du}{u^2}\int_1^\infty \frac{dv}{v^2}\{\min\,(u,v/a)+\min\,(1,1/a)-\min\,(1,v/a)-\min\,(u,1/a)\}$$

since

$$\int_0^\infty \frac{K_1(at)}{t}\frac{K_1(bt)}{b}\, dt = \min\,(a,b).$$

as $K_1(t)/t$ is the kernel of a Watson transform.

If we assume $0 < a < 1$, then $\min\,(1,1/a) = \min\,(1,v/a) = 1$ for $v \geqslant 1$. Therefore

$$I(a) = a\int_1^\infty \frac{du}{u^2}\int_1^\infty \frac{dv}{v^2}\{\min(u,v/a) - \min(u,1/a)\}.$$

Also, if $u < 1/a$ then $u < v/a$ in our range of integration and so $\min(u,v/a) - \min(u,1/a) = 0$ and

$$I(a) = a\int_{1/a}^\infty \frac{du}{u^2}\left(\int_1^{au} \frac{v}{a}\frac{dv}{v^2} + \int_{au}^\infty u\frac{dv}{v^2} - \frac{1}{a}\int_1^\infty \frac{dv}{v^2}\right)$$
$$= a.$$

A similar proof would show that $I(a) = 1$ if $1 \leqslant a$, and so

$$\int_0^\infty \frac{G_1(t)}{t}\frac{G_1(at)}{t}\,dt = \min(1,a).$$

Setting $a = b/c$ and $t = cu$,

$$\int_0^\infty \frac{G_1(cu)}{u}\frac{G_1(bu)}{u}\,du = c\min(1,b/c) = \min(c,b),$$

and therefore $G_1(t)/t$ is the kernel of a Watson transform (T., Theorem 131).

COROLLARY. *The condition that $F(x)$ be real may be replaced by the condition that $k(\frac{1}{2}+it)$ be bounded in $(-\infty,\infty)$.*

This follows from Corollary 1, Theorem 2.

The following example shows that Theorems 2 and 3 do not hold for all functions $F(x)$ satisfying (1) and that some extra condition must be imposed if $F(x)$ may be complex.

Let $F(x) = \alpha^{\frac{1}{2}}e^{-\alpha x}$ where $0 < \text{am}\,\alpha < \pi/2$. Then $F(x) \in L^2(0,\infty)$ and satisfies (1). Also

$$f(s) = \alpha^{\frac{1}{2}}\int_0^\infty e^{-\alpha x}x^{s-1}dx = \alpha^{\frac{1}{2}-s}\Gamma(s)$$

and $k(s) = f(1-s)/\Gamma(1-s) = \alpha^{s-\frac{1}{2}}$ or $k(\frac{1}{2}+it) = \alpha^{it}$. Thus $|k(\frac{1}{2}+it)| = e^{-t(\text{am}\,\alpha)}$, and so $k(s)/(1-s)$ does not belong to any integrable class in $(\frac{1}{2}-i\infty, \frac{1}{2}+i\infty)$. The arguments in Theorems 2 and 3 therefore break down completely.

4. Examples. (i). Let $F(x) = e^{-x}$. Then (1) is satisfied, $f(s) = \Gamma(s)$, and $k(s) = 1$. Therefore $K_1(t)/t = 0$ for $t < 1$, $K_1(t)/t = 1/t$ for $t > 1$ and $G_1(x)/x = \sqrt{(2/\pi)}\sin x/x$.

In this case the result (b) of the formal analysis holds and

$$G(x) = \sqrt{\left(\frac{2}{\pi}\right)}\frac{F(ix) + F(-ix)}{2} = \sqrt{\left(\frac{2}{\pi}\right)}\cos x$$

is a Fourier kernel.

