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Inequalities for Partial Derivatives and their Applications

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Abstract. We present various weighted integral inequalities for partial derivatives acting on products and compositions of functions that are applied in order to establish some new Opial-type inequalities involving functions of several independent variables. We also demonstrate the usefulness of our results in the field of partial differential equations.

1 Introduction

Let u be absolutely continuous on $[0,h]$ such that $u(0) = u(h) = 0$; then Opial's inequality [6] asserts that

$$\int_0^h |u(s)u'(s)|ds \le \frac{h}{4}\int_0^h |u'(s)|^2 ds. \tag{1.1}$$

In 1962, Bessack [2] showed the following result which implies (1.1) and is very useful in applications. Let u be absolutely continuous on $[0,h]$, and satisfy $u(0) = 0$; then

$$\int_0^h |u(s)u'(s)|ds \le \frac{h}{2}\int_0^h |u'(s)|^2 ds. \tag{1.2}$$

Inequalities (1.1) and (1.2), which are frequently used in the study of qualitative as well as quantitative properties of solutions of initial value problems for differential equations, have motivated a large number of extensions, generalizations, variants, and discrete analogues. A brief account of such inequalities can be found in [1].

In 1982, Yang [10] generalized Opial's inequality to the case of two variables. For $u \in C^{(1,1)}([a,T]\times[c,S])$ such that $u(a,s) = u(t,c) = 0$ for all $(t,s) \in [a,T]\times[c,S]$, one has

$$\int_a^T \int_c^S \Big| u(t,s)\frac{\partial^2 u}{\partial t \partial s}(t,s)\Big| dsdt \le \frac{(T-a)(S-c)}{2}\int_a^T \int_c^S \Big|\frac{\partial^2 u}{\partial t\partial s}(t,s)\Big|^2 dsdt. \tag{1.3}$$

For practical application purposes inequality (1.3) has been improved as well as generalized in several different directions [3,4,7–9,11–13]. Motivated by these works, in this paper we prove some new Opial-type inequalities involving functions of n variables. To achieve our goal, we consider some weighted integral inequalities for partial

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derivatives acting on products and compositions of functions. As applications, we study the uniqueness of the initial value problem and obtain upper bounds of solutions of certain partial differential equations.

Throughout the paper, we denote by $\mathbb{N}$ the set of natural numbers, $\mathbb{N} = \{1, 2, \dots\}$ and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let n and m be in $\mathbb{N}$ and let $p > 1$ and $q > 1$ be conjugate exponents $1/p + 1/q = 1$. Denote by $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$ the multi-index, *i.e.*, $\alpha_j \in \mathbb{N}_0$, $j = 1, \dots, n$, and $|\boldsymbol{\alpha}| = \alpha_1 + \cdots + \alpha_n$. Furthermore, we have the following abbreviations:

$$\boldsymbol{\alpha}! = \alpha_1! \cdots \alpha_n!, \qquad \boldsymbol{\alpha} \in \mathbb{N}_0^n,$$

$$\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_n} = \frac{\boldsymbol{\alpha}!}{\boldsymbol{\beta}_1! \cdots \boldsymbol{\beta}_n!}, \qquad \boldsymbol{\alpha}, \boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_n \in \mathbb{N}_0^n,$$

$$\sum_{\boldsymbol{\alpha}} f(\boldsymbol{\alpha}) = \sum_{k_1=0}^{\alpha_1} \cdots \sum_{k_n=0}^{\alpha_n} f(k_1, \dots, k_n), \qquad \boldsymbol{\alpha} \in \mathbb{N}_0^n.$$

As usual we denote by $\mathbb{R}^n$ the n-dimensional Euclidean space. This is the set of all n-tupels of real numbers $\boldsymbol{x} = (x_1, \dots, x_n)$, $x_j \in \mathbb{R}$, $j = 1, \dots, n$, with the linear operations $\boldsymbol{x} + \boldsymbol{y} = (x_1 + y_1, \dots, x_n + y_n)$ and $\lambda \boldsymbol{x} = (\lambda x_1, \dots, \lambda x_n)$ for $\boldsymbol{x} = (x_1, \dots, x_n)$, $\boldsymbol{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. In particular, let $\mathbf{0} = (0, \dots, 0)$ and $\mathbf{1} = (1, \dots, 1)$.

We shall write $\boldsymbol{x} < \boldsymbol{y}$ when $x_j < y_j$ for all $j = 1, \dots, n$. We interpret $\boldsymbol{x} \le \boldsymbol{y}, \boldsymbol{x} > \boldsymbol{y}$, and $\boldsymbol{x} \ge \boldsymbol{y}$, analogously. Let $\boldsymbol{a} = (a_1, \dots, a_n)$ and $\boldsymbol{b} = (b_1, \dots, b_n)$ be in $\mathbb{R}^n$ such that $\boldsymbol{a} < \boldsymbol{b}$. Then we set

$$Q = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{a} < \boldsymbol{x} \le \boldsymbol{b}\},$$
$$\Omega = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{a} \le \boldsymbol{x} \le \boldsymbol{b}\},$$
$$\Omega_{\boldsymbol{x}} = \{\boldsymbol{t} \in \mathbb{R}^n : \boldsymbol{a} \le \boldsymbol{t} \le \boldsymbol{x}\}, \quad \boldsymbol{x} \in \Omega.$$

We denote by $\mathrm{Vol}(\Omega)$ the volume of the region Ω.

For any continuous real-valued function u defined on Ω we denote by $\int_\Omega u(\boldsymbol{x}) d\boldsymbol{x}$ the n-fold integral $\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} u(x_1, \dots, x_n) dx_1 \cdots dx_n$, and for any $\boldsymbol{x} \in \Omega$, $\int_{\Omega_{\boldsymbol{x}}} u(\boldsymbol{t}) d\boldsymbol{t}$ is the n-fold integral $\int_{a_1}^{x_1} \cdots \int_{a_n}^{x_n} u(t_1, \dots, t_n) dt_1 \cdots dt_n$.

