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THE POLYCYCLIC LENGTH OF LINEAR AND FINITE POLYCYCLIC GROUPS

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1. Introduction. In what follows, a polycyclic series, for the group G, is any finite series

$$G = G_0 \geqq G_1 \geqq \ldots \geqq G_l = 1$$

of subgroups of G, such that $G_{i+1} \lhd G_i$ and G_i/G_{i+1} is cyclic, for all $i = 0, \ldots, l-1$. A group that has a polycyclic series is called a polycyclic group, and if G is a polycyclic group, then the polycyclic length of G, which we denote by $\rho(G)$, is the number of non-trivial factors of a polycyclic series for G of shortest length. Notice that a finite group is polycyclic if, and only if, it is solvable, while in general any polycyclic group is solvable, but the converse does not hold.

By a linear group of degree n over the field F, we shall mean any subgroup of the general linear group, $\mathrm{GL}(n, F)$, of all non-singular $n \times n$ matrices with entries from F.

In this paper it is shown that if G is a finite, completely reducible, solvable, linear group of degree n, then the polycyclic length of G is at most $\frac{1}{6}(19n - 8s)$, where s is the number of irreducible components of G. Moreover this bound can be attained for an infinite number of values of the degree. This theorem has two immediate corollaries. The first presents a bound on the polycyclic length of finite solvable linear groups of degree n, over fields of characteristic p. It states that for such a group, with Sylow p-subgroups of order p^m, the polycyclic length is at most $\frac{1}{6}(19n - 8) + m$. The second corollary shows that if G is an abstract finite solvable group, whose Sylow p-subgroups are of order p^m, then the polycyclic length of $G/0_{p'}(G)$ is bounded by $\frac{1}{6}(25m - 8)$.

In what follows, Fit G denotes the Fitting subgroup of the group G, and $0_p(G)$, and $0_{p'}(G)$, denote the unique maximal normal p-subgroup, and p'-subgroup, respectively.

The results presented here are taken from the author's Ph.D. thesis, presented at Carleton University under the direction of Professor J. D. Dixon.

2. Polycyclic linear groups. Most of the proof of the main theorem of this paper, on finite solvable linear groups, is contained in the following lemmas.

LEMMA 1. *If G is a primitive polycyclic linear group of degree n over an alge-*

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braically closed field, then

$$\rho(G/ZG) \leqq 3n - 2.$$

Proof. Since G is polycyclic, it is solvable. If G is equal to its centre, then the bound holds, trivially, and if G is not equal to its centre, then Fit $G \neq ZG$. By [**4**, Theorem 1],

$$|G : \text{Fit } G| \leqq 2^{n-1} \cdot 3^{(2n-1)/3},$$

and, by Suprunenko's theorem [**9**, Theorem 11],

$$|\text{Fit } G : ZG| \leqq n^2.$$

Since each factor in a polycyclic series of G/ZG accounts for at least one prime divisor of $|G : ZG|$, we see that $\rho(G/ZG)$ is bounded by $\log_2 |G : ZG|$, which is at most

$$\log_2(n^2 \cdot 2^{n-1} \cdot 2^{(\log_2 3)(2n-1)/3}).$$

By direct calculation, it is easily seen that

$$\log_2 n \leqq \tfrac{1}{2}(1 - \tfrac{1}{3} \log_2 3)(2n - 1)$$

for $n \geqq 6$, and hence

$$\rho(G/ZG) \leqq 3n - 2 \text{ for } n \geqq 6.$$

To deal with the exceptional cases, we note that, by Suprunenko's Theorem [**9**, Theorem 11], $|\text{Fit } G : ZG| = d^2$ for some divisor d of n, and if d has the prime decomposition

$$d = p_1^{\epsilon_1} \dots p_s^{\epsilon_s},$$

then $G/\text{Fit } G$ is isomorphic to a subgroup of the direct product of the symplectic groups Sp $(2\epsilon_i, p_i)$, $i = 1, \dots, s$.

Let $\nu(x)$ denote the number of prime divisors of the integer x, counted according to multiplicity. Then, for $n = 2, 3$ or 5, we have

$$\begin{aligned}\rho(G/ZG) &\leqslant \rho(G/\text{Fit } G) + \rho((\text{Fit } G)/ZG) \\ &\leqslant \nu\left(\prod_{i=1}^{s} |Sp(2\epsilon_i, p_i)| + \nu(n^2)\right) \\ &\leqslant \nu(n(n^2 - 1)) + 2\end{aligned}$$

since n is prime. Thus $\rho(G/ZG) \leqq 4, 6$ or 7 when $n = 2, 3$ or 5 respectively, and hence $\rho(G/ZG)$ is at most $3n - 2$.