(ii). If
$$F(x) = \sqrt{\left(\frac{2}{\pi}\right)}\frac{x}{1+x^2}$$

then $f(s) = \sqrt{\left(\frac{2}{\pi}\right)}\dfrac{1}{\cos\frac{1}{2}\pi s}$, and the Mellin transform of $G_1(x)/x$ is

$$\sqrt{\left(\frac{2}{\pi}\right)}f(s)\cos\tfrac{1}{2}\pi s/(1-s) = 1/(1-s).$$

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Therefore $$\frac{G_1(x)}{x} = \begin{cases} 0 & (0 < x < 1), \\ 1/x & (1 < x), \end{cases}$$

and the Watson transform of which $G_1(x)/x$ is the kernel transforms $h(t)$ into $h(1/t)/t$.

In this case $\frac{1}{2}\sqrt{(2/\pi)}\{F(ix) + F(-ix)\} = 0$ for all values of x except $\pm i$, and so the result (b) of the formal analysis does not hold.

Theorem 3 gives

$$\frac{G_1(x)}{x} = \frac{1}{x} \lim_{u\to 0} \frac{1}{2i} \sqrt{\left(\frac{2}{\pi}\right)} \int_{u-ix}^{u+ix} \sqrt{\left(\frac{2}{\pi}\right)} \frac{z}{1+z^2}\, dz$$
$$= \frac{1}{2\pi ix} \lim_{u\to 0} \log \frac{1+(u+ix)^2}{1+(u-ix)^2} = \frac{1}{2\pi x} \lim_{u\to 0} \operatorname{am}\left\{\frac{1+(u+ix)^2}{1+(u-ix)^2}\right\}$$
$$= \begin{cases} 0 & (0 < x < 1), \\ 1/x & (1 < x). \end{cases}$$

(iii). If $F(x) = \sqrt{\left(\frac{2}{\pi}\right)} \frac{1}{1+x^2}$ then all our conditions are satisfied and

$$f(s) = \sqrt{\left(\frac{\pi}{2}\right)} \operatorname{cosec} \tfrac{1}{2}\pi s,\quad k(s) = \sqrt{\left(\frac{2}{\pi}\right)} \sin \tfrac{1}{2}\pi s,\quad K_1(t)/t = \sqrt{\left(\frac{2}{\pi}\right)}(1-\cos x)/x$$

and $G_1(x)/x = \dfrac{1}{\pi x} \log \left|\dfrac{1+x}{1-x}\right|$.

In this case the formal result holds as well with $G(x) = 2/\pi(1-x^2)$.

(iv). In general, if any function $F(x)$ of the class $L^2(0,\infty)$ satisfies (1) then so does any Watson transform of $F(x)$.

For if the Watson transform with kernel $M_1(t)/t$ of $F(x)$ is $H(x)$, i.e.

$$H(x) = \frac{d}{dx}\int_0^\infty F(t)\, \frac{M_1(xt)}{t}\, dt,$$

then the transform of $F(ux)$ is

$$\frac{d}{dx}\int_0^\infty F(ut)\, \frac{M_1(xt)}{t}\, dt = \frac{d}{dx}\int_0^\infty F(t)\, \frac{M_1(xt/u)}{t}\, dt = \frac{1}{u} H\left(\frac{x}{u}\right).$$

Therefore, by the general Parseval relation for Watson transforms (a slight extension of T., 8.5.8)

$$\int_0^\infty F(x)F(ux)dx = \frac{1}{u}\int_0^\infty H(x)H\left(\frac{x}{u}\right) dx$$
$$= \int_0^\infty H(x)H(ux)dx$$

or $$\int_0^\infty H(x)H(ux)dx = \frac{1}{1+u}.$$

Taking $F(x) = \sqrt{\left(\frac{2}{\pi}\right)} \frac{1}{1+x^2}$ and $\frac{M_1(t)}{t} = \sqrt{\left(\frac{2}{\pi}\right)} \frac{1-\cos t}{t}$ we obtain

$$H(x) = \frac{2}{\pi}\frac{d}{dx}\int_0^\infty \frac{1-\cos xt}{t(1+t^2)}\, dt = \frac{2}{\pi}\int_0^\infty \frac{\sin xt}{1+t^2}\, dt.$$