Let $\partial = (\partial_1, \dots, \partial_n)$, $\partial_j = \partial/\partial x_j$, where $j = 1, \dots, n$. Then we set

$$\partial^{\boldsymbol{\alpha}} u(\boldsymbol{x}) = \frac{\partial^{|\boldsymbol{\alpha}|} u(\boldsymbol{x})}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}, \quad \boldsymbol{\alpha} \in \mathbb{N}_0^n, \boldsymbol{x} \in \mathbb{R}^n.$$

Let $\boldsymbol{\alpha} \ge \mathbf{1}$ be a multi-index, $p > 1$ and let $\rho\colon \Omega \to \mathbb{R}$ be a (positive and continuous) weight. We represent by $\mathcal{AC}_p^{\boldsymbol{\alpha}}(\Omega, \rho)$ the set of all functions $u\colon \Omega \to \mathbb{R}$ of class $C^{\boldsymbol{\alpha}}(\Omega)$ for which $\partial_j^{k_j} u|_{x_j = a_j} = 0$ for all $0 \le k_j \le \alpha_j - 1$, $j = 1, \dots, n$, and that $\int_\Omega |\partial^{\boldsymbol{\alpha}} u(\boldsymbol{x})|^p K_{\boldsymbol{\alpha}}(\boldsymbol{b}, \boldsymbol{x}) \rho(\boldsymbol{x}) d\boldsymbol{x} < \infty$, where

$$K_{\boldsymbol{\alpha}}(\boldsymbol{b}, \boldsymbol{x}) = \frac{(\boldsymbol{b} - \boldsymbol{x})^{\boldsymbol{\alpha} - \mathbf{1}}}{(\boldsymbol{\alpha} - \mathbf{1})!} = \prod_{j=1}^{n} \frac{(b_j - x_j)^{\alpha_j - 1}}{(\alpha_j - 1)!}.$$

2 Integral Inequalities for Partial Differential Operators Acting on Products and Compositions of Functions

Theorem 2.1 *Let ρ_j, where $j = 1, \dots, m$, be some weights on Ω. Then the function*

$$\omega(\boldsymbol{x}) := \Big[\partial^{\boldsymbol{\alpha}} \Big(\prod_{j=1}^{m} \int_{\Omega_x} K_{\boldsymbol{\alpha}}(\boldsymbol{x}, \boldsymbol{t}) \rho_j^{-q/p}(\boldsymbol{t}) d\boldsymbol{t} \Big) \Big]^{-p/q}$$

is a weight on Q. Moreover, if $u_j \in \mathcal{AC}_p^{\boldsymbol{\alpha}}(\Omega, \rho_j)$, where $j = 1, \dots, m$, then

$$\int_{\Omega} \Big| \partial^{\boldsymbol{\alpha}} \Big(\prod_{j=1}^{m} u_j(\boldsymbol{x}) \Big) \Big|^p K_{\boldsymbol{\alpha}}(\boldsymbol{b}, \boldsymbol{x}) \omega(\boldsymbol{x}) d\boldsymbol{x} \le \prod_{j=1}^{m} \int_{\Omega} \big| \partial^{\boldsymbol{\alpha}} u_j(\boldsymbol{x}) \big|^p K_{\boldsymbol{\alpha}}(\boldsymbol{b}, \boldsymbol{x}) \rho_j(\boldsymbol{x}) d\boldsymbol{x}. \tag{2.1}$$

Unless $m = 1$, equality holds in (2.1) if and only if

$$u_j(\boldsymbol{x}) = C_j \int_{\Omega_x} K_{\boldsymbol{\alpha}}(\boldsymbol{x}, \boldsymbol{t}) \rho_j^{-q/p}(\boldsymbol{t}) d\boldsymbol{t}, \quad \boldsymbol{x} \in \Omega, \tag{2.2}$$

where C_j, $j = 1, \dots, m$, are real constants.

Proof Observe first that ω is well defined, positive, and continuous on Q. Since ρ_j, where $j = 1, \dots, m$, are positive and continuous on Ω, this follows from

$$\partial^{\boldsymbol{\alpha}} \Big(\prod_{j=1}^{m} \int_{\Omega_x} K_{\boldsymbol{\alpha}}(\boldsymbol{x}, \boldsymbol{t}) \rho_j^{-q/p}(\boldsymbol{t}) d\boldsymbol{t} \Big) = \sum_{\boldsymbol{\beta}_1 + \dots + \boldsymbol{\beta}_m = \boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_m} \prod_{j=1}^{m} \partial^{\boldsymbol{\beta}_j} \Big(\int_{\Omega_x} K_{\boldsymbol{\alpha}}(\boldsymbol{x}, \boldsymbol{t}) \rho_j^{-q/p}(\boldsymbol{t}) d\boldsymbol{t} \Big)$$

and

$$\partial^{\boldsymbol{\beta}_j} \Big(\int_{\Omega_x} K_{\boldsymbol{\alpha}}(\boldsymbol{x}, \boldsymbol{t}) \rho_j^{-q/p}(\boldsymbol{t}) d\boldsymbol{t} \Big) = \begin{cases} \rho_j^{-q/p}(\boldsymbol{x}), & \text{if } \boldsymbol{\beta}_j = \boldsymbol{\alpha}, \\ \int_{\Omega_x} K_{\boldsymbol{\alpha} - \boldsymbol{\beta}_j}(\boldsymbol{x}, \boldsymbol{t}) \rho_j^{-q/p}(\boldsymbol{t}) d\boldsymbol{t}, & \text{otherwise}. \end{cases}$$

We next claim that for $\boldsymbol{0} \le \boldsymbol{\beta} \le \boldsymbol{\alpha}$ and $j = 1, \dots, m$, we have

$$|\partial^{\boldsymbol{\beta}} u_j(\boldsymbol{x})| \le \Big(\partial^{\boldsymbol{\beta}} \int_{\Omega_x} K_{\boldsymbol{\alpha}}(\boldsymbol{x}, \boldsymbol{t}) |\partial^{\boldsymbol{\alpha}} u_j(\boldsymbol{t})|^p \rho_j(\boldsymbol{t}) d\boldsymbol{t} \Big)^{1/p} \times \Big(\partial^{\boldsymbol{\beta}} \int_{\Omega_x} K_{\boldsymbol{\alpha}}(\boldsymbol{x}, \boldsymbol{t}) \rho_j^{-q/p}(\boldsymbol{t}) d\boldsymbol{t} \Big)^{1/q}, \quad \boldsymbol{x} \in \Omega. \tag{2.3}$$