The case $n = 1$ does not occur, since Fit $G \neq ZG$. If $n = 4$, then $G/\text{Fit } G$ is isomorphic to a subgroup of $\text{Sp}(4, 2)$, which in turn is isomorphic to the symmetric group S_6 [**1**, Chapter 5]. Now, instead of considering only S_6, we can apply the following result.

LEMMA 2. *If G is a permutation group on r letters with s orbits, then the composition length of G is at most $(4/3)$ $(r - s)$, and this bound can be attained by suitable permutation groups. Consequently, if G is solvable, then*

$$\rho(G) \leqq (4/3)(r - s).$$

Thus, when $n = 4$, we have

$$\rho(G/\text{Fit } G) \leqslant \left[\frac{4}{3} \cdot 5\right] = 6$$

(where $[x]$ denotes the integral part of x). By Suprunenko's Theorem

$$|\text{Fit } G : ZG| \leqq 16,$$

so we can conclude that

$$\rho(G/ZG) \leqq 6 + \nu(|\text{Fit } G : ZG|) \leqq 6 + 4 = 3n - 2,$$

which completes the proof of Lemma 1.

Proof of Lemma 2. For any finite group H, let $c(H)$ denote the composition length of H. To prove Lemma 2 we use induction or r, and we can assume immediately that $r \geqq 2$. If G is intransitive with orbits $\Omega_1, \ldots, \Omega_s$ then it is a subdirect product of the groups $G|\Omega_1, \ldots, G|\Omega_s$, where $G|\Omega_i$ is transitive on r_i letters, and $\sum_{i=1}^{s} r_i = r$. Hence, by induction,

$$c(G) \leqslant \sum_{i=1}^{s} c(G|\Omega_i) \leqslant \sum_{i=1}^{s} (4/3)(r_i - 1) = (4/3)(r - s).$$

If G is transitive and imprimitive, then there exists an intransitive normal subgroup H of G, with k orbits, and G/H is isomorphic to a subgroup of the symmetric group S_k, for some k satisfying $2 \leqq k \leqq n/2$. Therefore, by induction, and using the result in the intransitive case,

$$c(G) = c(H) + c(G/H) \leqq (4/3)(r - k) + (4/3)(k - 1) = (4/3)(r - 1).$$

If G is primitive, let M be a minimal normal subgroup of G, and $P \neq 1$ a Sylow p-subgroup of M. Since M is a direct product of isomorphic simple groups, P also is a direct product of groups and $c(M) \leqq c(P)$. If $N = N_G(P)$, then $G = MN$ by the Frattini argument, and hence

$$c(G) = c(M) + c(G/M) \leqq c(P) + c(N/(N \cap M)) \leqq c(P) + c(N/P) = c(N).$$

Now if N is intransitive or imprimitive, then $c(G) \leqq c(N) \leqq (4/3)(r - 1)$. If N is primitive, let M_1 be a minimal normal subgroup of N contained in ZP. Then M_1 is an abelian p-group. If N_α is the subgroup of N fixing α, then, since N is primitive, $N = M_1 N_\alpha$ and N_α acts as a permutation group of degree $r - 1$ on the set of letters different from α. If M_1 is cyclic of order p, then

$$c(N) \leqq c(M_1) + c(N_\alpha) \leqq 1 + (4/3)(r - 2) < (4/3)(r - 1),$$

so suppose $|M_1| = p^k$ with $k \geqq 2$. Then $\Phi(M_1) \neq M_1$, and the Frattini subgroup $\Phi(M_1)$ is normal in N, so $\Phi(M_1) = 1$ since M_1 is minimal. Therefore, M_1 is an elementary abelian p-group.