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In this case $h(s)$, the Mellin transform of $H(x)$ is $\Gamma(s)\tan\frac{1}{2}\pi s$, and applying Theorem 2 to $H(x)$ we obtain $k(s) = h(1-s)/\Gamma(1-s) = \cot\frac{1}{2}\pi s$ and

$$\frac{K_1(t)}{t} = \frac{1}{\pi t}\log\left|\frac{1+x}{1-x}\right|.$$

Therefore (5) becomes

$$H(x) = \frac{x}{\pi}\int_0^\infty e^{-xt}\log\left|\frac{1+t}{1-t}\right| dt$$
$$= \frac{2}{\pi}\int_0^\infty \frac{e^{-xt}}{1-t^2}\, dt$$

after integrating by parts and taking the final integral as a principal value at $t = 1$. This second expression for $H(x)$ can also be obtained from the first by the calculus of residues.

The first expression for $H(x)$ cannot be used in Theorem 3 for calculating $G_1(x)/x$ as it has no meaning unless x is real. Using the second, however, we obtain

$$\frac{G_1(x)}{x} = \sqrt{\left(\frac{2}{\pi}\right)}\,\frac{1-\cos x}{x}.$$

5. The three theorems above may be extended to include the case where there are two different functions involved in (1).

THEOREM 4. *Let $F(x)$ and $H(x)$ of the class $L^2(0,\infty)$ satisfy the equation*

$$\int_0^\infty F(x)H(ux)dx = \frac{1}{1+u} \qquad (u > 0) \tag{11}$$

and have Mellin transforms $f(s)$ and $h(s)$. Then $f(s)h(1-s) = \Gamma(s)\Gamma(1-s)$.

If $k(s) = f(1-s)/\Gamma(1-s)$ and $l(s) = h(1-s)/\Gamma(1-s)$ are bounded for s in the range $(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$ then (5) holds with $K_1(t)$ defined as in (4). A similar formula holds for $H(x)$ with $K_1(t)$ replaced by $L_1(t)$ where

$$\frac{L_1(x)}{x} = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{l(s)}{1-s}\,x^{-s}ds.$$

The functions $K_1(t)/t$ and $L_1(t)/t$ belong to the class $L^2(0,\infty)$ and are the kernels of conjugate Watson transforms.

Functions $G_1(x)/x$ defined as in Theorem 3 and $M_1(x)/x$ defined in the same way with $H(z)$ replacing $F(z)$ exist and are conjugate Watson kernels with Mellin transforms $\sqrt{\left(\frac{2}{\pi}\right)}\, f(s)\cos\frac{1}{2}\pi s$ and $\sqrt{\left(\frac{2}{\pi}\right)}\, h(s)\cos\frac{1}{2}\pi s$.

This Theorem is proved in almost the same way as Theorems 1, 2, and 3, and so no proof will be given.

In this Theorem it must be specified that $h(s)$ and $l(s)$ are bounded on $(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$ and the condition that $F(x)$ and $H(x)$ be real is not sufficient for the sections of the Theorem corresponding to Theorems 2 and 3.

This can be shown by taking $F(x) = xe^{-x}$ and $H(x) = (1-e^{-x})/x$. These functions are $L^2(0,\infty)$ and satisfy (11), but as $k(s) = 1 - s$ no function $K_1(t)/t$ can be defined and the argument breaks down. The functions

$$G(x) = \tfrac{1}{2}\sqrt{\left(\frac{2}{\pi}\right)}\{F(ix) + F(-ix)\} = \sqrt{\left(\frac{2}{\pi}\right)}\, x \sin x$$

and

$$M(x) = \tfrac{1}{2}\sqrt{\left(\frac{2}{\pi}\right)}\{H(ix) + H(-ix)\} = \sqrt{\left(\frac{2}{\pi}\right)} \sin x/x$$

obtained as in the initial formal argument are, however, conjugate Fourier kernels in a certain sense. In fact, if $q(t)$ is a function such that $tq(t) \in L^2(0,\infty)$, then its G transform is

$$r(x) = \sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty q(t)xt \sin xt dt$$
$$= xT_s\{tq(t)\}$$

where $T_s\{tq(t)\}$ is the sine transform of $tq(t)$. Further, the M transform of $r(x)$ is

$$\sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty tT_s\{tq(t)\}\frac{\sin xt}{xt}\, dt = \frac{1}{x}\, xq(x) = q(x).$$

Thus, in a certain sense, the original formal assertion (c) still holds even though our rigorous argument breaks down.