The case $\boldsymbol{\beta} = \boldsymbol{\alpha}$ in (2.3) is trivial. Suppose that $\boldsymbol{0} \le \boldsymbol{\beta} \le \boldsymbol{\alpha}$ and $\boldsymbol{\beta} \ne \boldsymbol{\alpha}$. Since $u_j \in \mathcal{AC}_p^{\boldsymbol{\alpha}}(\Omega, \rho_j)$, it follows that

$$|\partial^{\boldsymbol{\beta}} u_j(\boldsymbol{x})| = \Big| \int_{\Omega_x} K_{\boldsymbol{\alpha} - \boldsymbol{\beta}}(\boldsymbol{x}, \boldsymbol{t}) \partial^{\boldsymbol{\alpha}} u_j(\boldsymbol{t}) d\boldsymbol{t} \Big|$$

$$\le \int_{\Omega_x} K_{\boldsymbol{\alpha} - \boldsymbol{\beta}}(\boldsymbol{x}, \boldsymbol{t}) |\partial^{\boldsymbol{\alpha}} u_j(\boldsymbol{t})| d\boldsymbol{t} \tag{2.4}$$

$$\le \Big(\int_{\Omega_x} K_{\boldsymbol{\alpha} - \boldsymbol{\beta}}(\boldsymbol{x}, \boldsymbol{t}) |\partial^{\boldsymbol{\alpha}} u_j(\boldsymbol{t})|^p \rho_j(\boldsymbol{t}) d\boldsymbol{t} \Big)^{1/p} \tag{2.5}$$

$$\times \Big(\int_{\Omega_x} K_{\boldsymbol{\alpha} - \boldsymbol{\beta}}(\boldsymbol{x}, \boldsymbol{t}) \rho_j^{-q/p}(\boldsymbol{t}) d\boldsymbol{t} \Big)^{1/q}$$

by Hölder's inequality, so we have (2.3).

Now, by virtue of Leibniz's rule, (2.3), and Hölder's inequality, we have

$$\begin{aligned}
&\Big|\partial^{\boldsymbol{\alpha}}\Big(\prod_{j=1}^{m}u_j(\boldsymbol{x})\Big)\Big|\\
&\le \sum_{\boldsymbol{\beta}_1+\cdots+\boldsymbol{\beta}_m=\boldsymbol{\alpha}}\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}_1,\dots,\boldsymbol{\beta}_m}\prod_{j=1}^{m}\big|\partial^{\boldsymbol{\beta}_j}u_j(\boldsymbol{x})\big|\\
&\le \sum_{\boldsymbol{\beta}_1+\cdots+\boldsymbol{\beta}_m=\boldsymbol{\alpha}}\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}_1,\dots,\boldsymbol{\beta}_m}\prod_{j=1}^{m}\Big(\partial^{\boldsymbol{\beta}_j}\int_{\Omega_x}K_{\boldsymbol{\alpha}}(\boldsymbol{x},\boldsymbol{t})\big|\partial^{\boldsymbol{\alpha}}u_j(\boldsymbol{t})\big|^p\rho_j(\boldsymbol{t})\,d\boldsymbol{t}\Big)^{1/p}\\
&\qquad\times\Big(\partial^{\boldsymbol{\beta}_j}\int_{\Omega_x}K_{\boldsymbol{\alpha}}(\boldsymbol{x},\boldsymbol{t})\rho_j^{-q/p}(\boldsymbol{t})\,d\boldsymbol{t}\Big)^{1/q}\\
&\le\Big[\sum_{\boldsymbol{\beta}_1+\cdots+\boldsymbol{\beta}_m=\boldsymbol{\alpha}}\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}_1,\dots,\boldsymbol{\beta}_m}\prod_{j=1}^{m}\partial^{\boldsymbol{\beta}_j}\int_{\Omega_x}K_{\boldsymbol{\alpha}}(\boldsymbol{x},\boldsymbol{t})|\partial^{\boldsymbol{\alpha}}u_j(\boldsymbol{t})|^p\rho_j(\boldsymbol{t})\,d\boldsymbol{t}\Big]^{1/p}\\
&\qquad\times\Big[\sum_{\boldsymbol{\beta}_1+\cdots+\boldsymbol{\beta}_m=\boldsymbol{\alpha}}\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}_1,\dots,\boldsymbol{\beta}_m}\prod_{j=1}^{m}\partial^{\boldsymbol{\beta}_j}\int_{\Omega_x}K_{\boldsymbol{\alpha}}(\boldsymbol{x},\boldsymbol{t})\rho_j^{-q/p}(\boldsymbol{t})\,d\boldsymbol{t}\Big]^{1/q}\\
&=\Big[\partial^{\boldsymbol{\alpha}}\Big(\prod_{j=1}^{m}\int_{\Omega_x}K_{\boldsymbol{\alpha}}(\boldsymbol{x},\boldsymbol{t})\big|\partial^{\boldsymbol{\alpha}}u_j(\boldsymbol{t})\big|^p\rho_j(\boldsymbol{t})\,d\boldsymbol{t}\Big)\Big]^{1/p}\\
&\qquad\times\Big[\partial^{\boldsymbol{\alpha}}\Big(\prod_{j=1}^{m}\int_{\Omega_x}K_{\boldsymbol{\alpha}}(\boldsymbol{x},\boldsymbol{t})\rho_j^{-q/p}(\boldsymbol{t})\,d\boldsymbol{t}\Big)\Big]^{1/q},
\end{aligned}$$

and so

$$\Big|\partial^{\boldsymbol{\alpha}}\Big(\prod_{j=1}^{m}u_j(\boldsymbol{x})\Big)\Big|^p\omega(\boldsymbol{x})\le\partial^{\boldsymbol{\alpha}}\Big(\prod_{j=1}^{m}\int_{\Omega_x}K_{\boldsymbol{\alpha}}(\boldsymbol{x},\boldsymbol{t})\big|\partial^{\boldsymbol{\alpha}}u_j(\boldsymbol{t})\big|^p\rho_j(\boldsymbol{t})\,d\boldsymbol{t}\Big)\tag{2.6}$$

for all $\boldsymbol{x}\in\Omega$. Multiplying both sides of (2.6) by $K_{\boldsymbol{\alpha}}(\boldsymbol{b},\boldsymbol{x})$ and then integrating with respect to $\boldsymbol{x}$ over Ω yields (2.1), as required.