If $C = C_N(M_1)$, then $C \cap N_\alpha$ is normalized by M_1 and N_α, and is therefore normal in N. Consequently, since N_α contains no nontrivial normal subgroup of N, $C \cap N_\alpha = 1$. Now C contains M_1, so $CN_\alpha = M_1N_\alpha = N$, and since C is also contained in M_1 we have $C = M_1$. Hence $N_\alpha \cong N/C$ and

$$r = |N : N_\alpha| = |M_1| = p^k.$$

Since N_α is isomorphic to a subgroup of Aut M_1, and M_1 is elementary abelian of order p^k, $|N_\alpha|$ divides

$$|\mathrm{Aut}\ M_1| = (p^k - 1)(p^k - p) \dots (p^k - p^{k-1}) < p^{k^2}.$$

Consequently, $|G|$ divides $p^k|\mathrm{Aut}\ M_1| < p^{k+k^2}$, and hence $c(G)$ is bounded by $(k + k^2) \log_2 p$. Now it is easily verified that

$$k + k^2 \leqq (4/3)(p^{k-1} - p^{-1})$$

for all p and all $k \geqq 2$, except $p = 2$, $k = 2, 3, 4, 5$, and $p = 3$, $k = 2, 3$. Therefore, except in these cases, we have

$$\begin{aligned} c(G) \leqq (k + k^2) \log_2 p < (4/3)(p^{k-1} - p^{-1})\, p &= (4/3)(p^k - 1) \\ &= (4/3)(r - 1). \end{aligned}$$

We finally consider the exceptional cases. If $p = 2$ and $k = 2$, then G is a subgroup of the symmetric group S_4, and hence the composition length of G is at most $4 = (4/3)(r - 1)$. In the remaining cases, we know that $|G|$ divides

$$p^k|\mathrm{Aut}\ M_1| = p^k(p^k - 1)(p^k - p) \dots (p^k - p^{k-1}),$$

and hence $c(G)$ is bounded by the number of prime divisors of $p^k|\mathrm{Aut}\ M_1|$, counted according to multiplicity. Thus, for $p = 2$, $k = 3, 4, 5$, and for $p = 3$, $k = 2, 3$, it is easily verified that $c(G) < (4/3)(r - 1)$.

To see that the bound can be attained, let $H_0 = S_4$. In general H_{t-1} is a permutation group on 4^t letters and H_t is the wreath product of H_{t-1} and S_4, as defined in [**7**, p. 81]. Thus H_t has a normal subgroup K isomorphic to the direct product of four copies of H_{t-1}, and H_t/K is isomorphic to S_4. Therefore, H_t is a permutation group on 4^{t+1} letters, with one orbit, and by induction on t it is clear that

$$c(G) = 4^{t+1} + 4^t + \dots + 4 = (4/3)(4^{t+1} - 1) = (4/3)(r - 1).$$

The direct product of s copies of H_t is isomorphic to a permutation group on $4^{t+1}s$ letters, with s orbits, and it is easily seen that the composition length of such a group is $s.c(H_t) = (4/3)(r - s)$.

LEMMA 3. *If G is an irreducible polycyclic linear group of degree n over an algebraically closed field F, and the centre ZG is finite, then $\rho(G) \leqq \frac{1}{6}(19n - 8)$.*

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This bound is attained, by suitable finite groups, whenever $n = 2 \cdot 4^k$ for $k = 0, 1, 2, \ldots$.

Remark. If ZG is not finite, then, since it is isomorphic to a multiplicative subgroup of F, it is not necessarily cyclic.

Proof. When $n = 1$ the result holds, so we proceed by induction on n. Since G is irreducible, ZG consists of scalar matrices. Hence, since ZG is finite, by assumption, it is cyclic. Now if G is primitive, then by Lemma 1,

$$\rho(G) \leqq \rho(ZG) + \rho(G/ZG) \leqq 1 + (3n - 2) \leqq \tfrac{1}{6}(19n - 8)$$

for $n \geqq 2$. So suppose that G is imprimitive. Then there exists a reducible normal subgroup H of G, such that H has d irreducible components, each of degree n/d, and G/H is isomorphic to a permutation group on d letters. Consequently, by Lemma 2, $\rho(G/H) \leqq (4/3)(d - 1)$.

