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BIVECTORS OVER A FINITE FIELD

BY

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ABSTRACT. Let U be an n-dimensional vector space over a finite field of q elements. The number of elements of Λ^2U of each irreducible length is found using the isomorphism of Λ^2U with H_n, the space of $n \times n$ skew-symmetric matrices, and results due to Carlitz and MacWilliams on the number of skew-symmetric matrices of any given rank.

Let U be an n-dimensional vector space over a finite field $F = GF(q)$. We consider the elements of Λ^2U (called bivectors, or 2-vectors). The (irreducible) length of a 2-vector is well known. Any 2-vector can be expressed as a sum $\sum_1^r x_i \wedge y_i$ where $\{x_1, \ldots, x_r, y_1, \ldots, y_r\}$ is independent and then its length is r. The 2-vectors of length 1 are called decomposable.

Of the $q^{\binom{n}{2}}$ elements of Λ^2U, it is difficult to count directly the number having a fixed length, since there is no unique representation for a 2-vector as a sum of the minimal number of decomposables. However, we can make use of the isomorphism of Λ^2U with H_n, the space of $n \times n$ skew-symmetric matrices over F. This isomorphism, denoted ϕ, is shown by Marcus and Westwick [3] to have the property that $z \in \Lambda^2U$ has length r if and only if $\phi(z) \in H_n$ has rank $2r$. The number of skew-symmetric matrices of rank $2r$ has been determined by Carlitz [1] and MacWilliams [2]. Consequently, we have

THEOREM. *If U is a vector space of dimension n over $GF(q)$, the number of vectors in Λ^2U of length r is*

$$K(n, r) = \prod_1^r \frac{q^{2i-2}}{(q^{2i}-1)} \prod_0^{2r-1} (q^{n-i}-1)$$

This is valid even when $q = 2^s$, although then Λ^2U coincides with the symmetric product V^2U.

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References

1. L. Carlitz, *Representations by Skew Forms in a Finite Field*, Arch. Math. V, 1954, p. 19–31.

2. J. MacWilliams, *Orthogonal Matrices over Finite Fields*, Amer. Math. Monthly **76** (1969), 152–164.

3. M. Marcus and R. Westwick, *Linear Maps on Skew-Symmetric Matrices: The Invariance of the Elementary Symmetric Functions*, Pac. Jour. Math. **10** (1960), 917–924.

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