Unless $m=1$, the equality condition in (2.1) is implied by (2.3). So equalities in (2.4) and (2.5) hold for all $\boldsymbol{0}\le\boldsymbol{\beta}\le\boldsymbol{\alpha}$ and $\boldsymbol{x}\in\Omega$. Assume that $\partial^{\boldsymbol{\alpha}}u_j\not\equiv0$ on Ω (otherwise, we can take $C_j=0$). Then, from the equality condition of Hölder's inequality, we have

$$|\partial^{\boldsymbol{\beta}}u_j(\boldsymbol{x})|=\int_{\Omega_x}K_{\boldsymbol{\alpha}-\boldsymbol{\beta}}(\boldsymbol{x},\boldsymbol{t})|\partial^{\boldsymbol{\alpha}}u_j(\boldsymbol{t})|\,d\boldsymbol{t},\tag{2.7}$$

$$A_j|\partial^{\boldsymbol{\alpha}}u_j(\boldsymbol{t})|^p\rho_j(\boldsymbol{t})=B_j\rho_j^{-q/p}(\boldsymbol{t})\quad\text{a.e.}\quad\boldsymbol{t}\in\Omega_{\boldsymbol{x}},\tag{2.8}$$

where A_j and B_j are real constants. By the continuity of ρ_j on Ω, equations (2.7) and (2.8) imply that there exist some real constants $C_j\ne0$, where $j=1,\dots,m$, such that

$$\partial^{\boldsymbol{\alpha}}u_j(\boldsymbol{t})=C_j\rho_j^{-q/p}(\boldsymbol{t}),\qquad\boldsymbol{t}\in\Omega_{\boldsymbol{x}}.$$

We thus get (2.2). Conversely, direct computation shows that equality holds in (2.1) if u_j, $j=1,\dots,m$, are given by (2.2). ∎

Next, we establish integral inequalities for partial differential operators acting on compositions of functions. For $k\in\mathbb{N}$ and $0<R\le\infty$, we denote by $\mathcal{G}_R^k$ the class of all functions $G\colon(-R,R)\to\mathbb{R}$ satisfying the following conditions:

(a) $G\in C^k(-R,R)$,
(b) $G^{(j)}(0)=0$ for all $0\le j\le k-1$,
(c) $|G^{(k)}(x)|\le G^{(k)}(|x|)$ for all $x\in(-R,R)$, and

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(d) if $x \le y^{1/p} z^{1/q}$, $0 \le x, y, z < R$, then $0 \le G^{(k)}(x) \le [G^{(k)}(y)]^{1/p}[G^{(k)}(z)]^{1/q}$.

Remark 2.2 From [5, Remark 2.6] we notice that if $G \in \mathcal{G}_R^k$, then $G^{(j)}$, where $j = 1, \dots, k$, are non-negative and increasing on the interval $(0, R)$. Moreover, if $j \le k$, then $\mathcal{G}_R^k \subset \mathcal{G}_R^j$.

Example 2.3 The function $G(x) = |x|^p$, where $p \in [k, \infty) \cup \mathbb{N}_0$, belongs to $\mathcal{G}_\infty^k$.

Theorem 2.4 *Let G be a function of class $\mathcal{G}_R^{|\boldsymbol{\alpha}|}$ and let ρ be a weight on Ω such that there exists a new weight*

$$\theta(\boldsymbol{x}) = \Big[\partial^{\boldsymbol{\alpha}} G\Big(\int_{\Omega_x} K_{\boldsymbol{\alpha}}(\boldsymbol{x},\boldsymbol{t})\rho^{-q/p}(\boldsymbol{t})d\boldsymbol{t}\Big)\Big]^{-p/q}, \quad \boldsymbol{x} \in Q.$$

Let $u \in \mathcal{AC}_p^{\boldsymbol{\alpha}}(\Omega, \rho)$ be such that $\int_\Omega |\partial^{\boldsymbol{\alpha}} u(\boldsymbol{x})|^p K_{\boldsymbol{\alpha}}(\boldsymbol{b},\boldsymbol{x})\rho(\boldsymbol{x})d\boldsymbol{x} < R$; then

$$\int_\Omega \big|\partial^{\boldsymbol{\alpha}} G(u(\boldsymbol{x}))\big|^p K_{\boldsymbol{\alpha}}(\boldsymbol{b},\boldsymbol{x})\theta(\boldsymbol{x})d\boldsymbol{x} \le G\Big(\int_\Omega \big|\partial^{\boldsymbol{\alpha}} u(\boldsymbol{x})\big|^p K_{\boldsymbol{\alpha}}(\boldsymbol{b},\boldsymbol{x})\rho(\boldsymbol{x})d\boldsymbol{x}\Big). \tag{2.9}$$

Assume further that $G^{(|\boldsymbol{\alpha}|)}$ is strictly increasing on $(0, R)$; then equality holds in (2.9) only if there exists a real constant C such that

$$u(\boldsymbol{x}) = C \int_{\Omega_x} K_{\boldsymbol{\alpha}}(\boldsymbol{x},\boldsymbol{t})\rho^{-q/p}(\boldsymbol{t})d\boldsymbol{t}, \quad \boldsymbol{x} \in \Omega. \tag{2.10}$$

Proof For $u \in \mathcal{AC}_p^{\boldsymbol{\alpha}}(\Omega, \rho)$ and $\boldsymbol{0} \le \boldsymbol{\beta} \le \boldsymbol{\alpha}$, we see from (2.3) that

$$\begin{aligned} |\partial^{\boldsymbol{\beta}} u(\boldsymbol{x})| \le {} & \Big[\partial^{\boldsymbol{\beta}}\Big(\int_{\Omega_x} K_{\boldsymbol{\alpha}}(\boldsymbol{x},\boldsymbol{t})|\partial^{\boldsymbol{\alpha}} u(\boldsymbol{t})|^p \rho(\boldsymbol{t})d\boldsymbol{t}\Big)\Big]^{1/p} \\ & \times \Big[\partial^{\boldsymbol{\beta}}\Big(\int_{\Omega_x} K_{\boldsymbol{\alpha}}(\boldsymbol{x},\boldsymbol{t})\rho^{-q/p}(\boldsymbol{t})d\boldsymbol{t}\Big)\Big]^{1/q}, \quad \boldsymbol{x} \in \Omega. \end{aligned} \tag{2.11}$$

According to (2.11) and Remark 2.2, we get

$$\begin{aligned} |G^{(k)}(u(\boldsymbol{x}))| &\le G^{(k)}(|u(\boldsymbol{x})|) \\ &\le G^{(k)}\Big(\Big[\int_{\Omega_x} K_{\boldsymbol{\alpha}}(\boldsymbol{x},\boldsymbol{t})|\partial^{\boldsymbol{\alpha}} u(\boldsymbol{t})|^p \rho(\boldsymbol{t})d\boldsymbol{t}\Big]^{1/p}\Big[\int_{\Omega_x} K_{\boldsymbol{\alpha}}(\boldsymbol{x},\boldsymbol{t})\rho^{-q/p}(\boldsymbol{t})d\boldsymbol{t}\Big]^{1/q}\Big) \\ &\le \Big[G^{(k)}\Big(\int_{\Omega_x} K_{\boldsymbol{\alpha}}(\boldsymbol{x},\boldsymbol{t})|\partial^{\boldsymbol{\alpha}} u(\boldsymbol{t})|^p \rho(\boldsymbol{t})d\boldsymbol{t}\Big)\Big]^{1/p} \\ &\quad \times \Big[G^{(k)}\Big(\int_{\Omega_x} K_{\boldsymbol{\alpha}}(\boldsymbol{x},\boldsymbol{t})\rho^{-q/p}(\boldsymbol{t})d\boldsymbol{t}\Big)\Big]^{1/q} \end{aligned} \tag{2.12}$$