Now H is a subdirect product of groups $H_1, \ldots, H_d$. If

$$\prod_{i=1}^{d} H_i = K_0 > K_1 > \ldots > K_t = 1$$

is a polycyclic series of $\prod_{i=1}^d H_i$, then

$$(H \cap K_j)/(H \cap K_{j+1}) \cong (H \cap K_j)K_{j+1}/K_{j+1} \leqq K_j/K_{j+1}.$$

But K_j/K_{j+1} is cyclic for each j, so

$$H = H \cap K_0 \geqq H \cap K_1 \geqq \ldots \geqq H \cap K_t = 1$$

is a polycyclic series of H, and

$$\rho(H) \leqslant \sum_{i=1}^{d} \rho(H_i).$$

Since each H_i $(i = 1, \ldots, d)$ is linear of degree n/d, it follows from the induction assumption that $\rho(H)$ is at most $(d/6)(19(n/d) - 8)$. Consequently

$$\rho(G) \leqslant \frac{d}{6}\left(19\left(\frac{n}{d}\right) - 8\right) + (4/3)(d - 1) = (1/6)(19n - 8).$$

In section 4 *Examples*, we will construct a family of groups to show that this bound can be attained. The construction used also provides examples for the theorem below.

3. Solvable groups. Throughout this section, we consider only groups G that are finite and solvable.

THEOREM. *If G is a finite completely reducible solvable linear group of degree n, then $\rho(G) \leqq \frac{1}{6}(19n - 8s)$, where s is the number of irreducible components of G. This bound can be attained when $n = 2 \cdot 4^k \cdot s$ $\{k = 0, 1, 2, \ldots\}$.*

Proof. G is completely reducible, so there is no loss of generality in assuming that the underlying field is algebraically closed, since the number of irreducible components can only increase if the field is extended [**5**, 2.10].

If G is irreducible, then the result follows from Lemma 3. If G is reducible, then it is a subdirect product of irreducible linear groups $G_1, \ldots, G_s$ of degrees $n_1, \ldots, n_s$ respectively, where $\sum_{i=1}^{s} n_i = n$. Consequently

$$\rho(G) \leqslant \sum_{i=1}^{s} \rho(G_i) \leqslant (1/6)\left(19\left(\sum_{i=1}^{s} n_i\right) - 8s\right) = (1/6)(19n - 8s)$$

by Lemma 3. The construction of examples, to show that this bound can be attained, is deferred until section 4.

The group of all 2×2 upper-triangular matrices with determinant 1, over the field of order p^k, has polycyclic length k. Thus the assumption of complete reducibility, in the Theorem, is essential for the existence of a bound on $\rho(G)$ that depends only on n. However, the following result is immediate.

COROLLARY 1. *If G is a finite solvable linear group of degree n, over a field F of characteristic p, and the Sylow p-subgroups of G have order p^m, then $\rho(G) \leqq \frac{1}{6}(19n - 8) + m$.*

Proof. By [**5**, Theorems 2.4 and 2.8C], there exists a normal p-subgroup N of G, such that G/N is isomorphic to a completely reducible linear group, of degree n, over F. Thus, since $\rho(N) \leqq m$, it follows from the theorem that

$$\rho(G) \leqq \rho(N) + \rho(G/N) \leqq m + \tfrac{1}{6}(19n - 8).$$

The following result is an easy consequence of Corollary 1, and is, perhaps, of independent interest.

COROLLARY 2. *Let G be a finite solvable group. If p is a prime dividing the order of G, and the Sylow p-subgroups of G have order p^m, then*

$$\rho(G/0_{p'}(G)) \leqq \tfrac{1}{6}(25m - 8).$$

Proof. Let P be a maximal normal p-subgroup of $G_1 = G/0_{p'}(G)$, and $|P| = p^{m_1}$. Thus $m_1 \neq 0$. Now G_1/P has a faithful representation on $P/\Phi(P)$, as a vector space of dimension $\leqq m_1$, over the field of order p [**6**, 6.3.4]. Corollary 1 therefore implies that

$$\rho(G_1/P) \leqq \tfrac{1}{6}(19m_1 - 8) + (m - m_1),$$

and since $\rho(P) \leqq m_1$, it follows that

$$\rho(G/0_{p'}(G)) \leqq \tfrac{1}{6}(19m_1 - 8) + m \leqq \tfrac{1}{6}(25m - 8).$$

4. Examples. There exists a finite solvable primitive linear group H_0, of degree 2 over the complex numbers, having a finite (and therefore cyclic) centre, and $|H_0| = 24|ZH_0|$, [**2**, § 57, D]. In fact, H_0/ZH_0 is isomorphic to the group of all rotations of the octahedron, which is S_4. Therefore $\rho(H_0) = 5 = \frac{1}{6}(19n - 8)$.