Other extensions of Theorems 1, 2, and 3 are obtained by replacing equation (1) by slightly more general equations as in the following two theorems.

THEOREM 5. *Let $F(x)$ be a real function defined in $(0,\infty)$ such that*

$$\int_0^\infty \{F(x)\}^2 x^{c-1} dx < \infty \qquad (c > 0),$$

$$\int_0^\infty F(x)F(ux)x^{c-1}dx = \frac{\Gamma(c)}{(1+u)^c}, \tag{12}$$

and let $f(s)$ be the Mellin transform of $F(x)$ defined for $R(s) = c/2$.

Then $$f(s)f(c-s) = \Gamma(s)\Gamma(c-s) \qquad (R(s) = c/2)$$

and if $$k(s) = \frac{f(c-cs)}{\Gamma(c-cs)} \qquad (R(s) = 1/2)$$

then $$k(s)k(1-s) = 1 \qquad (R(s) = 1/2).$$

If $K_1(t)/t$ is the Watson kernel derived from $k(s)$ in the usual way, then

$$F(x) = x\int_0^\infty K_1(t^c)e^{-xt}dt. \tag{13}$$

Conversely, if $K_1(t)/t$ is any L^2 Watson kernel, then (13) *defines a function $F(x)$ that satisfies* (12).

The proof will not be given as the only difference between it and the proof of Theorems 1 and 2 is that the more general Mellin transform theory of T., Theorems 71, 72, and 73 is used.

Examples. (i). If $k(s) = 1$, then

$$\frac{K_1(t)}{t} = \begin{cases} 0 & (0 < t < 1) \\ 1/t & (1 < t) \end{cases}$$

and this leads to $F(x) = e^{-x}$ which obviously satisfies (12).

(ii). If $K_1(x)/x = \sqrt{\left(\frac{2}{\pi}\right)} \sin x/x$ then $F(x) = \sqrt{\left(\frac{2}{\pi}\right)}\, x \int_0^\infty e^{-xt} \sin t^c dt.$

This can be integrated in finite terms if $c = \frac{1}{2}$, giving

$$F(x) = \frac{1}{\sqrt{(2x)}}\, e^{-\frac{1}{4}x}$$

as a solution of (12) with $c = \frac{1}{2}$.

THEOREM 6. *Let $F(x)$ be a real function belonging to the class $L^2(0,\infty)$ and satisfying*

$$\int_0^\infty F(x)F(ux)dx = \frac{u^{\frac{c-1}{2}}\Gamma(c)}{(1+u)^c} \tag{14}$$

with $c > 0$ for all positive u.

If $f(s)$ is the Mellin transform of $F(x)$ and

$$k(s) = \frac{f(1-s)}{\Gamma\left(\frac{c}{2}+\frac{1}{2}-s\right)}$$

then $k(s)k(1-s) = 1$ for $R(s) = \frac{1}{2}$ and

$$F(x) = -x\frac{d}{dx}\, x^{\frac{c-1}{2}} \int_0^\infty \frac{K_1(t)}{t}\, t^{\frac{c-1}{2}} e^{-tx} dt \tag{15}$$

where $K_1(t)/t$ is the Watson kernel derived from $k(s)$ in the usual way.

This Theorem is proved in the same way as Theorems 1 and 2 and so the proof will not be given.

As a formal deduction from the formula for $F(x)$ we obtain

$$F(x) = x^{\frac{c-1}{2}} \int_0^\infty t^{\frac{c-1}{2}} e^{-xt} K(t) dt$$

where $K(t)$ is a Fourier kernel.

Example. If $\frac{K_1(t)}{t} = \sqrt{\left(\frac{2}{\pi}\right)}\, \frac{1-\cos t}{t}$, then (15) with $c = 3$ gives

$$F(x) = \sqrt{\left(\frac{2}{\pi}\right)}\, \frac{2x}{(1+x^2)^2}$$

as a function satisfying (14).

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