for $0 \le k \le |\boldsymbol{\alpha}|$ and $\boldsymbol{x} \in \Omega$. Notice that

$$\partial^{\boldsymbol{\alpha}} G(u(\boldsymbol{x})) = \prod_{i=1}^n \big[\partial_i u(\boldsymbol{x})\big]^{\alpha_i} G^{(|\boldsymbol{\alpha}|)}(u(\boldsymbol{x})) + \cdots + \partial^{\boldsymbol{\alpha}} u(\boldsymbol{x}) G'(u(\boldsymbol{x})), \tag{2.13}$$

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where all coefficients appearing in the sum are non-negative. Using (2.11) and (2.12) in (2.13) and applying Hölder's inequality we obtain

$$\big|\partial^{\boldsymbol{\alpha}} G(u(\boldsymbol{x}))\big|^p \le \partial^{\boldsymbol{\alpha}} G\Big(\int_{\Omega_{\boldsymbol{x}}} K_{\boldsymbol{\alpha}}(\boldsymbol{x},\boldsymbol{t})|\partial^{\boldsymbol{\alpha}} u(\boldsymbol{t})|^p \rho(\boldsymbol{t})\,d\boldsymbol{t}\Big) \times \Big[\partial^{\boldsymbol{\alpha}} G\Big(\int_{\Omega_{\boldsymbol{x}}} K_{\boldsymbol{\alpha}}(\boldsymbol{x},\boldsymbol{t})\rho^{-q/p}(\boldsymbol{t})\,d\boldsymbol{t}\Big)\Big]^{p/q},$$

and hence

$$\big|\partial^{\boldsymbol{\alpha}} G(u(\boldsymbol{x}))\big|^p K_{\boldsymbol{\alpha}}(\boldsymbol{b},\boldsymbol{x})\theta(\boldsymbol{x}) \le K_{\boldsymbol{\alpha}}(\boldsymbol{b},\boldsymbol{x})\partial^{\boldsymbol{\alpha}} G\Big(\int_{\Omega_{\boldsymbol{x}}} K_{\boldsymbol{\alpha}}(\boldsymbol{x},\boldsymbol{t})|\partial^{\boldsymbol{\alpha}} u(\boldsymbol{t})|^p \rho(\boldsymbol{t})\,d\boldsymbol{t}\Big), \quad \boldsymbol{x}\in\Omega. \tag{2.14}$$

Integrating both sides of (2.14) with respect to $\boldsymbol{x}$ over Ω and using the monotonicity of G we obtain (2.9).

If $G^{(|\boldsymbol{\alpha}|)}$ is strictly increasing on $(0,R)$, then so is $G^{(k)}$ for $0\le k\le|\boldsymbol{\alpha}|-1$. Thus, the case of equality follows from (2.11) and (2.12). Hence, by an argument similar to that used in the proof of Theorem 2.1 we derive u has the form (2.10), as required. ■

Combining Theorems 2.1 and 2.4 yields the following corollary.

Corollary 2.5 *Let $G_j\in\mathcal{G}_R^{|\boldsymbol{\alpha}|}$ and ρ_j, where $j=1,\dots,m$, be some weights on Ω such that there exists a new weight*

$$\eta(\boldsymbol{x}) := \Big[\partial^{\boldsymbol{\alpha}}\prod_{j=1}^m G_j\Big(\int_{\Omega_{\boldsymbol{x}}} K_{\boldsymbol{\alpha}}(\boldsymbol{x},\boldsymbol{t})\rho_j^{-q/p}(\boldsymbol{t})\,d\boldsymbol{t}\Big)\Big]^{-p/q}, \quad \boldsymbol{x}\in Q.$$

Let $u_j\in\mathcal{AC}_p^{\boldsymbol{\alpha}}(\Omega,\rho_j)$, where $j=1,\dots,m$, be such that

$$\int_\Omega\big|\partial^{\boldsymbol{\alpha}} u_j(\boldsymbol{x})\big|^p K_{\boldsymbol{\alpha}}(\boldsymbol{b},\boldsymbol{x})\rho_j(\boldsymbol{x})\,d\boldsymbol{x} < R;$$

then

$$\int_\Omega\Big|\partial^{\boldsymbol{\alpha}}\Big(\prod_{j=1}^m G_j(u_j(\boldsymbol{x}))\Big)\Big|^p K_{\boldsymbol{\alpha}}(\boldsymbol{b},\boldsymbol{x})\eta(\boldsymbol{x})\,d\boldsymbol{x} \le \prod_{j=1}^m G_j\Big(\int_\Omega\big|\partial^{\boldsymbol{\alpha}} u_j(\boldsymbol{x})\big|^p K_{\boldsymbol{\alpha}}(\boldsymbol{b},\boldsymbol{x})\rho_j(\boldsymbol{x})\,d\boldsymbol{x}\Big). \tag{2.15}$$

If, in addition, we assume that $G_j^{(|\boldsymbol{\alpha}|)}$, where $j=1,\dots,m$, are strictly increasing on $(0,R)$, then equality holds in (2.15) only if

$$u_j(\boldsymbol{x}) = C_j\int_{\Omega_{\boldsymbol{x}}} K_{\boldsymbol{\alpha}}(\boldsymbol{x},\boldsymbol{t})\rho_j^{-q/p}(t)\,d\boldsymbol{t}, \quad \boldsymbol{x}\in\Omega,$$

where C_j, $j=1,\dots,m$, are real constants.

By virtue of Corollary 2.5 one can derive a new Opial-type inequality involving functions of n variables as follows.