Assuming that we have defined the group H_k, and that it is irreducible with degree $2 \cdot 4^k$ and polycyclic length $\frac{1}{6}(19(2 \cdot 4^k) - 8)$, let $v_1^i, \ldots, v_{2.4^k}^i$ be a basis for a complex vector space V_i and

$$W_s = V_1 \oplus \cdots \oplus V_s.$$

By defining the action of an isomorph of H_k on each V_i, one obtains a completely reducible linear group H_k^s of degree $2 \cdot 4^k \cdot s$ and polycyclic length $\frac{1}{6}(19(2 \cdot 4^k \cdot s) - 8s)$.

For each permutation π on the set $\{1, 2, 3, 4\}$ one can define a linear transformation

$$\varphi_\pi : W_4 \to W_4 \quad \text{by} \quad \varphi_\pi(v_j{}^i) = v_j{}^{\pi i}$$

for all j and all i. Define H_{k+1} to be the subgroup of $\mathrm{GL}(W_4)$ which is generated by H_k^4 and all of the φ_π $(\pi \in S_4)$. Then H_{k+1} is solvable, imprimitive and has degree $2 \cdot 4^{k+1}$ and polycyclic length

$$4(\tfrac{1}{6}(19(2 \cdot 4^k) - 8)) + 4 = \tfrac{1}{6}(19(2 \cdot 4^{k+1}) - 8).$$

This gives an inductive construction of the examples required for Lemma 3, and the Theorem.

5. Nilpotent groups. In this section we restrict our attention even further and consider only finite nilpotent linear groups. We will use the following Lemma.

LEMMA 4. *If G is a finite completely reducible linear p-group of degree n, with s irreducible components, then*

$$\rho(G) \leqq (p - 1)^{-1} (pn - s) \leqq 2n - 1.$$

Proof. Since G is completely reducible, we can assume, without loss of generality, that the underlying field is algebraically closed [**5**, 2.8c]. We can also assume that $n > 1$.

If G is irreducible, then the underlying field is not of characteristic p [**8**, Theorem 10]. Therefore, since G is finite, $n = p^k$ for some integer $k > 0$ [**5**, 5.2], and we proceed by induction on k. It is well-known that, since G is a finite linear p-group of degree $n \neq 1$ over an algebraically closed field, it has a reducible normal subgroup H of index p (see for example, [**5**, 2.6 Exercise 5]). By Clifford's theorem [**2**, 49.2] H has p irreducible components, each of degree p^{k-1}. Therefore, by the induction hypothesis,

$$\begin{aligned} \rho(G) &\leqq 1 + \rho(H) \leqq 1 + p(p - 1)^{-1}(p \cdot p^{k-1} - 1) \\ &= (p - 1)^{-1}(pn - 1). \end{aligned}$$

If G is reducible, then it is a subdirect product of groups $G_1, \ldots, G_s$ where, for each i, G_i is an irreducible linear p-group of degree p^{k_i}, and $\sum_{i=1}^s p^{k_i} = n$.

Now, by induction,

$$\rho(G_i) \leqq (p-1)^{-1}(p \cdot p^{k_i} - 1)$$

for each i and, as we saw in the proof of Lemma 3,

$$\rho(G) \leqslant \sum_{i=1}^{s} \rho(G_i),$$

so it follows that

$$\rho(G) \leqslant (p-1)^{-1}\left(p\left(\sum_{i=1}^{s} p^{k_i}\right) - s\right) = (p-1)^{-1}(pn - s).$$

The following result is almost a triviality, and we omit the proof.

LEMMA 5. *Suppose that the finite solvable group G is the direct product of groups $G_i (i = 1, \ldots, s)$. If $|G_i|$ and $|G_j|$ are relatively prime for all i and j, $i \neq j$, then*

$$\rho(G) \leqslant \max_{i=1,\ldots,s} \rho(G_i).$$

THEOREM. *Let G be a finite nilpotent linear group of degree n, over a field F. If the characteristic of F is zero, or does not divide the order of G, then $\rho(G) \leqq 2n - 1$. If F has characteristic p, and the Sylow p-subgroup of G is of order p^m, then $\rho(G) \leqq \max(m, 2n - 1)$.*

Proof. If F has characteristic zero, or not dividing $|G|$, then G is completely reducible, by Maschke's theorem. The result then follows, from Lemmas 4 and 5, since G is the direct product of its Sylow subgroups. In the second case, the polycyclic length of the Sylow p-subgroup of G is clearly at most m, and the remaining Sylow subgroups are completely reducible, so the result again follows immediately, from Lemmas 4 and 5.

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