Corollary 2.6 *Let $r, s > 0$ be such that $1/p + 1/r = 1/s$. Suppose that the hypotheses in Corollary 2.5 are valid. Then, for a weight σ on Ω such that*

$$K := \Big(\int_\Omega K_{\boldsymbol{\alpha}}(\boldsymbol{b}, \boldsymbol{x}) \eta^{-r/p}(\boldsymbol{x}) \sigma^{r/s}(\boldsymbol{x}) d\boldsymbol{x} \Big)^{1/r} < \infty,$$

we have

$$\Big[\int_\Omega \Big| \partial^{\boldsymbol{\alpha}} \Big(\prod_{j=1}^m G_j(u_j(\boldsymbol{x})) \Big) \Big|^s K_{\boldsymbol{\alpha}}(\boldsymbol{b}, \boldsymbol{x}) \sigma(\boldsymbol{x}) d\boldsymbol{x} \Big]^{1/s} \leq K \prod_{j=1}^m \Big[G_j \Big(\int_\Omega \big| \partial^{\boldsymbol{\alpha}} u_j(\boldsymbol{x}) \big|^p K_{\boldsymbol{\alpha}}(\boldsymbol{b}, \boldsymbol{x}) \rho_j(\boldsymbol{x}) d\boldsymbol{x} \Big) \Big]^{1/p}. \tag{2.16}$$

Assume further that $G_j^{(|\boldsymbol{\alpha}|)}$, where $j = 1, \ldots, m$, are strictly increasing on $(0, R)$; then equality holds in (2.16) only if

$$u_j(\boldsymbol{x}) = C_j \int_{\Omega_{\boldsymbol{x}}} K_{\boldsymbol{\alpha}}(\boldsymbol{x}, \boldsymbol{t}) \rho_j^{-q/p}(t) d\boldsymbol{t}, \quad \boldsymbol{x} \in \Omega,$$

where C_j, $j = 1, \ldots, m$, are real constants.

Proof Using Hölder's inequality with conjugate exponents p/s and r/s yields

$$\Big[\int_\Omega \Big| \partial^{\boldsymbol{\alpha}} \Big(\prod_{j=1}^m G_j(u_j(\boldsymbol{x})) \Big) \Big|^s K_{\boldsymbol{\alpha}}(\boldsymbol{b}, \boldsymbol{x}) \sigma(\boldsymbol{x}) d\boldsymbol{x} \Big]^{1/s} \leq K \Big[\int_\Omega \Big| \partial^{\boldsymbol{\alpha}} \Big(\prod_{j=1}^m G_j(u_j(\boldsymbol{x})) \Big) \Big|^p K_{\boldsymbol{\alpha}}(\boldsymbol{b}, \boldsymbol{x}) \eta(\boldsymbol{x}) d\boldsymbol{x} \Big]^{1/p},$$

which, in view of (2.15), gives (2.16). ■

Remark 2.7 Corollary 2.5 gives rise to further Opial-type inequalities for functions of n variables:

- Let $\boldsymbol{\alpha} = \mathbf{1}$, $m = 1$, $s = 1$, $p = r = 2$, $\sigma = \rho_1 \equiv 1$, $G_1(x) = x^2$, and $u = u_1 \in \mathcal{AC}_2^{\mathbf{1}}(\Omega, 1)$. One has

$$\int_\Omega \big| \partial^{\mathbf{1}} u^2(\boldsymbol{x}) \big| d\boldsymbol{x} \leq \mathrm{Vol}(\Omega) \int_\Omega \big| \partial^{\mathbf{1}} u(\boldsymbol{x}) \big|^2 d\boldsymbol{x}. \tag{2.17}$$

  Equality holds in (2.17) if and only if u has the form

$$u(\boldsymbol{x}) = C\,\mathrm{Vol}(\Omega_{\boldsymbol{x}}), \quad \boldsymbol{x} \in \Omega,$$

  where C is a real constant.
- Let $\boldsymbol{\alpha} = \mathbf{1}$, $m = 1$, $s = 1$, $p = r = 2$, $\rho_1 \equiv 1$, $G_1(x) = x^2$, and $u = u_1 \in \mathcal{AC}_2^{\mathbf{1}}(\Omega, 1)$; then (2.16) reduces to

$$\int_\Omega \big| \partial^{\mathbf{1}} u^2(\boldsymbol{x}) \big| \sigma(\boldsymbol{x}) d\boldsymbol{x} \leq K \int_\Omega \big| \partial^{\mathbf{1}} u(\boldsymbol{x}) \big|^2 d\boldsymbol{x} \tag{2.18}$$

  provided

$$K := \Big(2^n \int_\Omega \mathrm{Vol}(\Omega_{\boldsymbol{x}}) \sigma^2(\boldsymbol{x}) d\boldsymbol{x} \Big)^{1/2} < \infty.$$

- If one sets $\boldsymbol{\alpha} = \mathbf{1}$, $m = 1$, $s = 1$, $\sigma = \rho_1 \equiv 1$, $G_1(x) = |x|^p$ and $u = u_1 \in \mathcal{AC}_p^{\mathbf{1}}(\Omega, 1)$, then we have

$$\int_\Omega \big| \partial^{\mathbf{1}} |u(\boldsymbol{x})|^p \big| \, d\boldsymbol{x} \le [\mathrm{Vol}(\Omega)]^{p-1} \int_\Omega \big| \partial^{\mathbf{1}} u(\boldsymbol{x}) \big|^p \, d\boldsymbol{x}. \tag{2.19}$$

Equality holds in (2.19) if and only if there is a real constant C such that $u(\boldsymbol{x}) = C\,\mathrm{Vol}(\Omega_{\boldsymbol{x}})$, $\boldsymbol{x} \in \Omega$.

3 Applications

This section deals with some applications of inequalities obtained in Remark 2.7 in studying the partial differential equation

$$\partial^{\mathbf{1}} u(\boldsymbol{x}) = \zeta(\boldsymbol{x}, \langle u \rangle), \quad \boldsymbol{x} \in \Omega, \tag{3.1}$$

with the initial conditions $u|_{x_j = a_j} = 0$ for $j = 1, \dots, n$, where

$$\langle u \rangle = \Big(u, \partial^{(1,0,\dots,0)} u, \partial^{(0,1,\dots,0)} u, \dots, \partial^{(0,1,\dots,1)} u \Big) \subset \mathbb{R}^{2^n - 1}.$$

3.1 Uniqueness of the Initial Value Problem

If the function ζ satisfies a Lipschitz condition, *i.e.,* there exists a non-negative constant M such that for all $(\boldsymbol{x}, \langle u \rangle), (\boldsymbol{x}, \langle \overline{u} \rangle) \in \Omega \times \mathbb{R}^{2^n - 1}$,

$$|\zeta(\boldsymbol{x}, \langle u \rangle) - \zeta(\boldsymbol{x}, \langle \overline{u} \rangle)| \le M |u(\boldsymbol{x}) - \overline{u}(\boldsymbol{x})|, \tag{3.2}$$

then equation (3.1) has at most one solution on Ω. Notice that the constant M in (3.2) can be replaced by a non-negative continuous function $f(\boldsymbol{x})$ on Ω. We refer to [1,12,13] for a more general and detailed discussion of this problem. However, we will see in the following theorem that the Lipschitz condition (3.2) is only a sufficient but not a necessary condition to prove the uniqueness of the solution of (3.1).

Theorem 3.1 *Suppose that for* $(\boldsymbol{x}, \langle u \rangle), (\boldsymbol{x}, \langle \overline{u} \rangle) \in \Omega \times \mathbb{R}^{2^n - 1}$,

$$\big| \zeta(\boldsymbol{x}, \langle u \rangle) - \zeta(\boldsymbol{x}, \langle \overline{u} \rangle) \big| \le f(\boldsymbol{x}) \big| u(\boldsymbol{x}) - \overline{u}(\boldsymbol{x}) \big|, \tag{3.3}$$

where f *is non-negative on* Ω *and there exists a positive number* M *such that*

$$\mathrm{Vol}(\Omega_{\boldsymbol{x}}) \Big(\int_{\Omega_{\boldsymbol{x}}} f^q(\boldsymbol{t}) d\boldsymbol{t} \Big)^{2q/p} \le M, \quad \boldsymbol{x} \in \Omega. \tag{3.4}$$

Then equation (3.1) *has at most one solution on* Ω.

Proof Assume that u and $\overline{u}$ are two solutions of (3.1). Then the function $v := u - \overline{u}$ satisfies

$$v(\boldsymbol{x}) = \int_{\Omega_{\boldsymbol{x}}} \big[\zeta\big(\boldsymbol{t}, \langle u \rangle\big) - \zeta\big(\boldsymbol{t}, \langle \overline{u} \rangle\big) \big] d\boldsymbol{t}, \quad \boldsymbol{x} \in \Omega.$$

By (3.3) and Hölder's inequality, one has

$$|v(\boldsymbol{x})| \le \int_{\Omega_{\boldsymbol{x}}} f(\boldsymbol{t}) |v(\boldsymbol{t})| d\boldsymbol{t} \le \Big(\int_{\Omega_{\boldsymbol{x}}} f^q(\boldsymbol{t}) d\boldsymbol{t} \Big)^{1/q} \Big(\int_{\Omega_{\boldsymbol{x}}} |v(\boldsymbol{t})|^p d\boldsymbol{t} \Big)^{1/p},$$

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and so

$$|v(\boldsymbol{x})|^{2p} \le \Big(\int_{\Omega_x} f^q(\boldsymbol{t})d\boldsymbol{t}\Big)^{p/q}|v(\boldsymbol{x})|^p\Big(\int_{\Omega_x}|v(\boldsymbol{t})|^p d\boldsymbol{t}\Big), \quad \boldsymbol{x}\in\Omega.$$

Since

$$|v(\boldsymbol{x})|^p\Big(\int_{\Omega_x}|v(\boldsymbol{t})|^p d\boldsymbol{t}\Big) \le \partial^{\mathbf{1}}\Big(\int_{\Omega_x}|v(\boldsymbol{t})|^p d\boldsymbol{t}\Big)^2,$$

it follows that

$$|v(\boldsymbol{x})|^{2p} \le \sigma(\boldsymbol{x})\partial^{\mathbf{1}}\Big(\int_{\Omega_x}|v(\boldsymbol{t})|^p d\boldsymbol{t}\Big)^2, \tag{3.5}$$

where

$$\sigma(\boldsymbol{x}) := \Big(\int_{\Omega_x} f^q(\boldsymbol{t})d\boldsymbol{t}\Big)^{p/q}, \quad \boldsymbol{x}\in\Omega.$$

Let $\boldsymbol{c}\in Q$ be such that $\mathrm{Vol}(\Omega_{\boldsymbol{c}}) < 2^{-n}/M$. Then we have

$$K := \Big(2^n\int_{\Omega_c}\mathrm{Vol}(\Omega_{\boldsymbol{x}})\sigma^2(\boldsymbol{x})d\boldsymbol{x}\Big)^{1/2} < 1, \quad \boldsymbol{x}\in\Omega_{\boldsymbol{c}},$$

by (3.4). Integrating both sides of (3.5) with respect to $\boldsymbol{x}$ over $\Omega_{\boldsymbol{c}}$ and applying (2.18), we obtain

$$\int_{\Omega_c}|v(\boldsymbol{x})|^{2p}d\boldsymbol{x} \le K\int_{\Omega_c}|v(\boldsymbol{x})|^{2p}d\boldsymbol{x},$$

which yields $v(\boldsymbol{x}) = 0$ on $\Omega_{\boldsymbol{c}}$. If $\boldsymbol{c}\le\boldsymbol{b}$ and $\boldsymbol{c}\ne\boldsymbol{b}$, we can repeat the above arguments to obtain $v(\boldsymbol{x}) = 0$ on Ω. Hence, we have $u = \overline{u}$ on Ω. ∎

Example 3.2 Let $\delta\in(0,1)$ and consider the partial differential equation

$$\partial^{\mathbf{1}}u(\boldsymbol{x}) = \frac{u(\boldsymbol{x})}{[\mathrm{Vol}(\Omega_{\boldsymbol{x}})]^\delta}, \quad \boldsymbol{x}\in Q, \tag{3.6}$$

together with the initial conditions $u|_{x_j=a_j} = 0$ for $j = 1,\dots,n$. Since the function $f(\boldsymbol{x}) = 1/[\mathrm{Vol}(\Omega_{\boldsymbol{x}})]^\delta$ is unbounded on Q, it follows that $\zeta(\boldsymbol{x},\langle u\rangle) = u(\boldsymbol{x})/[\mathrm{Vol}(\Omega_{\boldsymbol{x}})]^\delta$ does not satisfy the Lipschitz condition on Q. However, we see that $u\equiv 0$ is the unique solution of the problem. Indeed, let $q>1$ be such that $q\delta<1$. Then

$$\begin{aligned}\mathrm{Vol}(\Omega_{\boldsymbol{x}})\Big(\int_{\Omega_x} f^q(\boldsymbol{t})d\boldsymbol{t}\Big)^{2q/p} &= \frac{1}{(1-q\delta)^{2nq/p}}\big[\mathrm{Vol}(\Omega_{\boldsymbol{x}})\big]^{1+2(1-q\delta)q/p}\\ &\le \frac{1}{(1-q\delta)^{2nq/p}}\big[\mathrm{Vol}(\Omega)\big]^{1+2(1-q\delta)q/p}, \quad \boldsymbol{x}\in\Omega.\end{aligned}$$

Hence, by taking advantage of Theorem 3.1 we claim that equation (3.6) has the unique solution $u\equiv 0$ on Ω.

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3.2 Upper Bound of Solutions

Finally, we get an upper bound of the solutions of (3.1).

Theorem 3.3 *Let $\delta > 1$ and suppose that*

$$|\zeta(\boldsymbol{x}, \langle u \rangle)| \le f(\boldsymbol{x}) + h(\boldsymbol{x})|u(\boldsymbol{x})|^{\delta}, \quad \boldsymbol{x} \in \Omega, \tag{3.7}$$

where f and h are non-negative on Ω. We assume further that equation (3.1) has solutions $u \in \mathcal{AC}^{\mathbf{1}}_{\delta}(\Omega, 1)$. Then

$$u(\boldsymbol{x}) \le \int_{\Omega_x} \big[F^{1-\delta}(\boldsymbol{t}) - (\delta - 1) H(\boldsymbol{t}) \operatorname{Vol}(\Omega_{\boldsymbol{t}}) \big]^{1/(1-\delta)} d\boldsymbol{t} \tag{3.8}$$

as long as the right-hand side integral exists, where $F(\boldsymbol{x}) := \sup_{\boldsymbol{t} \in \Omega_x} f(\boldsymbol{t})$ and $H(\boldsymbol{x}) := \sup_{\boldsymbol{t} \in \Omega_x} h(\boldsymbol{t}) [\operatorname{Vol}(\Omega_{\boldsymbol{t}})]^{\delta-1}$ for $\boldsymbol{x} \in \Omega$.

Proof Making use of (2.19) with $p = \delta$ we observe that

$$|u(\boldsymbol{x})|^{\delta} = \int_{\Omega_x} \partial^{\mathbf{1}} |u(\boldsymbol{t})|^{\delta} d\boldsymbol{t} \le [\operatorname{Vol}(\Omega_{\boldsymbol{x}})]^{\delta-1} \int_{\Omega_x} \big| \partial^{\mathbf{1}} u(\boldsymbol{t}) \big|^{\delta} d\boldsymbol{t},$$

and so, in view of (3.7), we obtain

$$\big| \partial^{\mathbf{1}} u(\boldsymbol{x}) \big| \le f(\boldsymbol{x}) + h(\boldsymbol{x}) \big[\operatorname{Vol}(\Omega_{\boldsymbol{x}}) \big]^{\delta-1} \int_{\Omega_x} \big| \partial^{\mathbf{1}} u(\boldsymbol{t}) \big|^{\delta} d\boldsymbol{t}, \quad \boldsymbol{x} \in \Omega. \tag{3.9}$$

Let $\boldsymbol{s} \in \Omega$ be arbitrary, but fixed. Then inequality (3.9) gives

$$\big| \partial^{\mathbf{1}} u(\boldsymbol{t}) \big| \le F(\boldsymbol{s}) + H(\boldsymbol{s}) \int_{\Omega_t} \big| \partial^{\mathbf{1}} u(\boldsymbol{y}) \big|^{\delta} d\boldsymbol{y}, \quad \boldsymbol{t} \in \Omega_{\boldsymbol{s}}. \tag{3.10}$$

Next, let $R(\boldsymbol{t})$ be the right-hand side of (3.10), so that

$$\partial^{\mathbf{1}} R(\boldsymbol{t}) = H(\boldsymbol{s}) \big| \partial^{\mathbf{1}} u(\boldsymbol{t}) \big|^{\delta} \le H(\boldsymbol{s}) R^{\delta}(\boldsymbol{t}), \quad \boldsymbol{t} \in \Omega_{\boldsymbol{s}}, \tag{3.11}$$

where $R|_{t_j = a_j} = F(\boldsymbol{s})$ for $j = 1, \dots, n$. Since

$$\int_{\Omega_w} \frac{\partial^{\mathbf{1}} R(\boldsymbol{t})}{R^{\delta}(\boldsymbol{t})} d\boldsymbol{t} \ge \frac{1}{1-\delta} \big[R^{1-\delta}(\boldsymbol{w}) - F^{1-\delta}(\boldsymbol{s}) \big]$$

it follows from (3.11) that

$$\frac{1}{1-\delta} \big[R^{1-\delta}(\boldsymbol{w}) - F^{1-\delta}(\boldsymbol{s}) \big] \le H(\boldsymbol{s}) \operatorname{Vol}(\Omega_{\boldsymbol{w}}),$$

which, together with (3.10), yields

$$\big| \partial^{\mathbf{1}} u(\boldsymbol{w}) \big| \le R(\boldsymbol{w}) \le \big[F^{1-\delta}(\boldsymbol{s}) - (\delta - 1) H(\boldsymbol{s}) \operatorname{Vol}(\Omega_{\boldsymbol{w}}) \big]^{1/(1-\delta)}, \quad \boldsymbol{w} \in \Omega_{\boldsymbol{s}}.$$

In the above inequality replacing $\boldsymbol{w}$ by $\boldsymbol{s}$ and integrating both sides with respect to $\boldsymbol{s}$ over $\Omega_{\boldsymbol{x}}$ for $\boldsymbol{x} \in \Omega$, we obtain (3.8). ∎

We illustrate this in the following example.

Example 3.4 We consider the nonlinear partial differential equation

$$\partial^{\mathbf{1}} u(\boldsymbol{x}) = 1 + \frac{u^2(\boldsymbol{x})}{1 + [\operatorname{Vol}(\Omega_{\boldsymbol{x}})]^2}, \quad \boldsymbol{x} \in \Omega,$$

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with $u|_{x_j=a_j} = 0$ for all $j = 1, \dots, n$. Let $f(\boldsymbol{x}) = 1$ and $h(\boldsymbol{x}) = 1/(1 + [\mathrm{Vol}(\Omega_{\boldsymbol{x}})]^2)$. We see that $F(\boldsymbol{x}) = 1$ and $H(\boldsymbol{x}) = \mathrm{Vol}(\Omega_{\boldsymbol{x}})/(1 + [\mathrm{Vol}(\Omega_{\boldsymbol{x}})]^2)$. Therefore, as in (3.8),

$$u(\boldsymbol{x}) \le \int_{\Omega_x} \big(1 + [\mathrm{Vol}(\Omega_{\boldsymbol{t}})]^2\big)\, d\boldsymbol{t} = \mathrm{Vol}(\Omega_{\boldsymbol{x}}) + \frac{1}{3}[\mathrm{Vol}(\Omega_{\boldsymbol{x}})]^3, \quad \boldsymbol{x} \in \Omega.$$

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